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A Note on Minimax Inverse Generalized Minimum Cost Flow Problems

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Abstract

Given a generalized network and a generalized feasible flow f^0, we consider a problem finding a modified edge cost d such that f^0 is minimum cost with respect to d and the maximum deviation between the original edge cost and d is minimum. This paper shows the relationship between this problem and minimum mean circuit problems and analyze a binary search algorithm for this problem.

Keywords: inverse optimization, generalized networks, flow algorithms

1 Introduction

The inverse optimization problem is to find an objective function such that a given feasible solution is optimal and the variance between the original objective function and the finding one is minimum. With its rich applications, there is a good deal of literature on inverse optimization. Ahuja and Orlin wrote a survey of inverse optimization [1]. The same authors systematically studied inverse network flow problems [2]. They dealt with two types of objective functions; one is minimizing the sum of absolute deviations, that is, under the L_1 norms, and the other is minimizing the maximum absolute deviation, that is, under the L_∞ norms. We call inverse problems of these types the minisum inverse problem and the minimax inverse problem, respectively. Ahuja and Orlin [2] showed that the minisum inverse problem for each of shortest path, assignment, minimum s-t cut, and

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minimum cost flow reduces to solving the same kind of problem, and that the minimax inverse problem for each of shortest path, assignment, and minimum cost flow reduces to solving a minimum mean cycle problem. Moreover, they showed that the weighted version of a minimax inverse minimum cost flow problem reduces to solving a minimum cost-to-weight ratio cycle problem. In [9], the relationship between the minimax inverse minimum *s*-*t* cut problem and the maximum mean cut problem was elucidated. This paper treats the minimax inverse problem of generalized minimum cost flow.

We first define a generalized minimum cost flow problem. Let $G = (V, E)$ be a directed graph of n vertices and m edges with a non-negative capacity function $u : E \to \mathbb{R}_+$, a cost function $c : E \to \mathbb{R}$ and a positive gain function $\gamma : E \to \mathbb{R}_{++}$. For notational convenience, we assume that G has no multiple edges so that each edge can be uniquely specified by its endpoints. We also assume, without loss of generality, that the graph G is symmetric, that is,

$$\text{if } (v, w) \in E, \text{ then } (w, v) \in E,$$

and that the costs and gains are antisymmetric, that is,

$$c(v, w) = -c(w, v)/\gamma(w, v), \quad \forall (v, w) \in E, \tag{1}$$

$$\gamma(v, w) = 1/\gamma(w, v), \quad \forall (v, w) \in E.$$

For a function $f : E \to \mathbb{R}$, the excess $e_f(v)$ at a vertex v is defined by $e_f(v) := -\sum_{(v,w)\in E} f(v, w)$. Given a supply/demand function $b : V \to \mathbb{R}$, a *generalized flow* is a function $f : E \to \mathbb{R}$ that satisfies the capacity constraint:

$$f(v, w) \le u(v, w), \ \forall (v, w) \in E,$$

the flow antisymmetry constraint:

$$f(v, w) = -\gamma(w, v) f(w, v), \ \forall (v, w) \in E,$$

and the supply/demand constraint:

$$e_f(v) = b(v), \ \forall v \in V.$$

For a generalized flow f and a function $x : E \to \mathbb{R}$, we define

$$x(f) := \sum_{(v,w)\in E : f(v,w)>0} x(v, w) f(v, w).$$

The cost of a generalized flow f is given by $c(f)$. We call a generalized flow f a minimum-cost generalized flow, if $c(f) \leq c(f')$ holds for all generalized flows f'. The generalized minimum cost flow problem is that of finding a minimum-cost generalized flow.

Given a generalized flow f^0, the *minimax inverse generalized minimum cost flow problem* (MIGCF) is to find a modified cost function $d : E \rightarrow \mathbb{R}$ satisfying cost antisymmetry constraint (1) such that f^0 is minimum-cost generalized flow with respect to d, and $\max\{d(v,w)-c(v,w) \mid (v,w) \in E\}$ is minimum. When the gain of each edge is equal to one, from the cost antisymmetry constraint (1), minimizing $\max\{d(v,w)-c(v,w) \mid (v,w) \in E\}$ is equivalent to minimizing $\max\{|d(v,w) - c(v,w)| \mid (v,w) \in E\} = \|d - c\|_\infty$, and MIGCF coincides with the minimax inverse minimum cost flow problem. For a positive weight function $\tau : E \rightarrow \mathbb{R}_{++}$, the weighted version of MIGCF (wMIGCF) is to minimize $\max\{\tau(v,w)(d(v,w) - c(v,w)) \mid (v,w) \in E\}$. Note that, when $\tau(v,w) = \max\{1, 1/\gamma(v,w)\}$ for all $(v,w) \in E$, the objective of wMIGCF is equivalent to minimizing $\max\{|d(v,w) - c(v,w)| \mid (v,w) \in E\}$.

Throughout this paper, we assume that the costs c and gains γ are given by ratios of two relatively prime integers, and we denote the largest absolute value of these integers by B. Moreover, assume that the weights τ are also given by ratios of two relatively prime integers and denote the largest absolute value of these integers by W. Section 2 gives some definitions and basic properties. Section 3 shows that MIGCF is related to a *minimum mean circuit problem* which is a generalization of a minimum mean cycle problem, and wMIGCF is related to a *minimum cost-to-weight ratio circuit problem* which is a generalization of a minimum cost-to-weight cycle problem. Section 4 discusses polynomial time algorithms for these problems.

2 Preliminaries

This section reviews some basic definitions and properties.

The gain of a cycle (resp. path) is the product of the gains of edges participating in that cycle (resp. path). A *unit-gain cycle* is a cycle whose gain is equal to one. A *flow-generating (resp. flow-absorbing) cycle* is a cycle whose gain is greater (resp. less) than one. A *bicycle* is a flow-generating cycle, a flow-absorbing cycle, and a simple path from the first cycle to the second. We note that the edges in the flow-generating and flow-absorbing cycles need not be edge-disjoint.

Given a generalized flow f, the *residual capacity* function $u_f : E \to \mathbb{R}$ is defined by $u_f(v, w) := u(v, w) - f(v, w)$. The *residual graph* with respect to f is given by $G_f = (V, E_f)$, where $E_f = \{(v, w) \in E \mid u_f(v, w) > 0\}$. A *generalized circulation* on G_f is a flow $g : E \to \mathbb{R}$ that satisfies the capacity constraint given by the residual capacity u_f, the flow antisymmetry constraint, the supply/demand constraint for $b = 0$. A *circuit* on G_f is a generalized circulation g on G_f such that $\{(v, w) \in E \mid g(v, w) > 0\}$ is a single unit-gain cycle, a bicycle or the empty set. Any generalized circulation g on G_f can be decomposed into a collection of circuits $g_1, \ldots, g_k$ on G_f with $k \leq m$ such that $g = \sum_i g_i$ holds and that $g_i(v, w) > 0$ implies $g(v, w) > 0$ (e.g., see [4]).

The following theorem states the optimality criterion (e.g., see [4]).

Theorem 1 *A generalized flow f is minimum-cost if and only if G_f contains no circuit whose cost is negative.* □

A *potential function* π is a labeling of the vertices by real numbers. The *reduced cost* with respect to π is defined by $c^\pi(v, w) := c(v, w) + \pi(v) - \gamma(v, w)\pi(w)$. The next property comes from the linear programming duality theorem (also see [10, Section 6.2]).

Lemma 2 *Assume that the costs satisfy*

$$c(v, w) \geq -c(w, v)/\gamma(w, v), \quad \forall (v, w) \in E, \tag{2}$$

instead of the antisymmetricity (1). Any circuit on G_f has nonnegative cost if and only if there exists a potential function $\pi : V \to \mathbb{R}$ such that $c^\pi(v, w) \geq 0$, $\forall (v, w) \in E_f$. □

As a consequence of the above two properties, we obtain the second optimality criterion.

Theorem 3 *A generalized flow f is minimum-cost if and only if there exists a potential function $\pi : V \to \mathbb{R}$ such that $c^\pi(v, w) \geq 0$, $\forall (v, w) \in E_f$.* □

The following property is useful in our discussion.

Lemma 4 *For any potential π and any generalized circulation f, we have $c(f) = c^\pi(f)$.*

Proof.

$$
\begin{aligned}
c(f) &= c(f) + \sum_{v \in V} (\pi(v) \sum_{(v,w) \in E} f(v,w)) \\
&= \sum_{\substack{(v,w) \in E \\ f(v,w) > 0}} c(v,w) f(v,w) + \sum_{v \in V} \pi(v) (\sum_{\substack{(v,w) \in E \\ f(v,w) > 0}} f(v,w) - \sum_{\substack{(v,w) \in E \\ f(v,w) < 0}} \gamma(w,v) f(w,v)) \\
&= \sum_{\substack{(v,w) \in E \\ f(v,w) > 0}} (c(v,w) + \pi(v) - \gamma(v,w)\pi(w)) \ f(v,w) = c^\pi(f).
\end{aligned}
$$

□

3 Relationship to minimum mean/cost-to-weight ratio circuit problems

This section shows that MIGCF reduces to a minimum mean circuit problem in the residual graph G_{f^0} and wMIGCF reduces to a minimum cost-to-weight ratio circuit problem in G_{f^0}. Recall that f^0 is a given generalized flow for MIGCF and wMIGCF.

By using Theorem 3, we can formulate wMIGCF as follows:

$$
\text{(P1)} \left| \begin{array}{lll}
\text{Minimize} & \max\{\tau(v,w)(d(v,w) - c(v,w)) \mid (v,w) \in E\} & \\
\text{subject to} & d(v,w) = -d(w,v)/\gamma(w,v), & \forall (v,w) \in E, \\
& d^\pi(v,w) = d(v,w) + \pi(v) - \gamma(v,w)\pi(w) \geq 0, & \forall (v,w) \in E_{f^0}.
\end{array} \right.
$$

The cost antisymmetry constraint implies that the objective value of any feasible solution is nonnegative. Hence, if f^0 is minimum-cost for c, then an optimal solution d is equal to c. Problem (P1) can be reduced to solving the following problem:

$$
\text{(P2)} \left| \begin{array}{lll}
\text{Minimize} & \lambda & \\
\text{subject to} & c(v,w) + \frac{\lambda}{\tau(v,w)} + \pi(v) - \gamma(v,w)\pi(w) \geq 0, & \forall (v,w) \in E_{f^0}, \\
& \lambda \geq 0. &
\end{array} \right.
$$

Lemma 5 *When we have an optimal solution* $(\hat{\lambda}, \hat{\pi})$ *of* (P2)*, let* $\hat{d}$ *be given by:*

- $\hat{d}(v,w) = c(v,w) - \min\{0, c^{\hat{\pi}}(v,w)\}$ *and* $\hat{d}(w,v) = -\hat{d}(v,w)/\gamma(v,w)$*, if* $(v,w) \in E_{f^0}$ *and* $(w,v) \notin E_{f^0}$*.*
- $\hat{d}(v,w) = c(v,w) - c^{\hat{\pi}}(v,w)$ *and* $\hat{d}(w,v) = -\hat{d}(v,w)/\gamma(v,w)$*, if* $(v,w),(w,v) \in E_{f^0}$ *and* $c^{\hat{\pi}}(v,w) \leq 0$*.*

Then a pair $(\hat{d}, \hat{\pi})$ *is an optimal solution of* (P1)*. Moreover, the optimal values of* (P1) *and* (P2) *coincide with each other.*

Proof. It is clear that, for any feasible solution (d, π) of (P1) and any $\lambda \geq \max\{\tau(v,w)(d(v,w) - c(v,w)) \mid (v,w) \in E\}$, a pair of (λ, π) is feasible for (P2). Hence, $\hat{\lambda} \leq \lambda$ holds.

We now show that $(\hat{d}, \hat{\pi})$ is feasible for (P1) and $\hat{\lambda} \geq \max\{\tau(v,w)(\hat{d}(v,w) - c(v,w)) \mid (v,w) \in E\}$ holds. It is obvious that $\hat{d}$ satisfies the cost antisymmetry constraint. When $(v,w) \in E_{f^0}$ and $c^{\hat{\pi}}(v,w) \leq 0$, we have

$$\hat{d}^{\hat{\pi}}(v,w) = c(v,w) - c^{\hat{\pi}}(v,w) + \hat{\pi}(v) - \gamma(v,w)\hat{\pi}(w) = 0,$$

and

$$\hat{d}(v,w) - c(v,w) = -c^{\hat{\pi}}(v,w) \leq \frac{\hat{\lambda}}{\tau(v,w)},$$

where the last inequality comes from the condition of (P2). In this case,

$$\hat{d}^{\hat{\pi}}(w,v) = -\hat{d}^{\hat{\pi}}(v,w)/\gamma(v,w) = 0$$

and

$$\hat{d}(w,v) - c(w,v) = -(\hat{d}(v,w) - c(v,w))/\gamma(v,w) = c^{\hat{\pi}}(v,w)/\gamma(v,w) \leq 0 \leq \frac{\hat{\lambda}}{\tau(w,v)}$$

holds. When $(v,w) \in E_{f^0}$, $(w,v) \notin E_{f^0}$ and $c^{\hat{\pi}}(v,w) > 0$, $\hat{d}(v,w)$ is given by $c(v,w)$. Thus we have $\hat{d}^{\hat{\pi}}(v,w) = c^{\hat{\pi}}(v,w) > 0$ and $\hat{d}(v,w) - c(v,w) = 0 \leq \frac{\hat{\lambda}}{\tau(v,w)}$ and $\hat{d}(w,v) - c(w,v) = 0 \leq \frac{\hat{\lambda}}{\tau(w,v)}$. □

We are now in a position to discuss the relationship between wMIGCF and a minimum cost-to-weight ratio circuit problem. Given a positive weight $\omega : E \to \mathbb{R}_{++}$, the *minimum* *cost-to-weight ratio circuit problem* on G_{f^0} is defined as follows:

$$\min\{\frac{c(g)}{\omega(g)} \mid g : \text{circuit on } G_{f^0}\}, \tag{3}$$

where, for the zero circuit $g = 0$, we put $c(g)/\omega(g) = 0$. When $\omega(v, w) = 1$ for all $(v, w) \in E$, the minimum cost-to-weight ratio circuit problem is called the *minimum mean circuit problem.* When the gain of each edge is equal to one, the minimum cost-to-weight ratio circuit problem and the minimum mean circuit problem coincide with the minimum cost-to-weight cycle problem:

$$\min\{\frac{\sum_{(v,w)\in D} c(v, w)}{\sum_{(v,w)\in D} \omega(v, w)} \mid D : \text{cycle}\},$$

and the minimum mean cycle problem:

$$\min\{\frac{\sum_{(v,w)\in D} c(v, w)}{|D|} \mid D : \text{cycle}\},$$

respectively. Let $\hat{\xi}$ be the optimal value of the minimum cost-to-weight circuit problem (3).

Lemma 6 *Assume that* $\hat{\lambda}$ *is the optimal value of* (P2). *When* $\tau(v, w) = 1/\omega(v, w)$ *for all* $(v, w) \in E$*, we have* $\hat{\lambda} = -\hat{\xi}$*.*

Proof. Let $\hat{g}$ be an optimal circuit for minimum cost-to-weight circuit problem in G_{f^0}, and $\hat{\pi}$, together with $\hat{\lambda}$, be an optimal solution of (P2). When $\hat{g} = 0$, G_f contains no circuit with negative cost. From Lemma 2, we have $\hat{\lambda} = 0 = -\hat{\xi}$.

We now assume that $\hat{g} \neq 0$. Since

$$\begin{aligned} c(\hat{g}) &= c^{\hat{\pi}}(\hat{g}) \\ &= \sum_{\substack{(v,w)\in E \\ \hat{g}(v,w)>0}} (c(v, w) + \hat{\pi}(v) - \gamma(v, w)\hat{\pi}(w))\, \hat{g}(v, w) \\ &\geq \sum_{\substack{(v,w)\in E \\ \hat{g}(v,w)>0}} -\frac{\hat{\lambda}}{\tau(v, w)}\hat{g}(v, w) = -\hat{\lambda} \sum_{\substack{(v,w)\in E \\ \hat{g}(v,w)>0}} \omega(v, w)\hat{g}(v, w) \end{aligned}$$

holds, we have

$$\hat{\xi} = \frac{c(\hat{g})}{\omega(\hat{g})} \geq -\hat{\lambda}.$$

We replace each edge cost $c(v,w)$ by $\bar{c}(v,w) := c(v,w) - \hat{\xi}\omega(v,w)$. For any circuit g in G_{f^0}, we have

$$\begin{aligned} \bar{c}(g) &= \sum_{\substack{(v,w)\in E \\ g(v,w)>0}} (c(v,w) - \hat{\xi}\omega(v,w))g(v,w) \\ &= c(g) - \hat{\xi}\omega(g) \geq 0, \end{aligned}$$

since $c(g)/\omega(g) \geq \hat{\xi}$. Thus G_{f^0} contains no negative cost circuit with respect to $\bar{c}$. Since $\hat{\xi} \leq 0$, the cost $\bar{c}$ satisfies (2). Hence Lemma 2 implies that there exists a potential function π such that $\bar{c}^{\pi}(v,w) = c(v,w) - \hat{\xi}\omega(v,w) + \pi(v) - \gamma(v,w)\pi(w) = c(v,w) - \frac{\hat{\xi}}{\tau(v,w)} + \pi(v) - \gamma(v,w)\pi(w) \geq 0$ for all $(v,w) \in E_{f^0}$. Therefore, we obtain $\hat{\lambda} \leq -\hat{\xi}$. □

Theorem 7 *1. The optimal value of the minimum mean circuit problem is equal to the minus of the optimal value of MIGCF.*

2. The optimal value of the minimum cost-to-weight ratio problem is equal to the minus of the optimal value of wMIGCF, when $\omega(v,w) = 1/\tau(v,w)$ for all $(v,w) \in E$.

□

4 Algorithm

This section describes algorithms for wMIGCF. From Lemma 6, a minimum cost-to-weight circuit algorithm gives the optimal value $\hat{\lambda}$ of problem (P2). If we have $\hat{\lambda}$, an optimal solution of (P2) is obtained by the feasibility subroutine for linear constraints with at most two variables per inequalities (2VPI). Then we find an optimal solution of wMIGCF by Lemma 5.

In the following, we discuss an algorithm for the minimum cost-to-weight circuit problem. We assume that the weights ω for problem (3) are given by ratios of two integers which are no more than W.

For fractional optimization problems, the binary search is a standard strategy. The binary search algorithm maintains a search interval [LB, UB] containing $\hat{\xi}$, which is the optimal value of (3), and reduces the search interval without excluding $\hat{\xi}$ in each iteration. Each iteration checks there exists a negative circuit with respect to the cost function $c - \xi\omega$ for fixing ξ to $(\mathrm{UB} - \mathrm{LB})/2$. It is easy to see that $\xi \le \hat{\xi}$ holds if and only if there exists no negative cost circuit with respect to $c - \xi\omega$ in G_{f^0}. Let $c^{\pi}_{\xi\omega}(v, w) := c(v, w) - \xi\omega(v, w) + \pi(v) - \gamma(v, w)\pi(w)$ for all $(v, w) \in E$. When $\xi \le 0$, it follows from Lemma 2 that there exists $\pi : V \to \mathbb{R}$ with $c^{\pi}_{\xi\omega}(v, w) \ge 0$ for all $(v, w) \in E_{f^0}$ if and only if there exists no negative circuit with respect to $c - \xi\omega$ in G_{f^0}. Thus if there exists a potential π with $c^{\pi}_{\xi\omega}(v, w) \ge 0$ for all $(v, w) \in E_{f^0}$, then update $[\mathrm{LB}, \mathrm{UB}]$ to $[\xi, \mathrm{UB}]$; otherwise update $[\mathrm{LB}, \mathrm{UB}]$ to $[\mathrm{LB}, \xi]$. We can check such a potential π exists or not by algorithms to detect the feasibility of 2VPI. When the search interval is sufficiently small, we can obtain a minimum cost-to-weight ratio circuit. Wayne [10] has discussed the binary search algorithm solving the minimum cost-to-weight ratio circuit problem, and said the iteration number of the algorithm is $\mathrm{O}(m \log B)$. In the following lemma, we show that the binary search needs less iterations, and show how to obtain an optimal circuit when the binary search terminates.

Lemma 8 *Assume that* $\mathrm{UB} - \mathrm{LB} < 1/(B^{21n+4}W^{6n+4})$ *holds. For* $\xi = \mathrm{LB}$*, let* π *be a feasible potential with respect to* $c - \xi\omega$ *and* $\bar{E} = \{(v, w) \in E \mid c^{\pi}_{\xi\omega}(v, w) < 1/(B^{18n+4}W^{6n+3})\}$*. If* $\bar{E}$ *does not contain any unit-gain cycle and any bicycle, the zero flow is an optimal circuit. Otherwise, any circuit* $g \ne 0$ *with* $\{(v, w) \in E \mid g(v, w) > 0\} \subseteq \bar{E}$ *is an optimal circuit.*

Proof. Note that $|\{(v, w) \in E \mid g(v, w) > 0\}| \le \min\{3n, m\}$ holds for any circuit g. Hence for an edge $(v', w') \in E$ with $g(v', w') > 0$,

$$\omega(g) \le W \sum_{(v,w) \in E : g(v,w) > 0} g(v, w) < W \sum_{i=0}^{3n} B^i g(v', w') < W B^{3n+1} g(v', w')$$

holds. We also notice that we only have to consider circuits whose values are ratio. By g_d, we denote the denominator of a circuit g. Then, the least common denominator T_g of a circuit g is less than $B^{3n+1} g_d(v', w')$.

Let $\hat{g}$ be a minimum cost-to-weight ratio circuit. Then we obtain

$$c^{\pi}_{\xi\omega}(\hat{g}) = c^{\pi}(\hat{g}) - \xi\omega(\hat{g}) = c(\hat{g}) - \xi\omega(\hat{g}) = (\hat{\xi} - \mathrm{LB})\omega(\hat{g}) < \frac{\omega(\hat{g})}{B^{21n+4}W^{6n+4}}.$$

If there exists $(v', w') \in E \setminus \bar{E}$ with $\hat{g}(v', w') > 0$,

$$c^{\pi}_{\xi\omega}(\hat{g}) > c^{\pi}_{\xi\omega}(v', w')\hat{g}(v', w') > \frac{1}{B^{18n+4}W^{6n+3}} \cdot \frac{\omega(\hat{g})}{B^{3n}W} = \frac{\omega(\hat{g})}{B^{21n+4}W^{6n+4}}$$

holds. Therefore, $\hat{g}$ is a circuit in $\bar{E}$.

We assume that a circuit $g' \neq 0$ in $\bar{E}$ is not optimal. Since $\sum_{g'(v,w)>0} g'(v, w) \leq W\omega(g')$,

$$c^{\pi}_{\xi\omega}(g') < \sum_{(v,w)\in E: g'(v,w)>0} \frac{1}{B^{18n+4}W^{6n+3}} g'(v, w) \leq \frac{1}{B^{18n+4}W^{6n+3}} \cdot W\omega(g') = \frac{\omega(g')}{B^{18n+4}W^{6n+2}}$$

holds. On the other hand, we have

$$c^{\pi}_{\xi\omega}(g') = c^{\pi}(g') - \xi\omega(g') = c(g') - \xi\omega(g') \geq c(g') - \hat{\xi}\omega(g') = c(g') - \frac{c(\hat{g})}{\omega(\hat{g})} \cdot \omega(g').$$

We now evaluate the difference of $c(g')/\omega(g')$ and $c(\hat{g})/\omega(\hat{g})$.

$$\begin{aligned} \frac{c(g')}{\omega(g')} - \frac{c(\hat{g})}{\omega(\hat{g})} &= \frac{c(g')\omega(\hat{g}) - c(\hat{g})\omega(g')}{\omega(g')\omega(\hat{g})} \\ &\geq \frac{\left(T_{g'}T_{\hat{g}} \prod_{g'(v,w)>0} c_d(v, w) \prod_{\hat{g}(v,w)>0} \omega_d(v, w) \prod_{\hat{g}(v,w)>0} c_d(v, w) \prod_{g'(v,w)>0} \omega_d(v, w)\right)^{-1}}{\omega(g')\omega(\hat{g})} \\ &> \frac{1}{B^{18n+4}W^{6n+2}g'(v', w')\hat{g}(v'', w'')g'_d(v', w')\hat{g}_d(v'', w'')}, \end{aligned}$$

where c_d and ω_d stand for denominators of cost c and weight ω, respectively, and $g'(v', w') > 0$ and $\hat{g}(v'', w'') > 0$. Without loss of generality, we can assume $g'(v', w') = \hat{g}(v'', w'') = 1$. Hence we have

$$c^{\pi}_{\xi\omega}(g') \geq (\frac{c(g')}{\omega(g')} - \frac{c(\hat{g})}{\omega(\hat{g})})\omega(g') > \frac{\omega(g')}{B^{18n+4}W^{6n+2}},$$

which is contradiction. □

We can set the initial search interval as $[\mathrm{LB}, \mathrm{UB}] = [-B, 0]$. Hence the number of iterations is $\mathrm{O}(n \log(BW))$. Therefore, we obtain the following theorem.

Theorem 9 *The minimum cost-to-weight ratio circuit problem can be solved in* $\mathrm{O}(\mathrm{T}_{2\mathrm{VPI}} \cdot n \log(BW))$ *time, where* $\mathrm{T}_{2\mathrm{VPI}}$ *is a time bound for a 2VPI feasibility subroutine. Moreover, wMIGCF is also solved in the same running time.* □

Currently, the best known complexity bound of T_{2VPI} is $O(n^2 m \log n)$ due to Hochbaum and Naor [5]. Therefore, MIGCF and wMIGCF are solved in $O(n^3 m \log n \log B)$ and $O(n^3 m \log n \log(BW))$, respectively, by the binary search algorithm. When B and W are smaller than n, the binary search algorithm is superiority to the Megiddo's parametric search which runs in $O(n^3 m \log^4 n)$ time by incorporated with a 2VPI feasibility subroutine [3, 5] (see also [10]).

If G_{f^0} has no flow-generating cycles, we can check the existence of negative circuits, more efficiently. We start by constructing a graph $\bar{G}$ having vertices in V together with a new vertex s and edges in $\bar{E} = E_{f^0} \cup \{(s,v) \mid v \in V\}$. The cost and gain of every edge in $\{(s,v) \mid v \in V\}$ are zero and one, respectively. For each vertex $v \in V$, a label $\mu(v)$ is defined as the gain of the highest-gain simple s-v path in $\bar{G}$ and $\mu(s) = 1$. This label μ can be found by solving a shortest path problem, where edge lengths are defined as $-log(\gamma(v,w))$ for all $(v,w) \in \bar{E}$, because $\bar{G}$ contains no flow-generating cycles and every vertex is reachable from s. The relabeled gain, relabeled capacity, relabeled cost and relabeled weight of $(v,w) \in \bar{E}$ are defined as $\gamma_\mu(v,w) = \gamma(v,w)\mu(v)/\mu(w)$, $u_\mu(v,w) = u(v,w)/\mu(v)$, $c_\mu(v,w) = c(v,w)\mu(v)$ and $\omega_\mu(v,w) = \omega(v,w)\mu(v)$, respectively. Note that $\gamma_\mu(v,w) \leq 1$ for all $(v,w) \in \bar{E}$. Let $\bar{E}_\mu = \{(v,w) \in \bar{E} \mid \gamma_\mu(v,w) = 1\}$ and $\bar{G}_\mu = (V \cup \{s\}, \bar{E}_\mu)$. For all $v \in V$, there exists an s-v path in $\bar{G}_\mu$. Moreover, a cycle C is a unit-gain cycle in G_{f^0} if and only if C is a cycle in $\bar{G}_\mu$. Since G_{f^0} does not contain any bicycle, we have the following lemma.

Lemma 10 *Let G_{f^0} be a graph without flow-generating cycles. If $g(\neq 0)$ is a circuit on G_{f^0}, then $C = \{(v,w) \in E \mid g(v,w) > 0\}$ is a cycle in $\bar{G}_\mu$. On the other hand, when C is a cycle in $\bar{G}_\mu$, there exists a circuit $g \neq 0$ on G_{f^0} such that $(v,w) \in C$ if $g(v,w) > 0$. Moreover, we have*

$$\frac{c(g)}{\omega(g)} = \frac{\sum_{(v,w)\in C} c_\mu(v,w)}{\sum_{(v,w)\in C} \omega_\mu(v,w)}.$$

□

Therefore the minimum cost-to-weight ratio circuit problem (3) is reduced to

$$\min\{\frac{\sum_{(v,w)\in C} c_\mu(v,w)}{\sum_{(v,w)\in C} \omega_\mu(v,w)} \mid C : \text{cycle in } \bar{G}_\mu\}, \tag{4}$$

which is a minimum cost-to-weight ratio cycle problem on a graph without gains. Let $\bar{c}_{\xi\omega}(v,w) := c_\mu(v,w) - \xi\omega_\mu(v,w)$ for all $(v,w) \in \bar{E}$. Then we can check there exists a negative cycle on $\bar{G}_\mu$ with respect to $\bar{c}_{\xi\omega}$ in $\mathrm{O}(nm)$ time by Bellman-Ford's shortest path algorithm. In this case, since $\mathrm{T}_{\mathrm{2VPI}} = \mathrm{O}(nm)$ the binary search algorithm solves the minimum cost-to-weight ratio circuit problem and wMIGCF in $\mathrm{O}(n^2 m \log(BW))$ time. Moreover, using $\bar{G}_\mu$, an algorithm based on Megiddo's parametric search technique [6] solves the minimum cost-to weight ratio circuit problem and wMIGCF in $\mathrm{O}(n^2 m \log n)$ time, as the algorithm for generalized shortest path problems due to Oldham [8].

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