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# An Application of Stochastic Sensitivity Analysis to Financial Engineering

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## Abstract

In this paper, a brief introduction is presented for a stochastic sensitivity analysis in financial engineering. The analysis is essentially based on an integration by parts technique in the stochastic calculus of variations, traditionally known as Malliavin calculus. In particular, the present technique is applied to the simulation of the Greeks, i.e., option price sensitivities with respect to model parameters. We first describe a constructive approach to compute the Greeks using integration by parts formula in Malliavin calculus. Then, we apply the method first to European options where formulas can be computed explicitly. Later we study the case of Asian options where closed formulas are not available, and new estimators are derived for *Delta* sensitivities. It is demonstrated that the present technique enables the simulation of the Greeks without differentiation of the payoff functions.

## 1 Introduction

In a risk management of derivative securities, sensitivities of an option price are an important measure of the risk and there exists a great need for the efficient computation of sensitivities. Commonly referred to as the Greeks in finance, they are mathematically defined as the partial differential sensitivity coefficients of the option price with respect to underlying model parameters. In financial engineering, finite difference approximations are heavily used to simulate the Greeks by means of Monte Carlo or Quasi Monte Carlo procedures. However, it is well-known that the finite difference approximation soon becomes inefficient

particullary when payoff functions are complex and discontinuous. This is often the case when we deal with exotic options such as American options, Asian options, lookback options, etc. For instance, in American options, the execution time of options is not fixed but depend on a time interval, and in Asian options, the payoff depends on some average of the asset value in a given period of time.

To overcome this difficulty, Broadie and Glasserman (1996) proposed a method to take the differential of the payoff function inside the expectation operator required to compute the option price. But this idea (i.e., likelihood ratio method) is applicable only when the density of the random variable involved is explicitly known. Recently, Fournie et al. (1999) (2001) suggested the use of Malliavin calculus, by means of an integration by parts, to shift the differential operator from the payoff function to the underlying diffusion (e.g., Gaussian) kernel, introducing a weighting function. Benhamou (2000) extended the method by expressing the weighting function as a Skorohod integral to give a general description of solutions for the Malliavin weights. Some extensions to barrier and lookback options have been studied by Gobet and Kohatsu-Higa (2001). The real advantage of using Malliavin calculus is that it is applicable when we deal with random variables whose density is not explicitly known as the case of Asian options.

Another examples, which have been studied by Koda and Okano (2000) and Okano and Koda (2001) but that are not covered in this paper are models involving a step function and non-smooth objective functions. In these studies, the stochastic sensitivity analysis technique based on the Novikov's identity is used instead of Malliavin calculus.

In this paper, we present a brief introduction of Malliavin Calculus, and describe a constructive method for a stochastic sensitivity analysis in financial engineering. The present approach enables the simulation of the Greeks without resort to direct differentiation of the payoff function. As a result, new estimators are derived for *Delta* sensitivities of Asian options, where no closed form solutions are available. For more complete and rigorous treatment of Malliavin Calculus, interested readers are referred to Nualert (1995).

The remainder of the paper is organized as follows. In Section 2, we briefly review the essence of Malliavin calculus. In section 3, we present the constructive method to derive integration by parts formula. Section 4 shows some explicit formulas for the case of European options. In Section 5, we investigate the case of Asian options. We conclude in Section 6.

## 2 Malliavin Calculus

Following the standard notations that can be found in Nualert (1995), the most concise presentation of Malliavin Calculus may be as follows. Let  $W = \{W_t\}_{t \in [0,1]}$  be a standerd one-dimensional Brownian motion defined on a complete probability space  $(\Omega, \mathcal{F}, P)$ . Assume  $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,1]}$  is generated by  $W$ . Let  $R$  be the space

of random variables of the form  $F = f(W_{t_1}, \dots, W_{t_n})$ , where  $f$  is smooth. For  $F \in R$ ,  $D_t F = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(W_{t_1}, \dots, W_{t_n}) \mathbf{1}_{[0, t_i]}(t)$ . For  $k \in \mathbb{Z}_+$  and  $p \geq 1$ , let  $\mathbb{D}^{k,p}$  be the completion of  $R$  with respect to the norm

$$\|F\|_{k,p} = \left( E[|F|^p] + E\left[ \left( \sum_{j=1}^k \int_0^1 \dots \int_0^1 |D_{s_1, \dots, s_j}^j F|^2 ds_1 \dots ds_j \right)^{p/2} \right] \right)^{1/p},$$

where  $D_{t_1, \dots, t_j}^j F = D_{t_1} \dots D_{t_j} F$ . We let  $\|F\|_{0,p} = (E[F^p])^{1/p} = \|F\|_p$  and  $\mathbb{D}^\infty = \bigcap_{k,p} \mathbb{D}^{k,p}$ . For processes  $u = \{u_t\}_{t \in [0,1]}$  on  $(\Omega, \mathcal{F}, P)$ ,  $\mathbb{D}_{L^2([0,1])}^{k,p}$  is defined as  $\mathbb{D}^{k,p}$  but with norm  $\|u\|_{k,p,L^2([0,1])} = (E[\|u\|_{L^2([0,1])}^p] + E[(\sum_{j=1}^k \int_0^1 \dots \int_0^1 \|D_{s_1, \dots, s_j}^j u\|_{L^2([0,1])}^2 ds_1 \dots ds_j)^{p/2}])^{1/p}$ .

We denote by  $D^*(u)$  the Skorohod integral or the adjoint operator of  $D$ . This adjoint operator behaves like a stochastic integral. In fact, if  $u_t$  is  $\mathcal{F}_t$  adapted, then  $D^*(u) = \int_0^1 u_t dW_t$ , the Ito integral of  $u$  (see, e.g. Nualert (1995)). Here we write  $D^*(u) = \int_0^1 u_t dW_t$ , even if  $u_t$  is not  $\mathcal{F}_t$  adapted. Of the formulas we will use, the following are worth mentioning,

$$D^*(Fu) = \int_0^1 Fu_t dW_t = F \int_0^1 u_t dW_t - \int_0^1 (D_t F) u_t dt \quad (1)$$

for  $F \in \mathbb{D}^{1,2}$  and  $E[F^2 \int_0^1 u_t^2 dt] < \infty$ ; and

$$E\left[ \int_0^1 (D_t F) u_t dt \right] = E[F D^*(u)], \quad (2)$$

where  $E[\cdot]$  denotes a suitably defined expectation operator.

As a byproduct of all the above formulas one may obtain the integration by parts formula. However, in this exposition, it is enough to note that the idea behind the operator (i.e., Malliavin derivative)  $D$  is to differentiate a random variable with respect to the underlying noise generated by the Wiener process  $W_t$ . Accordingly, we just formally interpret that the relation  $D_s = \frac{\partial}{\partial (dW_s)}$  holds. With this in mind, we have some examples as follows:

$$\begin{aligned} D_t W_t &= 1, \\ D_t f(W_t) &= f'(W_t), \\ D_s \left( \int_0^1 f(W_t) dW_t \right) &= \int_s^1 f'(W_t) dW_t + f(W_s), \end{aligned}$$

etc., where  $f$  is a continuous differentiable function with at most polynomial growth at infinity.

### 3 Constructive Derivation of Integration by Parts Formula

In this Section, we will formally describe a constructive approach to derive an integration by parts formula proposed in Kohatsu-Higa and Montero (2001). To begin with, a way to understand any integration by parts formula is through the following general definition.

**Definition:** Given two random variables  $X$  and  $Y$ , we will say that the integration by parts formula is valid if for any smooth function  $f$  with bounded derivatives, we have the relation

$$E[f'(X)Y] = E[f(X)H],$$

for some random variable  $H \equiv H(X, Y)$ .

One can deduce an integration by parts formula through the duality principle given in Eq. (2). Let us denote  $Z = f(X)$  and using the chain rule of differentiation, we have

$$D_s Z = f'(X)D_s X.$$

Then, multiplying the above by  $Yh(s)$  where  $h$  is a process to be chosen appropriately, we obtain

$$D_s ZYh(s) = f'(X)D_s XYh(s).$$

Integrating this for  $s \in [0, 1]$ , we have

$$\int_0^1 D_s ZYh(s)ds = \int_0^1 f'(X)D_s XYh(s)ds = f'(X)Y \int_0^1 h(s)D_s Xds,$$

then, a simple manipulation of the terms in the above equation yields

$$\int_0^1 \frac{Yh(s)D_s Z}{\int_0^1 h(v)D_v Xdv} ds = f'(X)Y.$$

Therefore, by using the inner product notation and taking the expectation, we have

$$E[\langle DZ, u \rangle_{L^2([0,1])}] = E[f'(X)Y],$$

with

$$u_s = \frac{Yh(s)}{\int_0^1 h(v)D_v Xdv}.$$

Finally, if  $D^*$  is the adjoint operator of  $D$  (i.e., see Eq. (2)), then we have

$$E[\langle DZ, u \rangle_{L^2([0,1])}] = E[\langle Z, D^*u \rangle_{L^2([0,1])}] = E[ZD^*(u)] = E[f'(X)Y],$$

or equivalently,

$$E \left[ f(X)D^* \left( \frac{Yh}{\int_0^1 h(v)D_v Xdv} \right) \right] = E[f'(X)Y].$$

In particular for the case with  $h \equiv 1$ , we have

$$H \equiv H(X, Y) = D^* \left( \frac{Y}{\int_0^1 D_v X dv} \right). \quad (3)$$

If one has higher order derivatives then this procedure should be repeated iteratively. The use of the norms in the spaces  $\mathbb{D}^{n,p}$  is necessary in order to prove that the above expectations are finite (in particular the ones related to  $H$ ). Note that the integral  $\int_0^1 h(v) D_v X dv$  should not be degenerate with probability one. Otherwise the above argument is bound to fail. The process  $h$  that appears in the calculation is a parameter process that can be chosen so as to obtain this non-degeneracy. In particular, when  $h(v) = D_v X$ , one obtains the so-called Malliavin covariance matrix. It is important to note that we can construct different formulas depending on the way how we choose the process  $h$ .

## 4 The Greeks in European Options

European options are contracts that are signed between two parties (usually a bank and a customer) that allows to obtain possible monetary benefits if the price of certain asset falls above (call option) or below (put option) a predetermined fixed value, the strike price, at a certain fixed date, the expiration time. The Greeks are partial differential sensitivity coefficients of an option price with respect to underlying model parameters. In general, let  $X \equiv X(\alpha)$  be a random variable that depends on a parameter  $\alpha$ . Suppose that the option price is computed through a payoff function in the following form  $\mathcal{P}(\alpha) = E[\Phi(X(\alpha), \alpha)]$  where  $\Phi$  is generally non-smooth. The Greeks are therefore a measure of the sensitivity of this price with respect to its parameters. In particular, it can serve to avoid future risks in holding these options. If the Leibnitz rule of operator exchange between integration (i.e. expectation) and differentiation is applicable, we have

$$\frac{\partial \mathcal{P}(\alpha)}{\partial \alpha} = \frac{\partial E[\Phi(X(\alpha), \alpha)]}{\partial \alpha} = E \left[ \Phi'(X(\alpha), \alpha) \frac{\partial X(\alpha)}{\partial \alpha} + \frac{\partial \Phi(X(\alpha), \alpha)}{\partial \alpha} \right]. \quad (4)$$

In this Section, we would like to discuss the application of the integration by parts formula to European options to compute sensitivity derivatives without resort to direct differentiation of the payoff function.

### 4.1 The Malliavin expressions

The payoff function of European options depends only on the value of the underlying asset at the fixed expiration time  $T$ , i.e.,  $S_T$ . The interest in the European options lies in the fact that the Greeks can be computed explicitly in a closed form for a particular class of payoff functions. The reason is that the probability

density function is known for the random variable involved, i.e.,  $S_T$ , whereas in other options this is not the case.

First, we assume that our underlying asset  $S$  is described by a geometric Brownian motion under the risk neutral probability measure  $Q$  uniquely defined in complete market with no-arbitrage:

$$S_t = S_0 + \int_0^t r S_s ds + \int_0^t \sigma S_s dW_s, \quad (5)$$

where  $r$  is the riskless interest rate and  $\sigma$  denotes the volatility.

Second, from the previous arguments it follows that  $X(\alpha)$  must be in general a functional of  $S$ . In the case of European options, we have  $X(\alpha) = S_T$  and from Eq. (5) we obtain

$$S_T = S_0 \exp\{\mu T + \sigma W_T\}, \quad (6)$$

where  $\mu = r - \sigma^2/2$ . Note that Eq. (6) is frequently used in the subsequent development.

Now we compute *Delta*,  $\Delta$ ; the first-order partial differential sensitivity coefficient of the (discounted) expected outcome of the option, with respect to the present value of the asset:

$$\Delta = \frac{\partial}{\partial S_0} E[e^{-rT} \Phi(S_T)] = e^{-rT} E\left[\Phi'(S_T) \frac{\partial S_T}{\partial S_0}\right] = \frac{e^{-rT}}{S_0} E[\Phi'(S_T) S_T],$$

where  $E[\cdot]$  denotes the expectation under the risk neutral measure  $Q$ . Then, applying the formula given in Eq. (3) with  $X = Y = S_T$ , we may perform the integration by parts to give,

$$\begin{aligned} \Delta &= \frac{e^{-rT}}{S_0} E[\Phi(S_T) H(X, Y)] \\ &= \frac{e^{-rT}}{S_0} E\left[\Phi(S_T) D^* \left(\frac{Y}{\int_0^T D_v X dv}\right)\right] \\ &= \frac{e^{-rT}}{S_0} E\left[\Phi(S_T) D^* \left(\frac{S_T}{\int_0^T D_v S_T dv}\right)\right], \end{aligned}$$

which removes the derivative of  $\Phi$  in the expectation as desired.

In order to evaluate  $\int_0^T D_v S_T dv$ , we apply the rules of the stochastic derivative introduced above, i.e.,

$$D_v S_T = \sigma S_T D_v W_T = \sigma S_T \mathbf{1}_{\{v \leq T\}},$$

in which  $\mathbf{1}_{\{\cdot\}}$  denotes the indicator of the event in braces, and then we obtain the following result:

$$\int_0^T D_v S_T dv = \sigma T S_T, \quad (7)$$

which frequently appears in the subsequent development. Thus, using Eq. (7), it becomes possible to perform the stochastic integral,

$$D^* \left( \frac{S_T}{\int_0^T D_v S_T dv} \right) = D^* \left( \frac{S_T}{\sigma S_T \int_0^T dv} \right) = D^* \left( \frac{1}{\sigma T} \right) = \frac{W_T}{\sigma T},$$

with the help of Eq. (1) applied to  $F = \frac{1}{\sigma T}$ . Then the final expression for  $\Delta$  reads,

$$\Delta = E \left[ e^{-rT} \Phi(S_T) \frac{W_T}{S_0 \sigma T} \right]. \quad (8)$$

Let us move onto a new Greek, *Vega*, which is denoted by  $\mathcal{V}$  in this exposition. It measures how sensitive is the option price when the volatility changes, i.e.,

$$\mathcal{V} = \frac{\partial}{\partial \sigma} E[e^{-rT} \Phi(S_T)] = e^{-rT} E \left[ \Phi'(S_T) \frac{\partial S_T}{\partial \sigma} \right] = e^{-rT} E[\Phi'(S_T) S_T (W_T - \sigma T)].$$

We invoke again the recipe in Section 3, by using Eq. (3) with  $X = S_T$  and  $Y = S_T(W_T - \sigma T)$ , to derive

$$\mathcal{V} = e^{-rT} E \left[ \Phi(S_T) D^* \left( \frac{S_T(W_T - \sigma T)}{\int_0^T D_v S_T dv} \right) \right] = e^{-rT} E \left[ \Phi(S_T) D^* \left( \frac{W_T}{\sigma T} - 1 \right) \right],$$

where we have used Eq. (7). So the computation we must face is

$$D^* \left( \frac{W_T}{\sigma T} - 1 \right) = \frac{1}{\sigma T} D^*(W_T) - W_T.$$

Here a new instance of stochastic integral appears,  $D^*(W_T)$ . Applying again Eq. (1) with  $F = W_t$ , we have

$$D^*(W_T) = W_T^2 - \int_0^T D_s W_T ds = W_T^2 - T,$$

which leads to the following final expression for  $\mathcal{V}$ :

$$\mathcal{V} = E \left[ e^{-rT} \Phi(S_T) \left\{ \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right\} \right]. \quad (9)$$

The last example involves a second-order derivative: *Gamma*.  $\Gamma$  gives sensitivity information on the second-order dependence of the option price on the actual value of the underlying asset, i.e.,

$$\Gamma = \frac{\partial^2}{\partial S_0^2} E[e^{-rT} \Phi(S_T)] = \frac{e^{-rT}}{S_0^2} E[\Phi''(S_T) S_T^2].$$

After the first integration by parts, by applying Eq. (3) with  $X = S_T$  and  $Y = S_T^2$ , we derive

$$\Gamma = \frac{e^{-rT}}{S_0^2} E \left[ \Phi'(S_T) D^* \left( \frac{S_T^2}{\int_0^T D_v S_T dv} \right) \right] = \frac{e^{-rT}}{S_0^2} E \left[ \Phi'(S_T) D^* \left( \frac{S_T}{\sigma T} \right) \right].$$



The above stochastic integral may be simplified by using once more Eq. (1) with  $F = \frac{S_T}{\sigma T}$ , which leads to

$$D^*\left(\frac{S_T}{\sigma T}\right) = \frac{S_T}{\sigma T} D^*(1) - \frac{1}{\sigma T} \int_0^T D_s S_T ds = S_T \left\{ \frac{W_T}{\sigma T} - 1 \right\}.$$

Likewise, the second integration by parts yields

$$\begin{aligned} \Gamma &= \frac{e^{-rT}}{S_0^2} E \left[ \Phi'(S_T) S_T \left\{ \frac{W_T}{\sigma T} - 1 \right\} \right] \\ &= \frac{e^{-rT}}{S_0^2} E \left[ \Phi(S_T) D^* \left( \frac{S_T}{\int_0^T D_v S_T dv} \left\{ \frac{W_T}{\sigma T} - 1 \right\} \right) \right]. \end{aligned}$$

The stochastic integral involved is slightly cumbersome, but it does not endow any complexity and we have,

$$D^* \left( \frac{S_T}{\int_0^T D_v S_T dv} \left\{ \frac{W_T}{\sigma T} - 1 \right\} \right) = \frac{1}{\sigma T} D^* \left( \frac{W_T}{\sigma T} - 1 \right) = \frac{1}{\sigma T} \left\{ \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right\}.$$

If we bring together the previous partial results, we obtain the final expression,

$$\Gamma = \frac{1}{S_0^2 \sigma T} E \left[ e^{-rT} \Phi(S_T) \left\{ \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right\} \right]. \quad (10)$$

Comparing Eqs. (9) and (10), we find the following relationship between  $\mathcal{V}$  and  $\Gamma$ :

$$\Gamma = \frac{\mathcal{V}}{S_0^2 \sigma T}. \quad (11)$$

Since there exists closed expressions for all the Greeks, we may easily check the correctness of the above results. These formulas are already well-known although their proofs do not usually follow the integration by parts formula in the form we have presented here.

## 4.2 The *vanilla* options

For European options, there is a closed and tractable expression for the probability density function associated with  $S_T$ . This is the lognormal distribution written as

$$p(x) = \frac{1}{\sigma x \sqrt{2\pi T}} \exp\{-[\log(x/S_0) - \mu T]^2 / 2\sigma^2 T\}.$$

When  $p(x)$  is available, we can compute all the partial derivatives, starting from the explicit formulation for the security price,  $\mathcal{P}$ ,

$$\mathcal{P} = E[e^{-rT} \Phi(S_T)] = e^{-rT} \int_0^\infty \Phi(x) p(x) dx,$$

which is the Black-Scholes formula in integral form. Thus the knowledge of  $p(x)$  allows us, in principle, to compute all the Greeks once a payoff function has been selected.

One of the most popular choice is *vanilla* call option in which the payoff function reads,

$$\Phi(X) = \max(X - K, 0), \quad (12)$$

where  $K$  denotes the constant strike price. Then we can easily derive the following explicit expressions for the Greeks:

$$\begin{aligned} \Delta &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1(K)} e^{-x^2/2} dx, \\ \nu &= S_0 \sqrt{\frac{T}{2\pi}} e^{-[d_1(K)]^2/2}, \\ \Gamma &= \frac{1}{S_0 \sigma \sqrt{2\pi T}} e^{-[d_1(K)]^2/2}, \end{aligned}$$

where

$$d_1(K) = \frac{1}{\sigma \sqrt{T}} \left[ \log(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T \right],$$

as it can be found in any standard textbook on derivative securities, e.g., in Wilmot (1998). When we deal with European options, the constructive method presented here yields the equivalent results we attain if we directly differentiate the probability density function given above.

## 5 The Greeks in Asian Options

In this Section, we consider the Greeks for options written on the average value of the security price, i.e.,  $\frac{1}{T} \int_0^T S_s ds$ , instead of the final value  $S_T$ , as in European options. Note that the density function of the random variable does not have a known closed formula in this case. *Delta* for the Asian option is given by

$$\Delta = \frac{\partial}{\partial S_0} E \left[ e^{-rT} \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) \right] = \frac{e^{-rT}}{S_0} E \left[ \Phi' \left( \frac{1}{T} \int_0^T S_t dt \right) \frac{1}{T} \int_0^T S_t dt \right].$$

Then, applying the formula given in Eq. (3) with  $X = Y = \int_0^T S_t dt$ , we may perform the integration by parts to give,

$$\begin{aligned} \Delta &= \frac{e^{-rT}}{S_0} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) H(X, Y) \right] \\ &= \frac{e^{-rT}}{S_0} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) D^* \left( \frac{Y}{\int_0^T D_v X dv} \right) \right] \\ &= \frac{e^{-rT}}{S_0} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) D^* \left( \frac{\int_0^T S_t dt}{\sigma \int_0^T t S_t dt} \right) \right], \end{aligned}$$

where we have used the following calculation using the Malliavin derivative,

$$\begin{aligned}
\int_0^T D_v X dv &= \int_0^T \int_0^T D_v S_t dt dv \\
&= \int_0^T \int_0^T \sigma S_t D_v W_t dt dv \\
&= \int_0^T \int_0^T \sigma S_t \mathbf{1}_{\{v \leq t\}} dv dt \\
&= \int_0^T \int_{v=0}^t \sigma S_t dv dt = \sigma \int_0^T t S_t dt.
\end{aligned}$$

We obtain finally the following three formulas for *Delta*,  $\Delta_i (i = 1, 2, 3)$ :

$$\Delta_i = \frac{e^{-rT}}{S_0} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) H_i \right] \quad (i = 1, 2, 3) \quad (13)$$

where  $H_i (i = 1, 2, 3)$  are defined as follows:

$$H_1 = \frac{1}{\langle T \rangle} \left\{ \frac{W_T}{\sigma} + \frac{\langle T^2 \rangle}{\langle T \rangle} \right\} \quad [\text{Benhamou(2000)}] \quad (14)$$

$$H_2 = \frac{1}{\langle T \rangle} \left\{ \frac{W_T}{\sigma} + \frac{\langle T^2 \rangle}{\langle T \rangle} \right\} - 1 \quad [\text{Kohatsu - Higa(2001)}] \quad (15)$$

$$H_3 = \frac{1}{\langle T \rangle} \frac{W_T}{\sigma} - 1 \quad [\text{Koda(2002)}] \quad (16)$$

where

$$\begin{aligned}
\langle T \rangle &= \frac{\int_0^T t S_t dt}{\int_0^T S_t dt}, \\
\langle T^2 \rangle &= \frac{\int_0^T t^2 S_t dt}{\int_0^T S_t dt},
\end{aligned}$$

which are analogous to the first- and second-moment of  $t$  weighted by  $S_t$  over the finite time horizon  $t \in [0, T]$ , respectively. In the three formulas given above, last two are brand new, among which Eq. (16) is the simplest. Note that these estimators are obtained for  $H = D^* \left( \frac{\int_0^T S_t dt}{\sigma \int_0^T t S_t dt} \right)$ , corresponding to the different ways of decomposition of  $u$  and  $F$  in Eq. (1) as follows:

$$H_1 : \quad u = \frac{1}{\sigma} \int_0^T S_t dt, \quad F = \frac{1}{\int_0^T t S_t dt}$$

$$H_2 : \quad u = \frac{1}{\sigma}, \quad F = \frac{\int_0^T S_t dt}{\int_0^T t S_t dt}$$

$$H_3 : \quad u = \frac{1}{\sigma \int_0^T t S_t dt}, \quad F = \int_0^T S_t dt$$

Thus, the derivative of  $\Phi$  has been removed as desired.

There are other ways of performing the integration by parts. For instance, in Fournie et al. (1999), we have the following expression:

$$\Delta = \frac{e^{-rT}}{S_0} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) \left( \frac{2 \int_0^T S_t dW_t}{\sigma \int_0^T S_t dt} + 1 \right) \right],$$

whereas another variant of it can be found in Fournie et al. (2001):

$$\Delta = \frac{2e^{-rT}}{S_0 \sigma^2} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) \left( \frac{S_T - S_0}{\int_0^T S_t dt} - \mu \right) \right].$$

Although we have different expressions for  $\Delta$  as we have seen above, they are all statistically identical. Depending on the nature of the process of the underlying and associated volatility structures, it may be decided which one to use either of these different formulas.

The present method also applies to the computation of other Greeks by means of integration by parts technique. Then, *Vega* in this case becomes

$$\begin{aligned} \nu &= \frac{\partial}{\partial \sigma} E \left[ e^{-rT} \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) \right] \\ &= e^{-rT} E \left[ \Phi' \left( \frac{1}{T} \int_0^T S_t dt \right) \frac{1}{T} \int_0^T \frac{\partial S_t}{\partial \sigma} dt \right] \\ &= e^{-rT} E \left[ \Phi' \left( \frac{1}{T} \int_0^T S_t dt \right) \frac{1}{T} \int_0^T S_t (W_t - \sigma t) dt \right]. \end{aligned}$$

As before, following the recipe in Section 3 by applying the formula given in Eq. (3) with  $X = \int_0^T S_t dt$  and  $Y = \int_0^T S_t (W_t - \sigma t) dt$ , we have

$$\begin{aligned} \nu &= e^{-rT} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) D^* \left( \frac{Y}{\int_0^T D_v X dv} \right) \right] \\ &= e^{-rT} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) D^* \left( \frac{\int_0^T S_t W_t dt}{\sigma \int_0^T t S_t dt} - 1 \right) \right], \end{aligned}$$

which yields the following final expression:

$$\begin{aligned} \nu &= e^{-rT} E \left[ \Phi \left( \frac{1}{T} \int_0^T S_t dt \right) \right. \\ &\quad \left. \left\{ \frac{\int_{s=0}^T (\int_0^T S_t W_t dt) dW_s}{\sigma \int_0^T t S_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{(\int_0^T t S_t dt)^2} - W_T \right\} \right]. \end{aligned}$$

Note that this result is essentially identical to the one that is obtained by Benhamou (2000). Using Eq. (11), it is straightforward to compute *Gamma* as  $\nu$  divided by  $S_0^2 \sigma T$ .

## 6 Conclusion

In this paper, a constructive approach is presented to derive the Greeks in finance by means of integration by parts technique in Malliavin calculus. In particular, new estimators are derived for the simulation of *Delta* in the case of Asian options. This enables us to smoothen the payoff function to be estimated by the Monte Carlo or Quasi Monte Carlo procedure. Although the results presented in this study are theoretical, they will help us to obtain useful insights into the practical applications of the stochastic sensitivity analysis to financial engineering in general. Numerical validation of the present method will be a subject of the future study.

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