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Hypothesis Testing Based on Lagrange's Method:
Application to The Uniform Distribution.

by

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HYPOTHESIS TESTING BASED ON LAGRANGE'S METHOD:
APPLICATION TO THE UNIFORM DISTRIBUTION.

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Abstract. In this paper we deal with the uniform distribution as follows:

$$f(x|\theta)=\theta^{-1} \quad (0 < x < \theta; 0 < \theta).$$

The author proposes the tests which essentially have the acceptance regions derived from inverting the conditional minimum-length(CML) interval estimates for the function $\ln \theta$ of the parameter θ based on the Lagrange's method. She proposes the two-sided test for testing the hypothesis $H_0: \theta = \theta_0$ versus the alternative hypothesis $H_1: \theta \neq \theta_0$ with a positive constant θ_0 . Her test is unbiased. She also propose the uniformly most powerful one-sided test for testing $H'_0: \theta \leq \theta_0$ versus $H'_1: \theta > \theta_0$.

§1. Introduction.

The idea for the relation between the tests and interval estimates is seen in Neyman(1937) and recently, for example, Matusita(1951) and Matubara & Nogami (1982). For the hypothesis testing based on Lagrange's method we refer to Nogami(2002). Let $I_A(x)$ be the indicator function of the interval A such that $I_A(x)=1$ if $x \in A$; $=0$ if $x \notin A$. Here, we consider to test the parameter θ of the uniform distribution

$$(1) \quad f(x|\theta) = \theta^{-1} I_{(0, \theta)}(x) \quad (\theta > 0).$$

Let X_1, \dots, X_n be a random sample of size n taken from (1). We apply the similar analysis appeared in Sections 2.3 and 2.4 and Section 3 of Nogami(2002).

In Section 2 we consider the problem for testing the hypothesis $H_0: \theta = \theta_0$ versus the alternative hypothesis $H_1: \theta \neq \theta_0$ with a positive constant θ_0 . Let ξ be the defining property. Let $\theta^* = \ln \theta$ and $Y = \ln X$. We estimate θ^* by the unbiased estimate $U = \bar{Y} + 1 = n^{-1} \sum_{i=1}^n Y_i + 1$ and construct the conditional minimum-length (CML) interval estimate for θ^* based on the Lagrange's method. Then, we apply this interval estimate to get the two-sided test and show that our test is unbiased. Let α be a real number such that $0 < \alpha < 1$. For a reference on this problem there is a uniformly most powerful (UMP) test of size α in Ferguson(1967, p. 213) (as well as Lehmann(1986, p. 111)). However, this test cannot be applied to the test of $H'_0: \theta \leq \theta_0$ versus $H'_1: \theta > \theta_0$ because it takes probability of size α from the lower tail only.

In Section 3 we propose the one-sided unbiased test of $H'_0: \theta \leq \theta_0$ versus $H'_1: \theta > \theta_0$. As references for this problem we refer to Mood, Graybill and Boes(1988, p. 424) (as well as Ferguson(1967, p. 213) for a randomized test and Lehmann (1986, p. 111)). Since our test has the same power as that in Mood, Graybill & Boes(1988, p. 424) for $\theta > \theta_0$, our test is also UMP and of size α .

We call (S_1, S_2) a $(1-\alpha)$ interval estimate for the parameter η if $P_\eta[S_1 < \eta < S_2] = 1-\alpha$.

§2. The unbiased two-sided test.

Let $U = \bar{Y} + 1$. Let $Y_{(1)}$ be the smallest observation of Y_1, \dots, Y_n . Let $W = \sum_{i=1}^n Y_{(i)}$ ($= \sum_{i=1}^n Y_i$) and $V = Y_{(n)}$. We first find the density $h_{W, V}(w, v)$ of (W, V) .

Then, we find the density $g_W(w)$ of W . Furthermore, letting $T=2n(\theta^*+1-U)$ we obtain the density $h_T(t)$ of T to get the CML $(1-\alpha)$ interval estimate for θ^* based on U .

First of all we find the density of Y as follows:

$$(2) \quad g_Y(y) = \exp\{y - \theta^*\} I_{(-\infty, \theta^*)}(y).$$

Since, from (2), $g_{Y_1, \dots, Y_n}(Y_1, \dots, Y_n) = \exp\{\sum_{i=1}^n Y_i - n\theta^*\} I_{(-\infty, \theta^*)}(v)$, we can find the joint density of $W, V, Z_2=Y_{(2)}, Z_3=Y_{(3)}, \dots$, and $Z_{n-1}=Y_{(n-1)}$ as follows:

$$(3) \quad h(w, v, z_2, \dots, z_{n-1}) = n! \exp\{w - n\theta^*\},$$

for $-\infty < w - v - \sum_{i=2}^n z_i \leq z_2 \leq \dots \leq z_{n-1} \leq v \leq \theta^*$. To get $h_{W, V}(w, v)$ we integrate out (3) with respect to z_2, \dots, z_{n-1} . Then, we obtain

$$(4) \quad h_{W, V}(w, v) = \{n/\Gamma(n-1)\} \exp\{-(n\theta^* - w)\} (nv - w)^{n-2}, \quad \text{for } w \leq nv \leq n\theta^*.$$

Taking the marginal density $g_W(w)$ of W we have

$$(5) \quad g_W(w) = (\Gamma(n))^{-1} \exp\{-(n\theta^* - w)\} (n\theta^* - w)^{n-1}, \quad \text{for } w \leq n\theta^*.$$

Using a variable transformation $T=2n(\theta^*+1-U)=2\{n(\theta^*+1)-W\}$ we get, from (5), the density of T as follows:

$$(6) \quad h_T(t) = (\Gamma(n))^{-1} t^{n-1} e^{-t/2} 2^{-n} I_{(0, \infty)}(t).$$

Let r_1 and r_2 be real numbers such that $r_1 < r_2$. To find the CML $(1-\alpha)$ interval estimate for θ^* we want to minimize $r_2 - r_1$ subject to

$$(7) \quad P_\theta[r_1 < U - \theta^* < r_2] = 1 - \alpha.$$

But, by a variable transformation $t=2n(\theta^*+1-u)$ (7) is equal to

$$(8) \quad P[t_1 < T < t_2] = 1 - \alpha$$

with $t_1 = 2n(1 - r_2)$ and $t_2 = 2n(1 - r_1)$. Hence, we want to minimize $t_2 - t_1$ subject to the condition (8). Let λ be a Lagrange's multiplier and define

$$L = t_2 - t_1 - \lambda \left\{ \int_{t_1}^{t_2} h_T(t) dt - 1 + \alpha \right\}.$$

Then, $\partial L / \partial t_1 = 0 = \partial L / \partial t_2$ leads to

$$(9) \quad h_T(t_1) = h_T(t_2) \quad (= \lambda^{-1}).$$

Taking t_1 and t_2 which satisfy (9) and $\partial L / \partial \lambda = 0$, noticing that $r_1 = 1 - t_2 / (2n)$ and $r_2 = 1 - t_1 / (2n)$ we obtain the CML $(1 - \alpha)$ interval estimate for θ^* as follows:

$$(10) \quad (U - 1 + t_1 / (2n), U - 1 + t_2 / (2n)).$$

Hence, by letting $u_1^0 = \theta_0^* + 1 - t_2 / (2n)$ and $u_2^0 = \theta_0^* + 1 - t_1 / (2n)$ and inverting (10) for θ_0^* our test is to reject H_0 if $U \leq u_1^0$ or $u_2^0 \leq U$ or $\theta_0^* < V$ and to accept H_0 if $u_1^0 < U < u_2^0$ and $V \leq \theta_0^*$.

To check unbiasedness of this test we use (4) and obtain the power of the test as follows:

$$\pi(\theta) = P_\theta [U \leq u_1^0 \text{ or } u_2^0 \leq U \text{ or } \theta_0^* < V]$$

$$= P_\theta [\theta_0^* < V] + P_\theta [W \leq n\theta_0^* - 2^{-1}t_2 \text{ and } V \leq \theta_0^*] + P_\theta [n\theta_0^* - 2^{-1}t_1 \leq W \text{ and } V \leq \theta_0^*]$$

$$(11) \quad \pi(\theta) = \begin{cases} 1 - (1 - \alpha)(\theta_0 / \theta)^n, & \text{for } \theta_0 < \theta, \\ \int_0^{2n(\theta^* - \theta_0^*) + t_1} h_T(t) dt + \int_{2n(\theta^* - \theta_0^*) - t_2}^{\infty} h_T(t) dt, & \text{for } \theta_0 \exp\{-t_1 / (2n)\} < \theta \leq \theta_0, \\ 0, & \text{for } \theta < \theta_0 \exp\{-t_1 / (2n)\}. \end{cases}$$

$$\left\{ \begin{array}{ll} \int_{2n(\theta^* - \theta_0^*) + t_2}^{\infty} h_T(t) dt, & \text{for } \theta_0 \exp\{-t_2/(2n)\} < \theta \leq \theta_0 \exp\{-t_1/(2n)\}, \\ 1, & \text{for } 0 < \theta \leq \theta_0 \exp\{-t_2/(2n)\}. \end{array} \right.$$

Hence, $d\pi(\theta)/d\theta > 0$ for $\theta_0 < \theta$, $d\pi(\theta)/d\theta = 2n\theta^{-1} \{h_T(2n(\theta^* - \theta_0^*) + t_1) - h_T(2n(\theta^* - \theta_0^*) + t_2)\} < 0$ for $\theta_0 \exp\{-t_1/(2n)\} < \theta < \theta_0$ and $d\pi(\theta)/d\theta < 0$ for $\theta_0 \exp\{-t_2/(2n)\} < \theta \leq \theta_0 \exp\{-t_1/(2n)\}$. The second inequality above follows because of (9) and (6). Thus, we have $\pi(\theta) \geq \alpha = \pi(\theta_0)$ for real θ . Therefore, unbiasedness of the test is proved.

From the construction it is easily seen from (7) that our test is of size α .

We note that there is a UMP-size α test in Ferguson(1976, p. 213) (as well as Lehmann(1986, p. 111)) for this problem. The power of this test is the same as (11) for $\theta_0 < \theta$. However, since most of the time we have $\exp\{-2^{-1}t_2\} < \alpha^{1/n}$, our power for $\theta_0 \exp\{-t_2/(2n)\} < \theta < \theta_0 \alpha^{1/n}$ is no better than Ferguson(1976, p. 213). However, As I stated in Section 1, this (his) test is not applicable for the test of $H'_0: \theta \leq \theta_0$ versus $H'_1: \theta_0 < \theta$. In the next section we show that our test of $H'_0: \theta \leq \theta_0$ versus $H'_1: \theta_0 < \theta$ is UMP and of size α .

§3. The UMP one-sided test.

In this section we first consider the test of $H'_0: \theta \leq \theta_0$ versus $H'_1: \theta_0 < \theta$. As in Section 2 we let $\theta^* = \ln \theta$, $U = \bar{Y} + 1$ and $V = Y_{(n)}$. We furthermore define $u_2^* = \theta_0^* + 1 - t_1/(2n)$ where t_1 here is defined by

$$(12) \quad P[T < t_1] = \alpha.$$

Then, our one-sided test is to reject H'_0 if $u_2^* \leq U$ or $\theta_0^* < V$ and to accept H'_0 if $U < u_2^*$ and $V \leq \theta_0^*$. From Section 2 we can easily get the power of the test as follows:

$$\pi(\theta) = P_\theta [u_2^* \leq U \text{ or } \theta_0^* < V]$$

$$= \begin{cases} 1-(1-\alpha)(\theta_0/\theta)^n, & \text{for } \theta_0 < \theta \\ \int_{2n(\theta^*-\theta_0^*)+t_1}^{\infty} h_T(t) dt, & \text{for } \theta_0 \exp\{-t_1/(2n)\} < \theta \leq \theta_0 \\ 0, & \text{for } 0 < \theta \leq \theta_0 \exp\{-t_1/(2n)\}. \end{cases}$$

Since $d\pi(\theta)/d\theta > 0$ for $\theta_0 < \theta$ and $d\pi(\theta)/d\theta = 2n\theta^{-1}h_T(2n(\theta^*-\theta_0^*)+t_1) > 0$ for $\theta < \theta_0$, $\pi(\theta_0) = \alpha \leq \pi(\theta)$ for real θ such that $\theta_0 < \theta$. Hence, this test is unbiased.

It is immediate from (12) that our test is of size α .

Historically, there is a randomized test in Ferguson(1976, p. 213 #7(c)) which is better than our test in the sense of the power. However, it is more natural to compare our test with the test appeared in Mood, Graybill & Boes (1988, p.424) which rejects H_0 if $V > \theta_0(1-\alpha)^{1/n}$ and accepts H_0 if $V \leq \theta_0(1-\alpha)^{1/n}$. This test has the same power as our power for $\theta_0 < \theta$. Hence, our test is also UMP and of size α .

§ 4. Remark.

The term "interval estimate" used in this paper is due to Fabian & Hannan (1985).

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