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Imperfect Recall and Solution Concepts in Extensive Games

by

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Abstract

We consider some implications of imperfect recall for existence and subsetting results over various equilibrium concepts in extensive games. Our analysis focuses on commonly time-structured games which allow us to see more clearly in games of imperfect recall. We find that in a game of imperfect recall, a sequential equilibrium may not be a Nash equilibrium, and a perfect equilibrium may not be a sequential equilibrium. A perfect equilibrium exists in every one-player game, but a sequential equilibrium may not exist. We also give sufficiency conditions weaker than perfect recall, for the standard subsetting results between perfect equilibrium, sequential equilibrium and Nash equilibrium.

1 Introduction

We give some results for various equilibrium concepts when the perfect recall condition does not hold. We restrict our analysis to commonly time-structured games (Kline [5]). One reason for concentrating on these games rather than Kuhn's [7] more general class is to show that many of the same types of phenomena are obtained in the restricted class. A benefit of using commonly time-structured games is that it allows us to see more clearly in games of imperfect recall.

The equilibrium concepts considered in this paper are Nash equilibrium, sequential equilibrium, and perfect equilibrium. Our comparison is done in finite $n-$player games and one-player games. We relate our findings to those known for games of perfect recall.

In $n-$player games, a major finding is that the subsetting relationship between sequential equilibrium and the other two solution concepts breaks down when the game does not satisfy perfect recall. A sequential equilibrium may not be a Nash equilibrium, and a perfect equilibrium may not be a sequential equilibrium (Figure 1). On the other hand, every perfect equilibrium is a Nash equilibrium with or without the assumption of perfect recall (Theorem 3.2). The fact that a sequential equilibrium may not be a Nash equilibrium is shown to be related to the "local" nature of optimality of sequential equilibrium as opposed to the "global" optimality assumed by Nash equilibrium.

In one-player games, we find that a perfect equilibrium always exists (Theorem 4.1), but a sequential equilibrium may not exist if the player doesn't have perfect recall. We relate the non-existence of a sequential equilibrium to the fact that a player considering sequential rationality at an information set does not consider the implications of sequential rationality at his future information sets. This same feature is present in the notion of a "time-consistent" strategy introduced by Piccione and Rubinstein [10]. We show that every sequential equilibrium is time-consistent (Lemma 4.2), and thus non-existence of a time-consistent strategy implies non-existence of a sequential equilibrium. Battigalli [1] and Kline [4] showed that time-consistent strategies may not exist in one-player games with imperfect recall.

In Section 5 we extend the subsetting results for $n-$player games and the existence results for one-player games into some region of imperfect recall. The standard subsetting relationship between perfect equilibrium, sequential equilibrium, and Nash equilibrium holds in $n-$player games with a condition on memory weaker than perfect recall known as "occurrence memory" (Theorem 5.3). The subsetting relationship that every sequential equilibrium is a Nash equilibrium can be maintained even further to a weakening of occurrence memory known as "a-loss recall" (Theorem 5.1). By the existence of a perfect equilibrium in a one-player game, and the subsetting results of equilibria for occurrence memory, we obtain existence of a sequential equilibrium in a one-player game with occurrence memory (Corollary 5.4).

2 Extensive games and equilibrium concepts

In Section 2.1 we define a commonly time-structured game based on Kline [5]. The definition of an extensive game is based on Selten [11] and Kuhn [7]. In Section 2.2 we define the solution concepts to be explored.

2.1 Commonly time-structured games

A *finite extensive game* $\Gamma = ((K, \preceq), P, U, C, p, h)$ is defined for a finite set of players $\{0, 1, ..., n\}$. The *chance player* (*nature*) is player 0, and $N = \{1, ..., n\}$ is the set of *personal players*. $(K, \preceq)$ is a *finite tree* which means: (i) K is the finite set of nodes in the tree, (ii) $\preceq$ is a partial ordering in K describing weak precedence with a smallest element $x_0 \in K$ called the root node, and (iii) for each $x \in K$, the set $\{y \in K : y \preceq x\}$ is completely ordered by $\preceq$.

We write $x \npreceq y$ whenever it is not the case that $x \preceq y$. The *strict precedence relation* $\prec$ is defined by $x \preceq y$ and $y \npreceq x$. When $x \prec y$, we say x is a *predecessor* of y and also that y is a *successor* of x. The set K is partitioned into the set of nodes without successors called *terminal nodes* and denoted by Z, and the set of *decision nodes* denoted by X. P is a *player partition* describing the decision nodes where each player moves.

$U = \{U_0, U_1, ..., U_n\}$ is the *information pattern.* U_i is called player *i's information partition* and an element $u \in U_i$ is called an *information set* of player i. U_0 is assumed to be made up of singleton sets. We use $\mathcal{U} = \bigcup_{i \in N \cup \{0\}} U_i$ to denote the set of all information sets in the game.

C is a *choice partition* which describes the alternatives available to each player at each of his information sets. We use C_u to denote the set of alternatives at information set u. For an information set u, a choice $c \in C_u$ and a node $x \in K$ we write $u \prec_c x$ iff $y \prec x$ for some $y \in u$ and c is the choice at y leading to x. For $u, v \in U$ we write $u \prec v$ iff $x \prec y$ for some $x \in u$ and some $y \in v$. The last two elements of an extensive game are p, which assigns to each information set $u \in U_0$, a completely mixed[1] probability distribution over the choice set C_u, and h, which is a *payoff function* assigning a real vector $(h_1(z), ..., h_n(z))$ to each endnode $z \in Z$.

Commonly time-structured games: A finite extensive game Γ with the set of all information sets $\mathcal{U}$ is *commonly time-structured* iff there exists a natural number-valued function T on $\mathcal{U}$ such that:

$$\text{for all } u, v \in \mathcal{U},\ u \prec v \text{ implies } T(u) < T(v) \tag{2.1}$$

Kline [5] showed that every commonly time-structured game is an extensive game according to the definition of Kuhn [7]. In a commonly time-structured game, each player can order his moves across time with those of the other players. Such an ordering allows us to assign a time to each information set in the game. We use such an assignment in all the examples given in this paper. The time assignment allows us to make simple meaningful statements about a player's forgetfulness.

For example, the game of Figure 1 is a one player game with $U_1 = \{u, v\}$. Using the time assignment we can say that the player forgets at time $T = 3$ whether he moved at time $T = 2$.

Kuhn's definition of an extensive game did not require such an ordering. While all the results of this paper can be proved with Kuhn's definition, it is easier to reason through commonly time-structured games, and all the interesting findings of this paper occur in such games.

[1] By completely mixed we mean that every choice $c \in C_u$ is chosen with strictly positive probability.

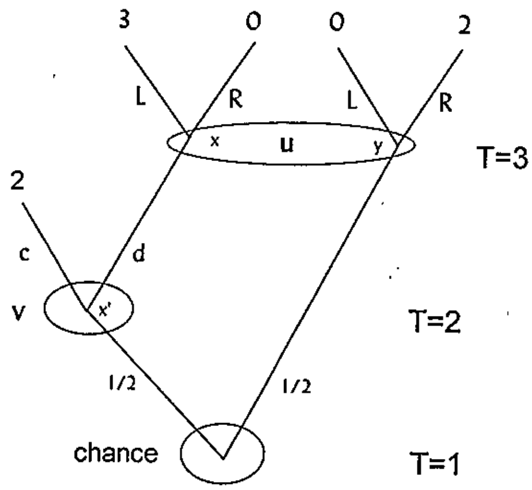

Figure 1

2.2 Equilibrium Concepts

We focus on behavior strategies, which are contingent plans that allow a player to randomize over the set of alternatives available to him at each information set. A *behavior strategy* of player i is a function b_i that assigns to each $u \in U_i$, a probability distribution b_{iu} over the set C_u of choices at u.We denote the set of behavior strategies of player i by B_i. Each b_{iu} is called a *local strategy* of player i at u. We denote the set of local strategies of player i at u by B_{iu}. An n-tuple $b = (b_1, ..., b_n)$ of behavior strategies, one for each player, is called a *strategy combination.* We will use (b'_i, b_{-i}) to denote the strategy combination obtained from b by replacing the behavior strategy b_i of player i by b'_i. We will also use (b'_{iu}, b_{-iu}) to denote the strategy combination obtained from b by replacing the local strategy of the player i moving at u by b'_{iu}.

The first equilibrium concept is a Nash equilibrium [8]. This equilibrium concept is based on the *ex ante* expected payoff of each player. For a strategy combination b, the *ex ante expected payoff* of player i is defined by $H_i(b) = \sum_{z \in Z} p(z, b)h_i(z)$, where $p(z, b)$ denotes the probability of reaching terminal node z when b is used in the game.

Nash Equilibrium: A strategy combination b^* is a *Nash equilibrium* iff for all $i \in N$, $H_i(b) \geq H_i(b'_i, b_{-i})$ for all $b'_i \in B_i$.

Next, we define a sequential equilibrium [6]. For this, we need the notion of a system of beliefs and the notion of expected payoff of a player conditional on being at an information set. A *system of beliefs* is a function μ on X satisfying: (a) $\mu(x) \in [0, 1]$ for all $x \in X$, and (b) $\sum_{x \in u} \mu(x) = 1$ for all $u \in \mathcal{U}$.

An ordered pair (b, μ) where b is a strategy combination and μ is a system of beliefs is called an *assessment.*

Given a strategy combination $b = (b_1, ..., b_n)$, a node $x \in X$, and a player $i \in N$, the *expected payoff of player i conditional on being at x* is defined by $H_{ix}(b) = \sum_{z \in Z_x} p(z \mid x, b)h_i(z)$ where $Z_x = \{z \in Z : x \prec z\}$ and $p(z \mid x, b)$ is the probability of reaching endnode z when we are currently at node x and b is being used in the continuation of the game. For an assessment (b, μ) and an information set u where a personal player i moves, the *expected payoff of player i conditional on being at u* is denoted by $H_{iu}(b, \mu) = \sum_{x \in u} \mu(x)H_{ix}(b)$. An assessment (b, μ) is *sequentially rational at information set u* of personal player i iff $H_{iu}(b, \mu) \geq H_{iu}((b'_i, b_{-i}), \mu)$ for all $b'_i \in B_i$. An assessment (b, μ) is called *sequentially rational* iff (b, μ) is sequentially rational at each personal information set in $\mathcal{U}$. An assessment (b, μ) is *consistent* iff there is a sequence of completely mixed[2] strategy combinations $\{b^k\}_{k=1}^{\infty}$ satisfying both $\lim_{k \to \infty} b^k = b$, and for each $u \in U$ and each $x \in u$, $\mu(x) = \lim_{k \to \infty} \frac{p(x, b^k)}{\sum_{y \in u} p(y, b^k)}$.

Sequential Equilibrium: An assessment (b, μ) is a *sequential equilibrium* iff (b, μ) is sequentially rational and consistent.

We say that a strategy combination b *supports a sequential equilibrium* iff there is a system of beliefs μ such that the assessment (b, μ) is a sequential equilibrium.

Finally, we define a perfect equilibrium [11]. It is based on the notion of a perturbed game. A *perturbed game* $\Gamma_\varepsilon = (\Gamma, \varepsilon)$ is a pair such that Γ is a finite extensive game and ε is a function assigning a minimum probability $\varepsilon_c > 0$ to each choice c at each personal information set u, and ε must satisfy the further restriction that for all personal information sets u, $\sum_{c \in C_u} \varepsilon_c < 1$. A behavior strategy $b_i \in B_i$ is *permissible* in the perturbed game Γ_ε iff at each $u \in U_i$, $b_{iu}(c) \geq \varepsilon_c$ for each $c \in C_u$. We let $B_{i\varepsilon}$ denote the set of permissible strategies of player i in the perturbed game Γ_ε. A strategy combination $b = (b_1, ..., b_n)$ is a *Nash equilibrium of the perturbed game* Γ_ε iff for all $i \in N$, $H_i(b) \geq H_i(b'_i, b_{-i})$ for all $b'_i \in B_{i\varepsilon}$.

Perfect Equilibrium: A strategy combination b is a *perfect equilibrium* in Γ iff there is a sequence of perturbed games $\{\Gamma^k\}_{k=1}^{\infty}$ of Γ, and a sequence of strategy combinations $\{b^k\}_{k=1}^{\infty}$ such that (i) for each k, b^k is a Nash equilibrium of the perturbed game Γ^k, and (ii) $\Gamma^k \to \Gamma$ and $b^k \to b$ as $k \to \infty$.

[2] A strategy combination $b^k = (b_1^k, ...b_n^k)$ is completely mixed if it assigns a strictly positive probability to each personal information set in the game. Since p completely mixes over chance moves, every node in the game tree is reached with positive probability under a completely mixed strategy combination.

3 N-player Games

Most of the analysis in game theory is on games with perfect recall. The perfect recall condition is interpreted as the condition under which each player remembers "what he observed" and "what he did." The precise condition on the information partition of a player in an extensive game is as follows.

Perfect recall: The information partition U_i satisfies *perfect recall* iff for all $u, v \in U_i$, all $x, y \in u$ and all $c \in C_v, v \prec_c x$ implies $v \prec_c y$.

The following results, listed in Theorem 3.1, are well known. We state them here just for comparison with the results we will give for games of imperfect recall.

Theorem 3.1 Let Γ be a game with perfect recall for each player.
(a) If (b, μ) is a sequential equilibrium, then b is a Nash equilibrium.
(b) If b is a perfect equilibrium, then (b, μ) is a sequential equilibrium for some system of beliefs μ.
(c) A perfect equilibrium b exists.

Without perfect recall, existence becomes problematic for each of the equilibrium concepts considered in this paper. However, regarding the subsetting relationships, perhaps less is known. We get the following result without the perfect recall assumption.

Theorem 3.2 If b is a perfect equilibrium, then b is a Nash equilibrium.

Selten ([11]; first Lemma 3) proved the result of Theorem 3.2 for a game of perfect recall. However, none of the arguments in his proof use the perfect recall assumption. Hence, Selten's proof is for games of imperfect recall.[3]

Notice that no mention of sequential equilibrium is made in Theorem 3.2. This is not a mistake, it is because there is no clear subsetting relationship between sequential equilibrium and the other concepts for a game of imperfect recall.

The game of Figure 1 has a perfect equilibrium that is not a sequential equilibrium, and a sequential equilibrium that is not a Nash equilibrium.

The unique Nash equilibrium and perfect equilibrium is $b_{1v}(c) = 1$ and $b_{1u}(R) = 1$. However, this is not a sequential equilibrium. Sequential rationality at v requires $b_{1v}(c) = 0$ and $b_{1u}(R) = 0$ for any system of beliefs. This alternative strategy combination when combined with the beliefs $\mu(x') = 1$,

[3]Incidentally, the results on subsetting of a "subgame perfect equilibrium" between perfect and Nash equilibrium also hold for games of imperfect recall. The interested reader is referred to Selten [11] for details on the solution concept called subgame perfect equilibrium.

$\mu(x) = \frac{1}{2}$, and $\mu(y) = \frac{1}{2}$, forms the unique sequential equilibrium of this game. The beliefs are obtained from any sequence of completely mixed strategies converging to the sequential equilibrium strategy, for example, the sequence defined by $b^k_{1v}(c) = \frac{1}{k+5}$ and $b^k_{1u}(R) = \frac{1}{k+5}$ for $k = 1, ..., \infty$.

One might wonder if any simple explanation can be given for our finding that a sequential equilibrium might not be a Nash equilibrium. One explanation is that sequential equilibrium is a concept of local optimality, since it requires optimization at each information set, but not necessarily for the whole game. On the other hand, both Nash equilibrium and perfect equilibrium are global optimality concepts since they require optimization over the whole game.

The distinction between local and global optimality can be related somewhat to one finding of Selten [11] regarding perfect equilibrium. He showed that when the perfect recall condition is satisfied, the global optimality of a perfect equilibrium can be obtained from the simultaneous satisfaction of many local optimality conditions, one at each information set. The local optimality conditions that suffice are optimality conditions for the "agent-normal form game" of the original extensive game.[4] Since every perfect equilibrium is a Nash equilibrium (Theorem 3.1), these local optimality conditions also suffice for the global optimality of a Nash equilibrium.

However, when we move to games of imperfect recall, these local optimality conditions might not be sufficient for the global optimality of a perfect equilibrium or Nash equilibrium, as is shown in Kline ([3], Theorem 3.4).

The local optimality conditions of a sequential equilibrium are different from the local optimality conditions defined by Selten [11] for the "agent normal form game". Nonetheless, the game of Figure 1 shows the insufficiency of the local optimality conditions of a sequential equilibrium for the global optimality of a Nash equilibrium in a game of imperfect recall.

4 One-player games

One-player games, or what are sometimes called decision problems, are simpler in the sense that we do not have to consider the reasoning of many players simultaneously. Thus, we might expect more results can be proved in those games. Indeed that is the case. The main improvement is about existence.

Theorem 4.1 A perfect equilibrium exists in every one-player game.

[4] We refer the reader to Selten (1975) for details on the agent normal form game. Basically, it is a game obtained from the original extensive game by attaching an independent decision maker to each information set, and giving him control over only choices at that information set, and assigning him the same payoffs as the player moving there in the original game.

Selten [11] proved the existence of a perfect equilibrium in an $n-$player game of perfect recall. Perfect recall was used to show the existence of a Nash equilibrium in each perturbed game. In a one player game, we have existence of a Nash equilibrium in every perturbed game with or without perfect recall. Hence the above theorem holds. By this theorem, and Theorem 3.2, we obtain the existence of a Nash equilibrium in every one-player game.

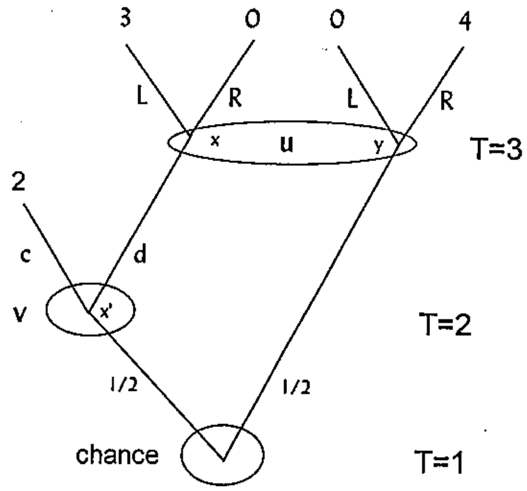

Figure 2

The existence of a sequential equilibrium, however, is not guaranteed in a one-player game of imperfect recall. The one-player game of Figure 2 is obtained from the game of Figure 1 by only changing the last payoff on the right from 2 to 4. In this new one-player game, a sequential equilibrium does not exist. At information set v, sequential rationality requires the player to choose as he did in Figure 1, that is, $b_{1v}(c) = 0$ and $b_{1u}(R) = 0$ for any beliefs. However, sequential rationality at u will now require him to choose $b_{1u}(R) = 1$ for any beliefs that might be part of a sequential equilibrium in this game. Hence, a sequential equilibrium does not exist.

The non-existence of a sequential equilibrium in this example can be related to the non-existence of what Piccione and Rubinstein [10] called a time-consistent strategy. They introduced this concept in a one-player game as a potential solution when a player can update his strategy as he moves through the game. Since we will compare this concept to a Nash equilibrium in $n-$player games in the next section, we give an $n-$player generalization of time-consistency here.

Time-Consistency: A strategy combination $b = (b_1, ..., b_n)$ is *time-consistent* iff for all $i \in N$, if $u \in U_i$ and $p(x,(b)) > 0$ for some $x \in u$, then $H_{iu}((b),\mu) \geq H_{iu}((b_i', b_{-i}),\mu)$ for all $b_i' \in B_i$, where μ is any system of beliefs that satisfies $\mu(x) = \frac{p(x,(b))}{\sum_{y \in u} p(y,(b))}$ for all $x \in u$.

Piccione and Rubinstein [10] noticed that for a one-player game of perfect recall, the set of Nash equilibria and time-consistent strategies are equivalent. They introduced the example of an "absent-minded" driver to show that with imperfect recall, the set of Nash equilibria and time-consistent strategies may be disjoint. Incidentally, the example they used was not commonly time-structured. However, Battigalli [1] gave a commonly time-structured example, and Kline ([4], Theorem 1) showed that this disjointedness result occurs in a large class of commonly time-structured games.

The following result is that all sequential equilibria are time-consistent. The converse need not hold, however, since time-consistency only restricts behavior at information sets that are reached by the strategy combination b, while a sequential equilibrium imposes restrictions at information sets throughout the game tree.

Lemma 4.2 If (b, μ) is a sequential equilibrium, then b is time-consistent.

Proof: Suppose that (b, μ) is a sequential equilibrium. Let u be an arbitrary information set of a personal player i. Suppose that $p(x, b) > 0$ for some $x \in u$. Consistency of (b, μ) implies that $\mu(x) = \frac{p(x,b)}{\sum_{y \in u} p(y,b)}$. Sequential rationality of (b, μ) implies that at u, $H_{iu}((b), \mu) \geq H_{iu}((b'_i, b_{-i}), \mu)$ for all $b'_i \in B_i$. Since u was chosen arbitrarily, b is time-consistent. □

In the game of Figure 2, we cannot find a time-consistent strategy. Time-consistency at v requires $b_{1v}(c) = 0$ and $b_{1u}(R) = 0$. However, time-consistency at u requires $b_{1u}(R) = 1$, and thus this game has no time-consistent strategy. The non-existence of a time-consistent strategy in this game implies the non-existence of a sequential equilibrium by Lemma 4.2. Kline ([4], Theorem 2) showed that the non-existence of a time-consistent strategy in a one player game occurs in a large class of games with imperfect recall. We return to this point and define this class of games in the next section.

The example of Figure 2 highlights another, perhaps unexpected, feature of a sequential equilibrium. The feature is that a player considering the sequential rationality of his strategy at an information set, does not consider the implications of sequential rationality at his future information sets.

In the game of Figure 2, the player considering the sequential rationality of a strategy combination at v, does not consider the implications of sequential rationality at his future information set u. Sequential rationality at v requires the player to choose $b_{1v}(d) = 1$ and $b_{1u}(L) = 1$ for any beliefs. This calculation ignores the fact that sequential rationality in the future, specifically at u, will require $b_{1u}(L) = 0$. If the player took sequential rationality at the future information set u into account, then he would clearly choose $b_{1v}(c) = 1$ to maximize his conditional expected payoff at v.

This same feature is shared by the concept of a time-consistent strategy. When I originally read Piccione and Rubinstein's [10] paper on time-consistency, I was under the impression that it incorporated the basic principles of backward induction and what is commonly regarded informally as "time-consistency".

In fact, in [4], I wrote that under the time-consistency view: "the player starts with a strategy, but each time his information set is reached he will decide whether or not to change the strategy. If he changes his strategy, then when he reaches a future information set he will consult the updated strategy and decide whether to make further changes." While I still believe that this is an accurate interpretation of the "time-consistency" view, it appears that this view is at odds with the concept of a time-consistent strategy [10] and the concept of a sequential equilibrium [6] as they are currently defined in the game theory literature. A player using either of these solution concepts does not take into account the fact that he optimizes at his future information sets.

Kline [4] showed by the same type of example as the one given in Figure 2 here, that the concept of a sequential equilibrium may not capture backward induction reasoning in a game of imperfect recall. On the other hand, a class of games including some region of imperfect recall for which the concept of sequential equilibrium does capture backward induction reasoning was also given in Kline ([4], Theorem 5.1).

5 Extending Results

In this section we show that the standard subsetting results of perfect equilibrium, subgame perfect equilibrium, and Nash equilibrium for $n-$player games can be extended to some region of imperfect recall.

We start with a condition on memory known as a-loss recall which was introduced by Kaneko and Kline [2] as a necessary and sufficient condition for mixed strategies[5] to fully compensate for a player's forgetfulness. This condition allows a player to forget things he did and learned.

A-loss Recall The information partition U_i satisfies *a-loss recall* iff for all $u, v \in U_i$, all $x, y \in u$, and all $c \in C_v$, if $v \prec_c x$, then either: (1) $v \prec_c y$ or (2) there exists $w \in U_i$ and distinct $d, e \in C_w$ satisfying $w \prec_d x$ and $w \prec_e y$.

The games of Figures 1 and 2 do not satisfy a-loss recall. The game of Figure 3 satisfies a-loss recall. It differs from the game of Figure 2 only because the chance move has been put under the control of player 1 and

[5]Mixed strategies differ from the behavior strategies used in the current paper. One difference is that the condition of a-loss recall does not guarantee that behavior strategies fully compensate for a player's forgetfulness.

renamed w with choices a and b.[6] In this game, the player still cannot recall at time $T = 3$ if he moved at time $T = 2$. However, in the current game, he also does not recall "what he did" at time $T = 1$.

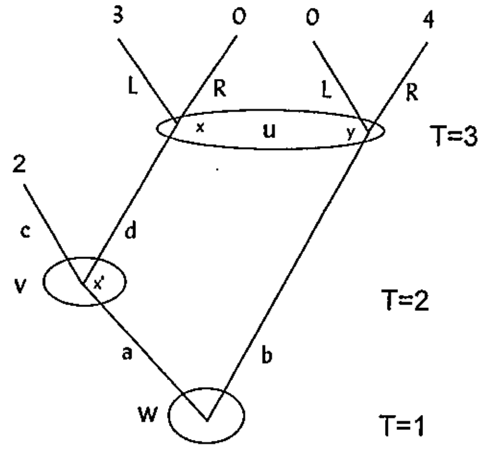

Figure 3

The next result is that a-loss recall is a sufficient condition for every sequential equilibrium to be a Nash equilibrium.

Theorem 5.1 Let Γ be a game with a-loss recall for each player. If (b, μ) is a sequential equilibrium, then b is a Nash equilibrium.

This theorem is proved by the combination of Lemma 4.2 and a result of Kline ([4], Theorem 1), that for every one-player game with a-loss recall, the set of *ex ante* optimal strategies and time-consistent strategies coincide. The generalization of *ex ante* optimal strategy to an $n-$player game results in a Nash equilibrium. We state the n-player game version of Kline's result in Lemma 5.2. It is proved by applying the result of Kline ([4], Theorem 1) to each personal player.

Lemma 5.2 Let Γ be a game with a-loss recall for each player. The following two statements are equivalent.
(a) b is a Nash equilibrium.
(b) b is time-consistent.

The reader can now verify that Theorem 5.1 follows immediately from Lemma 4.2 and Lemma 5.2.

We might try to extend the result that every perfect equilibrium is a sequential equilibrium to games with a-loss recall. The game of Figure 3, shows that we cannot go this far. The only perfect equilibrium in this game

[6] Every one-player commonly time-structured game without chance moves satisfies a-loss recall.

is $b_{1w}(b) = 1$, $b_{1v}(c) = 1$, and $b_{1u}(R) = 1$. This is not supported by a sequential equilibrium, however, since it is not sequentially rational at v. This is, however, a time-consistent strategy since v is not reached by the strategy, and thus time-consistency has no restriction there.

As was mentioned earlier in Section 4, Kline [4] showed that the non-existence of a time-consistent strategy in a one player game occurs in a large class of games with imperfect recall. Specifically, Kline ([4], Theorem 2) showed that for every one player game Γ that does not satisfy a-loss recall, there is another game Γ' that differs from Γ at most in terms of the payoffs, and Γ' does not have a time-consistent strategy. In the other direction, the existence of a time-consistent strategy in every one player game with a-loss recall was also shown by Kline([4], Theorem 2).

We find now, by the game of Figure 3, that the non-existence of a sequential equilibrium in a one player game is, in fact, a more serious problem then the non-existence of a time consistent strategy.

We can extend the result on subsetting between perfect equilibrium and sequential equilibrium to games that satisfy a stronger condition on memory than a-loss recall, known as occurrence memory.

Occurrence Memory: The information partition U_i of a player i satisfies *occurrence memory* iff for all $u, v \in U_i$, and all $x, y \in u$, if $v \prec x$ then $v \prec y$.

This condition is due to Okada [9] and is interpreted as requiring a player to recall everything he observed, though he might forget what he did. A player with a-loss recall, on the other hand, may forget both things he observed, and things he did. Every player with perfect recall has occurrence memory, and every player with occurrence memory has a-loss recall ([4], Lemma 4).

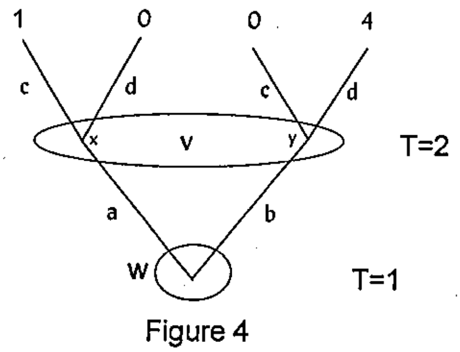

Figure 4

The game of Figure 4 with $U_1 = \{w, v\}$ is a simple example of a player with imperfect recall who has occurrence memory. All the other Figures given in the paper involve players who have imperfect recall, but do not have occurrence memory. In the example of Figure 4, the player at time $T = 2$ forgets only "what he did" at time $T = 1$.

Theorem 5.3 Let Γ be a game with occurrence memory for each player.
(a) If (b, μ) is a sequential equilibrium, then b is a Nash equilibrium.
(b) If b is a perfect equilibrium, then (b, μ) is a sequential equilibrium for some system of beliefs μ.

Since every player with occurrence memory satisfies a-loss recall, we obtain part (a) of Theorem 5.3 from Theorem 5.1. We prove part (b).

Proof of Theorem 5.3 (b): Suppose that b is a perfect equilibrium. There exists a sequence of perturbed games $\{\Gamma^k\}_{k=1}^{\infty}$ of Γ, and a sequence of strategy combinations $\{b^k\}_{k=1}^{\infty}$ such that (i) for each k, b^k is a Nash equilibrium of the perturbed game Γ^k, and (ii) $\Gamma^k \to \Gamma$ and $b^k \to b$ as $k \to \infty$.

For each $x \in X$, define the beliefs $\mu(x) = \lim_{k\to\infty} \frac{p(x,b^k)}{\sum_{y\in u} p(y,b^k)}$. Then (b, μ) is consistent. It suffices to show (b, μ) is sequentially rational.

Let u be an arbitrary information set of personal player i. For any perturbed game k, since b^k is a Nash equilibrium in Γ^k, it follows that:

$$H_i(b^k) \geq H_i(b'_i, b^k_{-i}) \text{ for all } b'_i \in B^k_i. \tag{5.1}$$

Here, B^k_i denotes the set of permissible strategies of player i in Γ^k. If we let $Z_{-u} = \{z \in Z : u \not\prec z\}$, then we can rewrite (5.1) as follows:

$$\sum_{z\in Z_{-u}} p(z, b^k)h_i(z) + \sum_{y\in u} p(y, b^k)H_{iy}(b^k) \geq$$

$$\sum_{z\in Z_{-u}} p(z, (b'_i, b^k_{-i}))h_i(z) + \sum_{y\in u} p(y, (b'_i, b^k_{-i}))H_{iy}(b'_i, b^k_{-i}),$$

$$\text{for all } b'_i \in B^k_i. \tag{5.2}$$

Consider any strategy $b'_i \in B^k_i$ that coincides with b^k_i everywhere except possibly at u and on $S(u) = \{v \in U_i : u \prec v\}$. Let $B^k_i(u, S(u))$ denote the set of all such strategies. By occurrence memory, it follows that for any $b'_i \in B^k_i(u, S(u))$, we have $\sum_{z\in Z_{-u}} p(z, (b'_i, b^k_{-i}))h_i(z) = \sum_{z\in Z_{-u}} p(z, b^k)h_i(z)$, and $p(y, (b'_i, b^k_{-i})) = p(y, b^k)$ for all $y \in u$. By this we obtain from (5.2) that:

$$\sum_{y\in u} p(y, b^k)H_{iy}(b^k) \geq \sum_{y\in u} p(y, b^k)H_{iy}(b'_i, b^k_{-i}) \text{ for all } b'_i \in B^k_i(u, S(u)). \tag{5.3}$$

Now if we define $\mu(x, b^k) = \frac{p(x,b^k)}{\sum_{y\in u} p(y,b^k)}$ at each $x \in u$, and use the fact that $\sum_{y\in u} p(y, b^k) > 0$ since this is a perturbed game, then we find that (5.3) is equivalent to:

$$\sum_{y\in u} \mu(x, b^k) H_{iy}(b^k) \geq \sum_{y\in u} \mu(x, b^k) H_{iy}(b'_i, b^k_{-i}) \text{ for all } b'_i \in B^k_i(u, S(u)). \quad (5.4)$$

Now take the limit as $k \to \infty$ to obtain from (5.4) by continuity of the payoff functions and continuity of μ that:

$$\sum_{y\in u} \mu(x) H_{iy}(b) \geq \sum_{y\in u} \mu(x) H_{iy}(b'_i, b_{-i}) \text{ for all } b'_i \in B_i(u, S(u)) \quad (5.5)$$

Observe that for any node $y \in u$, $H_{iy}(b'_i, b_{-i}) = H_{iy}(b''_i, b_{-i})$ for any two strategies b'_i and b''_i that agree at both u and everywhere on $S(u)$. Hence, from (5.5) we obtain:

$$\sum_{y\in u} \mu(x) H_{iy}(b) \geq \sum_{y\in u} \mu(x) H_{iy}(b'_i, b_{-i}) \text{ for all } b'_i \in B_i. \quad (5.6)$$

Since u was chosen arbitrarily, (b, μ) is sequentially rational. □

By Theorem 5.2 (b) and the perfect equilibrium existence result of Theorem 4.1 we obtain existence of a sequential equilibrium for every one-player game with occurrence memory. This is stated as the following corollary.

Corollary 5.4 Let Γ be a one-player game with occurrence memory. A sequential equilibrium exists.

6 Conclusions

We discussed some of the implications of imperfect recall for subsetting and existence results over various solution concepts. Perhaps the most interesting findings had to do with the concept of a sequential equilibrium. We showed that in a game without perfect recall, a perfect equilibrium may not be a sequential equilibrium, and a sequential equilibrium may not be a Nash equilibrium. Existence of a sequential equilibrium is problematic even in a one-player game.

These findings were used to point out two perhaps unexpected features of a sequential equilibrium. First, unlike Nash equilibrium and perfect equilibrium, sequential equilibrium is not a "global optimality" concept, but rather a series of "local optimality" conditions. Second, sequential equilibrium does not require a player at an information set to take into account that he will optimize at future information sets. This second feature leads to existence problems in games of imperfect recall.

References

[1] Battigalli, P. (1997), Dynamic consistency and imperfect recall, *Games and Economic Behavior* **20**, 31-50.

[2] Kaneko, M., and J. Kline (1995), Behavior strategies, mixed Strategies and perfect recall, *International Journal of Game Theory*, **24**, 127-145.

[3] Kline J. (1997), Information Structures and Decentralizability of Equilibria, *Economic Theory* **9**, 81-96.

[4] Kline, J. (2001), Minimum memory for equivalence between ex ante optimality and time-consistency, forthcoming in *Games and Economic Behavior.*

[5] Kline, J. (2002), Extensive games with time structures, mimeo.

[6] Kreps, D. and R. Wilson (1982), Sequential equilibria, *Econometrica* **50**, 863-894

[7] Kuhn, H. W. (1953), Extensive games and the problem of information, in Contributions to the Theory of Games II, Kuhn, H. W., and A. W. Tucker (eds), 193-216, Princeton University Press.

[8] Nash, J. F. (1951), Noncooperative games, *Annals of Mathematics* **54**, 286-295.

[9] Okada, A. (1987), Complete inflation and perfect recall in extensive games. *International Journal of Game Theory*, **16**, 85-91.

[10] Piccione, M., and A. Rubinstein (1997), On the interpretation of decision problems with imperfect recall, *Games and Economic Behavior*, **20**, 3-24.

[11] Selten, R. (1975), Reexamination of the perfectness concept for equilibrium points in extensive games, *International Journal of Game Theory*, **4**, 25-55.