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SUBDIVISIONS AND TRIANGULATIONS INDUCED BY A PAIR OF SUBDIVIDED MANIFOLDS

Yoshitsugu YAMAMOTO

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University of Tsukuba
Sakura, Ibaraki 305
JAPAN

Abstract

We consider methods for constructing subdivisions of subsets in $R^n \times [0,1]$ from a pair of subdivided manifolds L and M one of which lies in $R^n \times \{1\}$ and the other in $R^n \times \{0\}$. We introduce a one-to-one correspondence between cells (including lower dimensional cells) of L and cells of M, and make a join of cerresponding cells to obtain a new cell of higher dimension. We also show that any simplicial refinements of L and M provide triangulations of a certain subset of $R^n \times [0,1]$ under some conditions. The class of triangulations thus obtained embraces various known triangulations, especially triangulations used in interpretations of some variable dimension algorithms.

Key words: triangulations, fixed points, subdivided manifolds

Author's address: Institute of Socio-Economic Planning, University of Tsukuba, Sakura, Ibaraki 305, JAPAN

1. INTRODUCTION

Since the variable dimension algorithm was originally proposed by van der Laan and Talman [4 ∿ 10] a lot of interpretations and outgrowthes of the algorithm have been presented. Todd [12, 13], Todd and Wright [14] and Wright [15] have shown that some variable dimension algorithms can be viewed as following a path of zeroes of a piecewise linear map from R^{n+1} to R^n by constructing a triangulation of a certain subset of $R^n \times [0, 1]$. Kojima and Yamamoto [2, 3] have introduced a primal-dual pair of subdivided manifolds and by utilizing it presented several new variable dimension algorithms as well as a unifying interpretation of existing variable dimension algorithms. They also presented a new method of generating triangulations of $R^n \times (0, 1]$ with arbitrary reduction factor of mesh size. The class of triangulations generated includes J_3 and K_3 (see Todd [11]). Independently, Bárány [1] provided the same idea of generating triangulations.

In this paper we shall present a sufficient condition under which we can construct a subdivision and a triangulation in $R^n \times [0, 1]$ from a pair of subdivided manifolds in R^n and their simplicial refinements. In section 3 we introduce paired subdivided manifolds and general position, which is slightly different from that in Bárány [1]. In section 4 two additional conditions are imposed to guarantee the desired property. In section 5 we show that any simplicial refinements of paired subdivided manifolds satisfying the conditions imposed in the previous sections induce a triangulation of a certain subset of $R^n \times [0, 1]$ The last section is devoted to illustration of some examples.

2. PRELIMINARIES

A convex polyhedral set C in some Euclidean space R^n is called a cell and its dimension, denoted by $\dim C$, is defined to be the dimension of $\text{aff}(C)$, the affine subspace spanned by C. We denote the linear subspace $\text{aff}(C) - C = \{x - y : x \in \text{aff}(C), y \in C\}$ by $\text{tng}(C)$. $B \subseteq C$ is said to be a face of a cell C if B is a convex subset of C such that $x, y \in C$, $\alpha \in (0, 1)$ and $\alpha x + (1 - \alpha)y \in B$ always imply that $x, y \in B$. We also employ a covention that an empty set is a face of any cell. We write $B < C$ if B is a face of C.

Let L be a finite or countable collection of m dimensional cells (abbreviated by m-cells) in R^n. We denote the collection $\{B : B < C \text{ for some } C \in L\}$ by $\overline{L}$. L is called a subdivided manifold if the following conditions are satisfied:

(2.1) For every pair $B, C \in L$, $B \cap C$ is a face of B and C.

(2.2) Each $(m-1)$-cell of $\overline{L}$ lies in at most two m-cells of L.

The condition (2.2) together with the definition of $\overline{L}$ implies that every $(m-1)$-cell of $\overline{L}$ lies in one or two m-cells of L. Then the boundary ∂L of L is defined to be a collection of all $(m-1)$-cells of $\overline{L}$ which lie in exactly one m-cell of L. Let L and L' be subdivided m-manifolds in R^k with $|L| = |L'|$, where $|L| = \cup\{C : C \in L\}$. If each m-cell $C' \in L'$ is contained in some m-cell $C \in L$, L' is said to be a refinement of L. If, in addition, all cells of L' are simplices, the refinement L' is said to be simplicial.

Let X and Y be subsets of R^n. We define the join XY of X and Y as $XY = \{\lambda x + (1-\lambda)y : x \in X, y \in Y, \lambda \in [0, 1]\}$ with the convention that $XY = X$ if $Y = \phi$. In this paper we shall also abbreviate $(X \times \{1\})(Y \times \{0\})$ by $X \# Y$.

<u>Lemma 2.1.</u> Let X and Y be cells in R^n. Then

$$Z = cl(X \# Y)$$

is a cell in $R^n \times [0, 1]$, where cl means closure.

Proof. Let X' (or Y') and X'' (or Y'') be the convex hull of all extreme points of X (or Y) and the set of all rays in X (or Y). Define a subset W of R^{n+1} as

$$W = \left\{ w \in R^{n+1} : \begin{array}{l} w = \lambda\begin{pmatrix} x' \\ 1 \end{pmatrix} + (1-\lambda)\begin{pmatrix} y' \\ 0 \end{pmatrix} + \begin{pmatrix} x' \\ 1 \end{pmatrix} + \begin{pmatrix} y'' \\ 0 \end{pmatrix} \\ \text{for some } x' \in X',\ x'' \in X'',\ y' \in Y',\ y'' \in Y'' \\ \text{and } \lambda \in [0, 1] \end{array} \right\},$$

then W is a cell in R^{n+1}. We shall show that $W = Z$. Since Z is clearly contained in W, it suffices to show the reverse relation. Let w be an arbitrary point in W.

Case 1: Suppose that $x'' = 0$ ($y'' = 0$) whenever $\lambda = 0$ ($\lambda = 1$). Then w can be rewritten as

$$w = \lambda(\begin{pmatrix} x' \\ 1 \end{pmatrix} + (1/\lambda)\begin{pmatrix} x'' \\ 1 \end{pmatrix}) + (1-\lambda)(\begin{pmatrix} y' \\ 0 \end{pmatrix} + (1/(1-\lambda))\begin{pmatrix} y'' \\ 0 \end{pmatrix}),$$

which is clearly in Z.

Case 2: Suppose that $\lambda = 0$ and $x'' \neq 0$. Then w can be rewritten as

$$w = \begin{pmatrix} y \\ 0 \end{pmatrix} + \begin{pmatrix} x'' \\ 1 \end{pmatrix},$$

for some $y \in Y$ and $x'' \in X''$. Let x be an arbitrary point in X, and let

$$w(\varepsilon) = (1 - \varepsilon)w + \varepsilon\begin{pmatrix} x \\ 1 \end{pmatrix}$$

for $\varepsilon \in (0, 1)$. Then

$$\begin{aligned}\begin{pmatrix} y \\ 0 \end{pmatrix} + (1/\varepsilon)\left(w(\varepsilon) - \begin{pmatrix} y \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} x \\ 1 \end{pmatrix} + ((1/\varepsilon) - 1)\left(w - \begin{pmatrix} y \\ 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} x \\ 1 \end{pmatrix} + ((1/\varepsilon) - 1)\begin{pmatrix} x'' \\ 1 \end{pmatrix} \in X \times \{1\}.\end{aligned}$$

Hence for any $0 < \varepsilon < 1$, $w(\varepsilon)$ is in $X \# Y$, which implies the desired reverse relation.

The case where $\lambda = 1$ and $y'' \neq 0$ can be proved in exactly the same way. Q.E.D.

3. PAIRED SUBDIVIDED MANIFOLDS IN GENERAL POSITION

Let X and Y be cells in R^n. We say that X and Y are in general position if $\text{tng}(X) \cap \text{tng}(Y) = \{0\}$. This definition is slightly different from that in Bárány [1], which has a drawback that the general position property introduced is not invariant under the transfer of origin.

For two cells X and Y in general position let

$$Z = X \# Y,$$

then it is straightforward to see that

$$\dim Z = \dim X + \dim Y + 1$$

and that for each $z \in Z$ a pair of points $x \in X$ and $y \in Y$, and a scalar λ satisfying

$$z = \lambda \begin{pmatrix} x \\ 1 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

are uniquely determined in the sense that x (or y) is unique whenever $\lambda > 0$ (or $\lambda < 1$). As shown in Lemma 2.1 Z is not always a cell; we, however, define its face in the same way as for cells, and also use the symbol $<$.

<u>Lemma 3.1.</u> Let $X, Y \subseteq R^n$ be cells in general position. Then a convex set $Z \subseteq R^{n+1}$ is a face of $X \# Y$ if and only if $Z = X' \# Y'$ for some $X' < X$ and $Y' < Y$.

Proof. We first show "only if" part. If $Z \subseteq R^n \times \{0, 1\}$, the assertion is immediate. Then we shall consider the case where $Z \nsubseteq R^n \times \{0, 1\}$.

Let X' and Y' be subsets of R^n such that

$$X' \times \{1\} = Z \cap (R^n \times \{1\}) \subseteq X \times \{1\}$$

$$Y' \times \{0\} = Z \cap (R^n \times \{1\}) \subseteq Y \times \{0\},$$

then X' and Y' are clearly convex sets of X and Y, respectively. We shall show that $X' < X$ and $Y' < Y$. Suppose $x \in X'$ has a representation of the form

$$x = \alpha x^1 + (1 - \alpha)x^2$$

for some $x^1, x^2 \in X$ and $\alpha \in (0, 1)$. Since $\binom{x}{1} \in Z$ and $\binom{x^1}{1}, \binom{x^2}{1} \in X \# Y$, we obtain that $\binom{x^1}{1}, \binom{x^2}{1} \in Z$, or equivalently $x^1, x^2 \in X^1$. This implies that $X' < X$. We have $Y' < Y$ in exactly the same way.

By the convexity of Z and the definitions of X' and Y' it is clear that

$$Z \supseteq X' \# Y'.$$

Let z be an arbitrary point in Z, then there are $x \in X$, $y \in Y$ and $\lambda \in [0, 1]$ such that

$$z = \lambda\binom{x}{1} + (1 - \lambda)\binom{y}{0}.$$

Hence we have that

$$z = \begin{cases} \binom{x}{1} \in Z \cap (R^n \times \{1\}) = X' \times \{1\} \subseteq X' \# Y' & \text{if } \lambda = 1, \\ \binom{y}{1} \in Z \cap (R^n \times \{0\}) = Y' \times \{0\} \subseteq X' \# Y' & \text{if } \lambda = 0. \end{cases}$$

If $\lambda \in (0, 1)$, since $Z < X \# Y$, both $\begin{matrix} x \\ 1 \end{matrix}$ and $\begin{matrix} y \\ 0 \end{matrix}$ must be in Z, or equivalently $x \in X'$ and $y \in Y'$. Therefore $Z \subseteq X' \# Y'$.

To show the "if" part, now suppose $z \in Z$ has a representation of the form

$$z = \lambda z^1 + (1 - \lambda)z^2$$

for some $z^1, z^2 \in X \# Y$ and $\lambda \in (0, 1)$. Since there exist x^k, y^k and α_k such that

$$z^k = \alpha_k \begin{pmatrix} x^k \\ 1 \end{pmatrix} + (1 - \alpha_k) \begin{pmatrix} y^k \\ 0 \end{pmatrix}$$

$$x^k \in X, \quad y^k \in Y, \quad \alpha_k \in [0, 1]$$

for $k = 1, 2$, z is rewritten as

$$z = \{\lambda\alpha_1 \begin{pmatrix} x^1 \\ 1 \end{pmatrix} + (1 - \lambda)\alpha_2 \begin{pmatrix} x^2 \\ 1 \end{pmatrix}\} + \{\lambda(1 - \alpha_1)\begin{pmatrix} y^1 \\ 0 \end{pmatrix} + (1 - \lambda)(1 - \alpha_2)\begin{pmatrix} y^2 \\ 0 \end{pmatrix}\}.$$

Let $\beta = \lambda\alpha_1 + (1 - \lambda)\alpha_2$. It is clear that z^1 and z^2 lie in Z if $\beta = 0$ or 1. Now suppose $\beta \in (0, 1)$, then z is rewritten as

$$z = \beta(\gamma \begin{pmatrix} x^1 \\ 1 \end{pmatrix} + (1 - \gamma)\begin{pmatrix} x^2 \\ 1 \end{pmatrix}) + (1 - \beta)(\delta\begin{pmatrix} y^1 \\ 0 \end{pmatrix} + (1 - \delta)\begin{pmatrix} y^2 \\ 0 \end{pmatrix}),$$

where

$$\gamma = \lambda\alpha_1/\beta,$$

$$\delta = \lambda(1 - \alpha_1)/(1 - \beta).$$

From the uniqueness of the expression of z, we obtain that

$$\gamma x^1 + (1 - \gamma)x^2 \in X',$$

$$\delta y^1 + (1 - \delta)y^2 \in Y'.$$

Then

$$x^k \in X' \quad \text{whenever} \quad \alpha_k > 0,$$

$$y^k \in Y' \quad \text{whenever} \quad \alpha_k < 1.$$

Hence $z^k \in X' \# Y' = Z$ for $k = 1, 2$. This implies $Z < X \# Y$.

Q.E.D.

Now let L and M be subdivided manifolds in R^n and m be a nonnegative integer. We say that a triplet $(L, M; c)$ is paired subdivided manifolds with degree m if the following conditions (3.1) ∿ (3.3) are satisfied.

(3.1) For any $X \in \overline{L}$, $X^c \in \overline{M}$.

(3.1)' For any $Y \in \overline{M}$, $Y^c \in \overline{L}$.

(3.2) If $Z \in \overline{L} \cup \overline{M}$ and $Z^c \neq \phi$, then $(Z^c)^c = Z$.

(3.3) If $Z \in \overline{L} \cup \overline{M}$ and $Z^c \neq \phi$, then $\dim Z + \dim Z^c = m$.

Note that the condition (3.1) does not exclude the case where $X^c = \phi$ and $Y^c = \phi$. If, in addition,

(3.4) Z and Z^c are in general position for any $Z \in \overline{L} \cup \overline{M}$ with $Z^c \neq \phi$,

we say $(L, M; c)$ to be in general position. For paired subdivided manifolds in general position $(L, M; c)$, define a collection of (m+1)-dimensional convex sets

$$N = \{X \# X^c : X \in \overline{L},\ X^c \neq \phi\}$$

and its k-skeleton

be two parallel hyperplanes satisfying (4.1) and (4.2). Letting

$$\hat{a} = (a^T, b_0 - a_0)^T \in R^{n+1}$$

and consider the hyperplane

$$L = \{z \in R^{n+1}: \hat{a}^T z = b_0\}.$$

Suppose $z \in (X_1 \# X_1^c) \cap (X_2 \# X_2^c)$. Then we find $x^1 \in X_1$, $y^1 \in X_1^c$, $x^2 \in X_2$, $y^2 \in X_2^c$ and $\lambda \in [0, 1]$ such that

$$z = \lambda\begin{pmatrix} x^1 \\ 1 \end{pmatrix} + (1 - \lambda)\begin{pmatrix} y^1 \\ 0 \end{pmatrix} = \lambda\begin{pmatrix} x^2 \\ 1 \end{pmatrix} + (1 - \lambda)\begin{pmatrix} y^2 \\ 0 \end{pmatrix}.$$

Multiply both sides by $\hat{a}$, then we have

$$\lambda a^T(x^1 - x^2) + (1 - \lambda)a^T(y^1 - y^2) = 0.$$

Hence $a^T x^1 = a^T x^2 = a_0$ whenever $\lambda > 0$ and $a^T y^1 = a^T y^2 = b_0$ whenever $\lambda < 1$. Therefore we can conclude

$$z \in (X_1 \cap X_2) \# (X_1^c \cap X_2^c).$$

Since it is clear that

$$(X_1 \cap X_2) \# (X_1^c \cap X_2^c) \subseteq (X_1 \# X_1^c) \cap (X_2 \# X_2^c),$$

we have proved the lemma. Q.E.D.

The readers might have the criticism that the separation condition is too stringent as a sufficient condition for (4.3). Needless to say the separation condition is not necessary to guarantee (4.3) alone; it is however, necessary condition in the sense of the lemma below.

Lemma 4.2. Let X_1, X_2, Y_1 and Y_2 be cells in R^n and $Z_1 = X_1 \# Y_1$, $Z_2 = X_2 \# Y_2$. Suppose Z_1 and Z_2 are cells in R^{n+1} and

$$Z_1 \cap Z_2 < Z_1 \text{ and } Z_2$$

holds. Then there exist parallel heperplanes H and K satisfying (4.1) $\sim$ (4.2).

Proof. Let $L = \{x \in R^{n+1}: a^T x = a_0\}$ be a separating hyperplane of Z_1 and Z_2 such that

$$Z_1 \subseteq L^+, \quad Z_2 \subseteq L^-,$$

$$Z_1 \cap Z_2 = L \cap Z_1 = L \cap Z_2$$

(see Lemma A.1 and A.2 in Appendix for the existence of L), where $a = (a_1, a_2, \ldots, a_n, a_{n+1})^T \in R^{n+1}$. Now define two parallel hyperplanes in R^n:

$$H = \{x \in R^n: (a_1, a_2, \ldots, a_n)x = a_0 - a_n + 1\},$$

$$K = \{x \in R^n: (a_1, a_2, \ldots, a_n)x = a_0\}.$$

Then it is straightforward to see that

$$X_1 \subseteq H^+, \quad X_2 \subseteq H^-,$$

$$Y_1 \subseteq K^+, \quad Y_2 \subseteq K^- .$$

Furthermore

$$(X_1 \cap X_2) \times \{1\} = Z_1 \cap Z_2 \cap R^n \times \{1\} = L \cap Z_i \cap R^n \times \{1\} = H \times \{1\} \cap Z_i,$$

$$(Y_1 \cap Y_2) \times \{0\} = Z_1 \cap Z_2 \cap R^n \times \{0\} = L \cap Z_i \cap R^n \times \{0\} = K \times \{0\} \cap Z_i.$$

for $i = 1, 2$. Thus

$$X_1 \cap X_2 = H \cap X_i, \quad Y_1 \cap Y_2 = K \cap Y_i \quad \text{for} \quad i = 1, 2. \qquad \text{Q.E.D.}$$

<u>Lemma 4.3.</u> Let $(L, M; c)$ be paired subdivided manifolds and $N = \{X \# X^c: X \in \overline{L}, X^c \neq \phi\}$. If $(L, M; c)$ is in general position and satisfies the separation condition, then

$$Z_1 \cap Z_2 < Z_1 \text{ and } Z_2$$

for any pair $Z_1, Z_2 \in N$.

Proof. Let $X_1, X_2 \in \overline{L}$ be cells satisfying

$$Z_1 = X_1 \# X_1{}^c, \quad Z_2 = X_2 \# X_2{}^c,$$

then by Lemma 4.1 we have

$$Z_1 \cap Z_2 = (X_1 \cap X_2) \# (X_1{}^c \cap X_2{}^c).$$

If we note $X_1 \cap X_2 < X_1, X_2$ and $X_1{}^c \cap X_2{}^c < X_1{}^c, X_2{}^c$, then the assertion follows from Lemma 3.1. Q.E.D.

<u>Theorem 4.4.</u> Let $(L, M; c)$ be paired subdivided manifolds with degree m. Suppose $(L, M; c)$ is in general position and satisfies the separation condition. Furthermore suppose each cell in $L \cup M$ is bounded. Then $N = \{X \# X^c: X \in \overline{L}, X^c \neq \phi\}$ is a subdivided $(m+1)$-manifold.

proof. As can be seen in the proof of Lemma 2.1, $X \# X^c$ forms an $(m+1)$-cell when both X and X^c are bounded cells. Thus the theorem follows from Lemma 3.2 and Lemma 4.3. Q.E.D.

Theorem 4.5. Suppose the same assumptions as in Lemma 4.3 and let

$$\mathcal{B}_1 = \{X \times \{1\}: X \in \overline{L},\ \ X^c \neq \phi,\ \dim X = n\},$$

$$\mathcal{B}_0 = \{Y \times \{0\}: Y \in \overline{M},\ \ Y^c \neq \phi,\ \dim Y = m\},$$

$$\mathcal{B} = \left\{ X \# Y : \begin{array}{l} X \in \overline{L},\ \ Y \in \overline{M},\ \ \dim X + \dim Y = m-1, \\ \text{exactly one of the following two cases occurs:} \\ \text{(i) } X^c > Y, \\ \text{(ii) } Y^c > X. \end{array} \right\}.$$

Then

$$\partial N = \mathcal{B}_1 \cup \mathcal{B}_0 \cup \mathcal{B}.$$

Proof. Let $Z = X \# Y \in \mathcal{B}$. We can assume without loss of generality that $X^c > Y$ holds. Then

$$X \# X^c > Z,$$

which implies that $Z \in \overline{N}_m$. As we have seen in the proof of Lemma 3.2, $X \# Y^c$ and $Y^c \# Y$ are all candidates for members of N having Z as a face. Hence $Z \in \partial N$. Since $\mathcal{B}_1$ and $\mathcal{B}_0$ are clearly subclasses of ∂N, we have proved that

$$\partial N \supseteq \mathcal{B}_1 \cup \mathcal{B}_0 \cup \mathcal{B}.$$

To show the reverse relation, let $Z \in \partial N \backslash (\mathcal{B}_1 \cup \mathcal{B}_0)$. Then $Z = X \# Y$

for some $X \in \overline{L}$, $Y \in M$ with $\dim X + \dim Y = m-1$, and there exist a member, say $X \# X^c$, of N having Z as a face. Furthermore we have $Y^c \not> X$ since otherwise $Y^c \# Y > Z$. which is contrary to $Z \in \partial N$. Hence we have shown $Z \in B$. Q.E.D.

It is not preferable that N has a boundary point in $R^n \times (0, 1)$ when we intend to use N as a subdivision in piecewise linear continuation methods. To avoid this undesirable case we impose the following inclusion reversing condition (4.4):

(4.4) For $X \in \overline{L}$, $Y \in \overline{M}$ with $\dim X + \dim Y = m-1$, $X^c > Y$ and $Y^c > X$ imply each other.

Corollary 4.6. Suppose the same assumptions as in Lemma 4.3. Then $(L, M; c)$ satisfies the inclusion reversing condition (4.4) if and only if

$$|\partial N| \subseteq R^n \times \{0, 1\}.$$

Proof. If we define B_1 and B_0 as in Theorem 4.5, it is straightforward to see that $(L, M; c)$ satisfies the inclusion reversing condition (4.4) if and only if $\partial N = B_1 \cup B_0$. Q.E.D.

Paired subdivided manifolds $(L, M; c)$ is not necessarily the primal-dual pair of subdivided manifolds introduced in Kojima and Yamamoto [2]; however, if the inclusion reversing condition is satisfied, $(L, M; c)$ turns out to be a special one of the primal-dual pair of subdivided manifolds.

5. TRIANGULATIONS INDUCED BY SIMPLICIAL REFINEMENTS OF PAIRED SUBDIVIDED MANIFOLDS

Let L^* and M^* be simplicial refinements of L and M, respectively, and define the restriction $\overline{L}^*|X$ of $\overline{L}^*$ on X as

$$\overline{L}^* \mid X = \{\sigma \in \overline{L}^*: \ \sigma \subseteq X\}.$$

Since $\sigma \# \tau$ forms a simplex if two simplices σ and τ are in general position, we now define a collection of simplices K by using paired subdivided manifolds $(L, M; c)$ in general position as follows:

$$K = \{\sigma \# \tau: \ \sigma \in \overline{L}^* \mid X, \ \sigma \in \overline{M}^* \mid X^c, \ \ X \in \overline{L}, \ \ X^c \neq \phi\}. \tag{5.1}$$

<u>Theorem 5.1.</u> Suppose the same assumption as in Lemma 4.3. Then K in (5.1) is a triangulation of $|N|$.

Proof. Let η_1 and η_2 be simplices of K. Then we have $\sigma_1 \in \overline{L}^* \mid X_1$, $\sigma_2 \in \overline{L}^* \mid X_2$, $\tau_1 \in \overline{M}^* \mid X_1{}^c$ and $\tau_2 \in \overline{M}^* \mid X_2{}^c$ such that

$$\eta_1 = \alpha_1 \# \tau_1, \quad \eta_2 = \sigma_2 \# \tau_2.$$

In order to prove that $\eta_1 \cap \eta_2 < \eta_1$ and η_2 it suffices to show that

$$\eta_1 \cap \eta_2 = (\sigma_1 \cap \sigma_2) \# (\tau_1 \cap \tau_2)$$

because the general position condition implies that

$$(\sigma_1 \cap \sigma_2) \# (\tau_1 \cap \tau_2) < \sigma_1 \# \tau_1 \ \text{ and } \ \sigma_2 \# \tau_2.$$

It is immediate to see that

$$\eta_1 \cap \eta_2 \supseteq (\sigma_1 \cap \sigma_2) \# (\tau_1 \cap \tau_2).$$

To prove the reverse relation choose an arbitrary point $z \in \eta_1 \cap \eta_2$. Then we can find $x^1 \in \sigma_1$, $y^1 \in \tau_1$, $x^2 \in \sigma_2$ and $y^2 \in \tau_2$ such that

$$z = \lambda_1\begin{pmatrix} x^1 \\ 1 \end{pmatrix} + (1 - \lambda_1)\begin{pmatrix} y^1 \\ 0 \end{pmatrix} = \lambda_2\begin{pmatrix} x^2 \\ 1 \end{pmatrix} + (1 - \lambda_2)\begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$

for some $\lambda_1, \lambda_2 \in [0, 1]$. Since $\eta_1 \cap \eta_2 \subseteq (X_1 \# X_1^c) \cap (X_2 \# X_2^c)$, it follows from Lemma 4.1 that there exist $x \in X_1 \cap X_2$, $y \in X_1^c \cap X_2^c$ and $\lambda \in [0, 1]$ such that

$$z = \lambda\begin{pmatrix} x \\ 1 \end{pmatrix} + (1 - \lambda)\begin{pmatrix} y \\ 0 \end{pmatrix}.$$

We immediately have that $\lambda_1 = \lambda_2 = \lambda$,

$$\lambda(x^1 - x) = (1 - \lambda)(y - y^1)$$

and

$$\lambda(x^2 - x) = (1 - \lambda)(y - y^2).$$

Note that $x^1 - x \in \text{tng}(X_1)$, $y - y^1 \in \text{tng}(X_1^c)$, $x^2 - x \in \text{tng}(X_2)$ and $y - y^2 \in \text{tng}(X_2^c)$, then

$$x^1 = x^2 = x \qquad \text{whenever} \quad \lambda > 0,$$

$$y^1 = y^2 = y \qquad \text{whenever} \quad \lambda < 1.$$

Thus we have shown that

$$z \in (\sigma_1 \cap \sigma_2) \# (\tau_1 \cap \tau_2).$$

Let m be the degree of $(L, M; c)$ and $\rho = \sigma' \# \tau'$ be an m-simplex of $\overline{K}$. Suppose $\rho < \eta \in K$, then by Lemma 3.1 η is written as $\eta = \sigma \# \tau$ for some $\sigma > \sigma'$ and $\tau > \tau'$. Thus by the similar argument as in the proof of Lemma 3.2 we have either one of the two cases:

case 1: $\sigma = \sigma'$ and $\tau > \tau'$.

case 2: $\sigma > \sigma'$ and $\tau = \tau'$.

Hence we have shown that ρ lies in at most two (m+1)-simplices of K. Since it is clear that $|K| = |N|$, the assertion follows. Q.E.D.

Corollary 5.2. Suppose the same condition as in Lemma 4.3. If, in addition, the inclusion reversing condition (4.4) holds, then the triangulation K has no boundary points in $R^n \times (0, 1)$.

It should be noted that the three conditions (general position, separation and inclusion reversing) are all invariant under the transfer of L and M. Thus based on the same paired subdivided manifolds, transfer of origin together with various simplicial refinements of L and M generates various triangulations.

6. EXAMPLES

We shall present a few examples of paired subdivided manifolds inducing subdivisions and triangulations in $R^n \times [0, 1]$. Some of them were utilized to show that some variable dimension algorithms can be considered as a method of following piecewise linear path of solutions.

(i) Let e^i be the i-th unit vector of R^n, and define p^i, d^i $(i = 0, 1, \ldots, n)$ as

$$p^0 = -\sum_{j=1}^{n} e^j,$$

$$p^i = e^i \qquad (i = 1, 2, \ldots, n),$$

$$d^0 = \sum_{j=1}^{n} e^j,$$

$$d^i = \sum_{j=1}^{n} e^j - (n + 1)e^i \qquad (i = 1, 2, \ldots, n).$$

For $I \subseteq N^* = \{0, 1, \ldots, n\}$ we define

$$X(I) = \{\sum_{i \in I} \alpha_i p^i: \ \alpha_i \geqq 0 \quad (i \in I)\}$$

$$Y(I) = \{\sum_{i \in I} \beta_i d^i: \ \beta_i \geqq 0 \quad (i \in I), \quad \sum_{i \in I} \beta_i = 1\},$$

and let

$$\mathcal{L} = \{X(I): I \subseteq N^*, \ |I| = n\},$$

$$\mathcal{M} = \{Y(N^*)\}.$$

Since

$$\overline{L} = \{X(I): \ I \subseteq N^*, \ |I| \leqq n\},$$

$$\overline{M} = \{Y(I): \ I \subseteq N^*\},$$

we define c to pair L and M as follows:

$$X(I)^c = Y(N^*\backslash I) \qquad \text{for } I \subseteq N^*, \ |I| \leqq n,$$

$$Y(I)^c = X(N^*\backslash I) \qquad \text{for } I \subseteq N^*.$$

Thus $(L, M; c)$ forms paired subdivided manifolds with degree n. Fig. 1 depicts the two dimensional case. It is not difficult to see that $(L, M; c)$ is in general position and satisfies the separation condition and the inclusion reversing condition. Hence $(L, M; c)$ induces a subdivision with its boundary in $R^n \times \{0, 1\}$.

(ii) Let $N = \{1, 2, \ldots, n\}$ and S be the set of all n-dimensional vectors $s = (s_1, s_2, \ldots, s_n)$ such that $s_i \in \{-1, 0, 1\}$ for every $i \in N$. For each $s \in S$, we define

$$X(s) = \{x \in R^n: \ \begin{array}{ll} x_i \geqq 0 & (i \in I^+(s)), \\ x_i \leqq 0 & (i \in I^-(s)), \\ x_i = 0 & (i \in I^0(s))\}, \end{array}$$

$$Y(s) = \{y \in R^n: \ \begin{array}{ll} y_i = 1 & (i \in I^+(s)), \\ y_i = -1 & (i \in I^-(s)), \\ -1 \leqq y_i \leqq 1 & (i \in I^0(s))\}, \end{array}$$

where $I^{\pm}(s) = \{i \in N: \ s_i = \pm 1\}$ and $I^0(s) = \{i \in N: \ s_i = 0\}$. Let

$$L = \{X(s): \ s \in S, \ \ I^0(s) = \phi\},$$

$$m = \{Y(0)\},$$

then

$$\bar{L} = \{X(s): \ s \in S\},$$

$$\bar{m} = \{Y(s): \ s \in S\}.$$

we define c as

$$X(s)^c = Y(s), \ \ Y(s)^c = X(s) \qquad \text{for every} \ \ s \in S,$$

then $(L, M; c)$ forms paired subdivided manifolds with degree n. And we can see $(L, M; c)$ also satisfies the general position, separation and inclusion reversing conditions. We illustlate the two dimensional case in Fig. 2. This $(L, M; c)$ was implicitly used in van der Laan and Talman [6].

(iii) Let N and S be the same as defined in (ii). For each vector $s \in S$ define

$$T_+(s) = \{t \in S: \ t_i = s_i \ \text{ whenever } \ s_i \neq 0\},$$

$$T_-(s) = \{s_i e^i: \ s_i \neq 0\} = \{e^i: \ s_i = 1\} \cup \{-e^i: \ s_i = -1\},$$

$$X(s) = \{ \sum_{t \in T_+(s)} \alpha_t \cdot t : \alpha_t \geqq 0 \ (t \in T_+(s)) \},$$

$$Y(s) = \{ \sum_{t \in T_-(s)} \beta_t \cdot t : \beta_t \geqq 0 \ (t \in T_-(s)), \sum_{t \in T_-(s)} \beta_t \cdot t = 1 \}.$$

Here we employ the convention that

$$T_+(0) = \{0\}, \ T_-(0) = \{e^i : i \in N\} \cup \{-e^i : i \in N\},$$

and let

$$L = \{X(s): \; s \in S, \; s \text{ has only one nonzero component}\}$$

$$M = \{Y(0)\}.$$

Fig. 3 illustrates the two dimensional case. We see

$$\bar{L} = \{X(s): s \in S\},$$

$$\bar{M} = \{Y(s): s \in S\}.$$

Let c be difined as

$$X(s)^c = Y(s), \quad Y(s)^c = X(s) \qquad \text{for each } s \in S,$$

then $(L, M; c)$ forms paires subdivided manifolds. It is not difficult to see the three conditions are satisfied. Wright [15] proposed a simplicial algorithm by using a triangulation induced by this $(L, M; c)$.

(iv) The last one is the checkerboard subdivision proposed by Kojima and Yamamoto [3] and independently by Bárány [1]. Let Q be the set of all integral vectors $q = (q_1, q_2, \ldots, q_n)$. For each $q \in Q$, let

$$I_e(q) = \{i \in N: \; q_i \text{ is even}\},$$

$$I_o(q) = \{i \in N: \; q_i \text{ is odd}\}.$$

$$X(q) = \{x \in R^n: \quad x_i = q_i \quad (i \in I_e(q)),$$
$$q_i - 1 \leqq x_i \leqq q_i + 1 \quad (i \in I_o(q))\},$$

$$y(q) = \{y \in R^n: \; q_i - 1 \leqq y_i \leqq q_i + 1 \quad (i \in I_e(q)),$$
$$y_i = q_i \quad (i \in I_o(q))\}.$$

Let

$$L = \{X(q): \quad q \in Q, \quad I_e(q) = \phi\},$$

$$M = \{Y(q): \quad q \in Q, \quad I_0(q) = \phi\}.$$

Then L and M are subdivisions of R^n into n-dimensional hypercubes. We see

$$\bar{L} = \{X(q): \quad q \in Q\},$$

$$\bar{M} = \{Y(q): \quad q \in Q\},$$

then we define c as

$$X(q)^c = Y(q), \quad Y(q)^c = X(q) \qquad \text{for every} \quad q \in Q,$$

to obtain paired subdivided manifolds $(L, M; c)$. This also satisfies the three conditions: general position, separation and inclusion reversing.

Appendix

Lemma A. 1. Let B and C be polyhedral cones in R^k having the origin as a vertex. If $B \cap C = \{0\}$, then there exists a separating hyperplane H satisfying $H \cap B = H \cap C = \{0\}$.

Proof. Let $D = B - C = \{x - y: x \in B, \ y \in C\}$, then D is also a polyhedral cone. For $z^1, z^2 \in D$ and $\alpha \in (0, 1)$ suppose

$$0 = \alpha z^1 + (1 - \alpha) z^2$$

holds. Since z^i is written as $z^i = x^i - y^i$ for some $x^i \in B$ and $y^i \in C$, we have

$$\begin{aligned} 0 &= \alpha(x^1 - y^1) + (1 - \alpha)(x^2 - y^2) \\ &= \{\alpha x^1 + (1 - \alpha)x^2\} - \{\alpha y^1 + (1 - \alpha)y^2\} \\ &= x - y \end{aligned}$$

for some $x \in B$ and $y \in C$, which implies that $x = y = 0$. If we note that the origin is a vertex of both B and C, then $x^1 = x^2 = y^1 = y^2 = 0$, or $z^1 = z^2 = 0$. Hence the origin is also a vertex of D. Let $r^1, r^2, \ldots, r^p$ be extreme rays of D and R be the matrix consisting of column vectors r^i $(i = 1, 2, \ldots, p)$. Then there is no such a semipositive vector $\beta \in R^p$ as

$$R\beta = 0.$$

Applying Motzkin's alternative theorem we have $a \in R^k$

$$a^T R > 0.$$

Hence

$$a^T z > 0 \qquad \text{for any} \qquad z \in D \setminus \{0\}.$$

Now define a hyperplane

$$H = \{x \in R^k: \ a^T z = 0\},$$

then for any $x \in B$ and $y \in C$, with $x \neq y$

$$a^T x > a^T y.$$

Moreover we can easily see that

$$a^T x = a^T y, \quad x \in B, \quad y \in C$$

if and only if $x = y = 0$. Q.E.D.

Lemma A. 2. Let B and C be cells in R^k such that $B \cap C < B$ and C. Then there exists a separating hyperplane H of B and C satisfying

$$B \cap C = H \cap B = H \cap C.$$

Proof. Let $D = B \cap C$ and assume, without loss of generality, $0 \in$ rel int D, relative interior of D with respect to aff (D). We can write

$$B = \{x \in R^k: \ p^i x \geqq p_0^{\ i} \qquad i \in I\},$$

$$C = \{y \in R^k: \ q^j x \geqq q_0^{\ j} \qquad j \in J\}.$$

Now let

$$I_A = \{i \in I : \ p_0^{\ i} = 0\},$$

$$J_A = \{j \in J : \ q_0^{\ j} = 0\},$$

$$B' = \{x \in R^k: \ p^i x \geqq 0 \qquad i \in I_A\},$$

$$C' = \{y \in R^k: \quad q^j x \geqq 0 \qquad j \in J_A\},$$

then trivially $B' \supseteq B$ and $C' \supseteq C$. Let K be the orthogonal complement of tng (D). Then $0 \in$ rel int D implies that

(A-1) $\quad p^i, q^j \in K \qquad$ for $i \in I_A, \; j \in J_A$.

Now let

$$B'' = B' \cap K, \quad C'' = C' \cap K.$$

Then B'' and C'' are polyhedral cones in K. We shall show the origin is a vertex of both B'' and C''. Suppose

$$0 = \alpha x'' + (1 - \alpha)y''$$

for some $x'', y'' \in B''$ and $\alpha \in (0, 1)$. Then we can find $x, y \in B$ such that

$$x'' = \beta x, \quad y'' = \gamma y$$

for some $\beta, \gamma \geqq 0$. If either β or γ is zero, we have $x'' = y'' = 0$ and the desired result. Suppose $\beta, \gamma \neq 0$, then

$$0 = \frac{1}{\alpha\beta + (1-\alpha)\gamma}\{\alpha\beta x + (1 - \alpha)\gamma y\}.$$

This together with $D < B$ implies that $x, y \in D \subseteq$ tng (D). On the other hand $x, y \in B \cap K \subseteq K$. Therefore $x = y = 0$ and $x'' = y'' = 0$. In the same way we see that the origin is a vertex of C''. Clearly $B'' \cap C'' = \{0\}$, then applying Lemma A.1, we have a separating hyperplane

$$H' = \{x \in K: \quad a^T x = 0\}$$

defined by $a \in K$. We extend H' as

$$H = \{x \in R^k: \quad a^T x = 0\}$$

Let $x \in B'$, then x is rewritten as

$$x = x^1 + x^2 \qquad x^1 \in \text{tng}\ (D), \quad x^2 \in K.$$

Then by (A-1)

$$0 \geqq p^i x = p^i x^1 + p^i x^2 = p^i x^2$$

for any $i \in I_A$, or equivalently $x^2 \in B''$. Hence we obtain

$$a^T x = a^T x^1 + a^T x^2 = 0 + a^T x^2 \geqq 0.$$

In the same way as above we can see that

$$a^T x \leqq 0 \qquad \text{for any} \quad x \in C'.$$

Since $a \in K$, we have easily

$$D \subseteq H \cap B, \quad D \subseteq H \cap C.$$

Choose $x \in H \cap B$, and decompose x as

$$x = x^1 + x^2 \qquad x^1 \in \text{tng}\ (D), \quad x^2 \in B'' \subseteq K,$$

then

$$0 = a^T x = a^T x^1 + a^T x^2 = 0 + a^T x^2,$$

consequently

$$x^2 \in B'' \cap H' = \{0\}.$$

Therefore we obtain that

$$x \in \text{tng}\ (D) \cap B = D,$$

and

$$H \cap B = D.$$

In exactly the same way we have that

$$H \cap C = D. \qquad \text{Q.E.D.}$$

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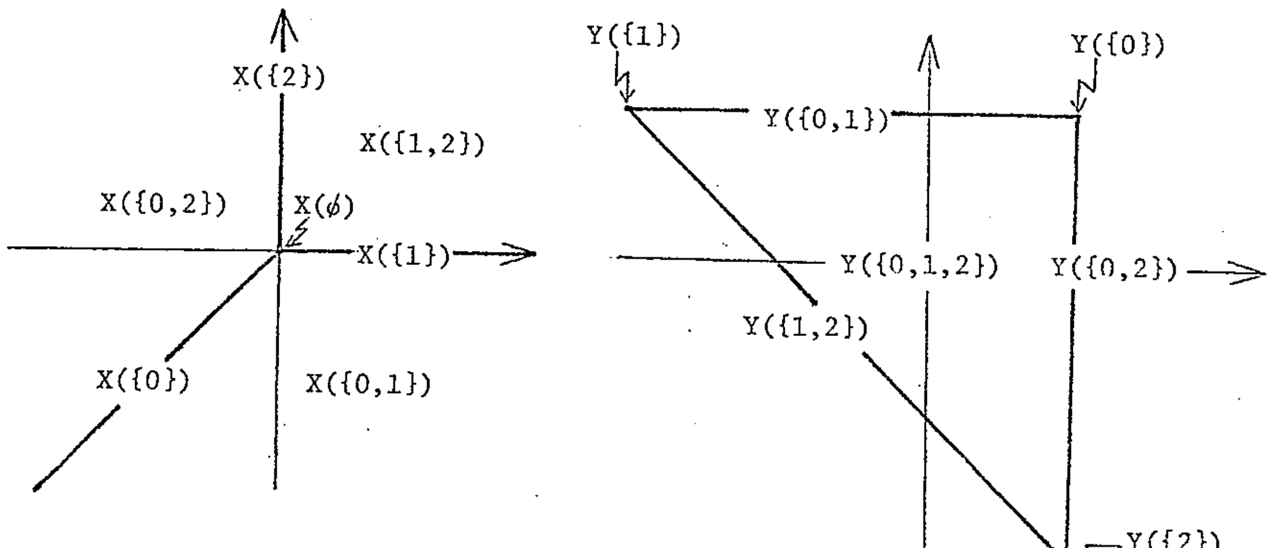

Fig. 1

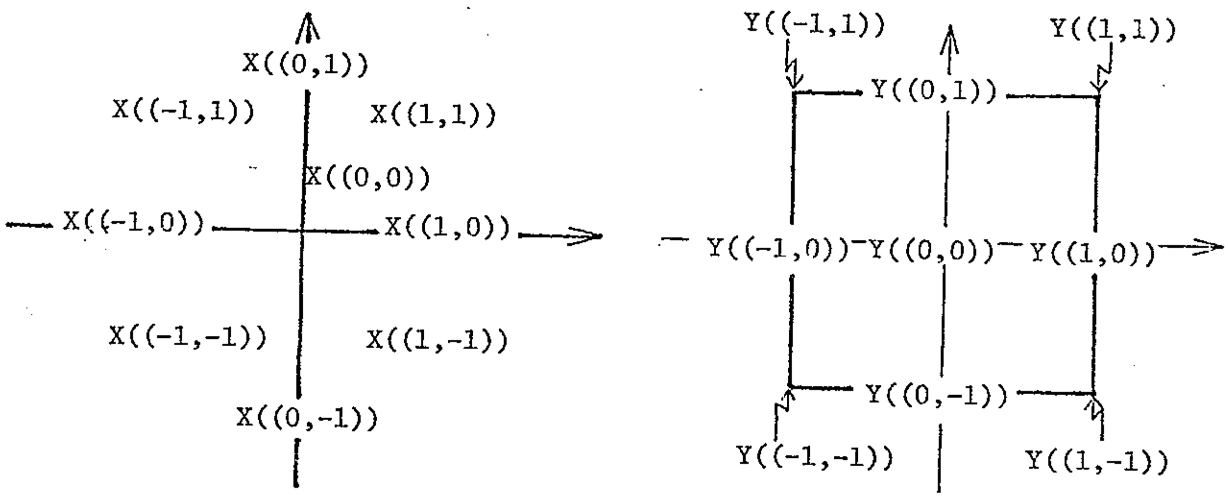

Fig. 2

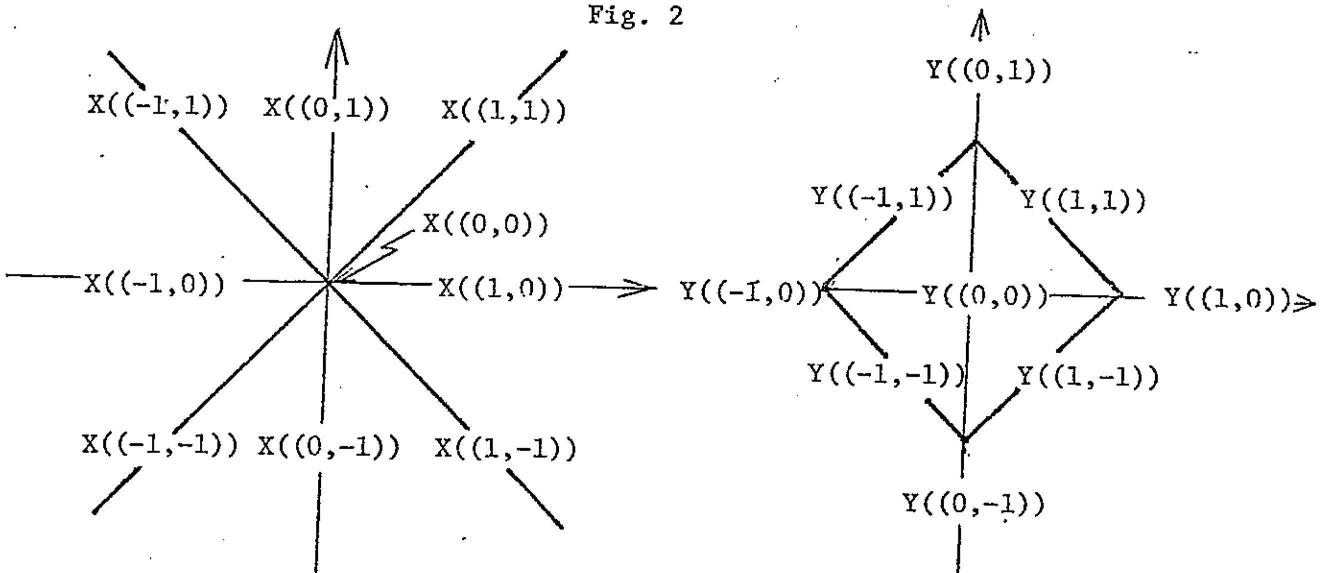

Fig. 3

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