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# RELATIVE EFFICIENCY OF SIMPLE RANDOM, STRATIFIED RANDOM AND SYSTEMATIC SAMPLING FOR ESTIMATING AN AREA OF A CERTAIN LAND USE

by

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## (1) INTRODUCTION

In urban land use analysis the first step is to measure an area occupied by a certain land use in a city. This task, however, becomes very costly if land use in a city is completely surveyed. To make this task more economical, a point sampling method is often employed in the related literature. (For example, see Berry (1962), Birch (1960, 1964), Blaut (1959), Haggett (1964), Haggett and Board (1964), Matalas (1963), Steiner (1957) and Wood (1955)). In statistics the major sampling methods are: simple random sampling, stratified random sampling and systematic sampling. In these methods, estimators for a certain parameter (such as a mean) are respectively proposed and relative efficiency of those estimators are theoretically examined. (See for example Cochran (1954)). Those general theorems are extended to spatial sampling by Das (1950), Obsorn (1942), Quensuille (1949), Williams (1956) and others, and their results are adopted in land use sampling. It should be noted, however, that the most of those sampling methods assume finite population $y_1$, $y_2$, ..., $y_n$ being distributed over a space $S$. (Some methods assume infinite super population for $y_i$, but the number of $y_i$, $i=1, 2, \ldots, n$ is still finite). Obviously population in land use is infinite because any point in a city $S$ can be a sample point, i.e., $y(\underline{x})$, $\underline{x} = (x_1, x_2) \in S$. Generally it is not guaranteed that the theorems obtained under finite population are applicable to infinite population. The objective of this paper is to explicitly deal with the estimation of land use in infinite population $S$ and to show several theorems with respect to accuracy and relative efficiency of the estimators used in simple random, stratified random, systematic sampling, which are respectively discussed in Sections 2, 3 and 4.

As a general setting, let $X$ be two dimensional real Cartesian space $X = \{\underline{x} = (x_1, x_2): -\infty \leq x_1, x_2 \leq \infty\}$, $S$ be a given compact subset of $X$, and $s$ be a given compact subset of $S$, i.e., $s \subseteq S \subset X$. (Imagine that $S$ is an urban area and $s$ is a certain land use area). The aim of point sampling is to estimate $|s| = \int_{\underline{x}\in s} d\underline{x}$ provided that $|S| = \int_{\underline{x}\in S} d\underline{x}$ is known. Let $y(\underline{x})$ be a variable taking values 1 or 0 indicating that if $\underline{x} \in s$, $y(\underline{x}) = 1$ and if $\underline{x} \notin s$, $y(\underline{x}) = 0$. Obviously

$$|s| = \int_{\underline{x}\in s} d\underline{x} = \int_{\underline{x}\in S} y(\underline{x})\, d\underline{x}. \qquad (1)$$

The method of sampling is to place a number, say $n$, of sample points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ on $S$ in a certain manner and to estimate $|s|$ by a function of $y(\underline{x}_i)$, $i = 1, 2, \ldots, n$.

## (2) SIMPLE RANDOM SAMPLING

In simple random sampling (or unrestricted random sampling), $n$ sample points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ are independently distributed over $S$ according to the uniform distribution given by

$$f(\underline{x}) = \begin{cases} 1/|S| & \text{if } \underline{x} \in S, \\ 0 & \text{if } \underline{x} \notin S. \end{cases} \quad (2)$$

Let

$$y_i = \begin{cases} 1 & \text{if } \underline{x}_i \in s, \\ 0 & \text{if } \underline{x}_i \notin s. \end{cases} \quad (3)$$

Then an estimator $|\hat{s}|_{sm}$ of $|s|$ used in simple random sampling is given by

$$|\hat{s}|_{sm} = |S| \sum_{i=1}^{n} y_i/n. \quad (4)$$

Concerning expected value and variance of this estimator, it is readily seen that:

<u>Theorem 1.</u> For simple random sampling,

$$E(|\hat{s}|_{sm}) = |s| \quad (5)$$

$$\text{Var } (|\hat{s}|_{sm}) = |s|(|S| - |s|)/n \quad (6)$$

<u>Proof.</u> Since $y_i$ follows the Bernoulli distribution with parameter $|s|/|S|$, the expected value and variance are obtained from

$$E(|\hat{s}|_{sm}) = |S| \sum_{i=1}^{n} E(y_i)/n = |S|n|s|/n|S|$$

$$\text{Var } (|\hat{s}|_{sm}) = |S|^2 \sum_{i=1}^{n} \text{Var}(y_i)/n^2$$

$$= |S|^2 \, n \, (|s|/|S|)(1 - |s|/|S|)/n^2,$$

which lead to equations (5) and (6).

Equation (5) shows that the estimator $|\hat{s}|_{sm}$ is unbiased. From equation (6) it follows that

$$0 \leq \mathrm{Var}(|\hat{s}|_{sm}) \leq |S|^2/4n, \tag{7}$$

where the maximum value is achieved when $|s| = |S|/2$. This implies that high efficiency is obtained when the area $|s|$ to be estimated is either small or large relative to $|S|$ and that efficiency becomes the lowest when the area is the half of $|S|$.

(3) STRATIFIED RANDOM SAMPLING

In stratified random sampling $S$ is divided into $m$ compact subsets, (or strata), $S_1, S_2, \ldots, S_m$, that satisfy $|S_i \cap S_j| = 0$, $i \neq j$ and $|S| = \sum_{i=1}^{m} |S_i|$. (Note that $|S_i|$ is known). On each stratum $S_i$, $n_i$ random sample points, $\underline{x}_{i1}, \underline{x}_{i2}, \ldots, \underline{x}_{in_i}$ are independently distributed according to the uniform distribution given by

$$f_i(\underline{x}) = \begin{cases} 1/|S_i| & \text{if } \underline{x} \in S_i, \\ 0 & \text{if } \underline{x} \notin S_i, \ i=1, 2, \ldots, m. \end{cases} \tag{8}$$

(Note that all points $\underline{x}_{11}, \underline{x}_{12}, \ldots, \underline{x}_{mn_m}$ are statistically independent and that $\sum_{i=1}^{n} n_i = n$). Let $s_i = s \cap S_i$ (a fraction of $s$ in $S_i$) and

$$y_{ij} = \begin{cases} 1 & \text{if } \underline{x}_{ij} \in s_i \\ 0 & \text{if } \underline{x}_{ij} \notin s_i,\ j=1,\ 2,\ \ldots,\ n_i, \end{cases} \tag{9}$$

Then an estimator $|\hat{s}|_{st}$ used in stratified random sampling is given by

$$|\hat{s}|_{st} = \sum_{i=1}^{m} |\hat{s}_i| \tag{10}$$

where

$$|\hat{s}_i| = |S_i| \sum_{j=1}^{n_i} y_{ij}/n_i. \tag{11}$$

<u>Theorem 2.</u> For stratified random sampling,

$$E(|\hat{s}|_{st}) = |s|, \tag{12}$$

$$\mathrm{Var}(|\hat{s}|_{st}) = \sum_{i=1}^{m} |s_i|(|S_i| - |s_i|)/n_i. \tag{13}$$

<u>Proof.</u> Since $|\hat{s}_i|$ is the estimator of $|s_i|$ used in simple random sampling, equation (12) and (13) are immediately obtained from Theorem 1.

Equation (11) says, like the case of simple random sampling, that the estimator $|\hat{s}|_{st}$ is unbiased.

To see relative efficiency with respect to $\mathrm{Var}(|\hat{s}|_{sm})$ and $\mathrm{Var}(|\hat{s}|_{st})$, consider the case in which $|S_i| = |S|/4$, $|s_i| = |S_i|/2$, $n_1=5$, $n_i=1$, $i=2, 3, 4$. Upon substituting these values into equations (6) and (13), it is obtained that $\mathrm{Var}(|\hat{s}|_{sm}) = 10\ |S|^2/320$ and

$\mathrm{Var}(|\hat{s}|_{st}) = 16\ |S|^2/320$. These values show that there exists a case in which stratified random sampling is less efficient than simple random sampling. However, if the number of sample points $n_i$ is carefully chosen, this case will not occur.

Theorem 3. If the number of sample points is given by

$$n_i = n\sqrt{|s_i|(|S_i| - |s_i|)} \Big/ \sum_{j=1}^{m} \sqrt{|s_j|(|S_j| - |s_j|)}\ , \tag{14}$$

then

$$\mathrm{Var}(|\hat{s}|_{st}) \leq \mathrm{Var}(|\hat{s}|_{sm}). \tag{15}$$

The equality holds if and only if

$$|s_i|/|S_i| = |s|/|S|,\ i=1,\ 2,\ \ldots,\ m \tag{16}$$

Proof. The proof is two-hold. First an optimal allocation of $n$ sample points to $m$ strata that minimize the variance given by equation (13) is to be considered. Second this minimized variance is to be compared with the variance given by equaiton (6). The first problem is mathematically written as

$$\text{Minimize} \quad \sum_{i=1}^{m} |s_i|\ (|S_i| - |s_i|)/n_i, \tag{17}$$

$$\text{subject to} \quad \sum_{i=1}^{m} n_i = n. \tag{18}$$

The Lagrange function $\mathcal{L}$ of this minimization problem is given by

$$\mathcal{L} = \sum_{i=1}^{m} |s_i|(|S_i| - |s_i|)/n_i + \lambda(n - \sum_{i-1}^{m} n_i). \tag{19}$$

By differentiating $\mathcal{L}$ with respect to $n_i$ and equating it to zero, the necessary condition for the minimum variance is written as

$$\sqrt{\lambda}\ n_i = \sqrt{|s_i|(|S_i| - |s_i|)} \tag{20}$$

Taking summation over $i=1, 2, \ldots, m$, it is obtained

$$\sqrt{\lambda}\ n = \sum_{i=1}^{m} \sqrt{|s_i|(|S_i| - |s_i|)} \tag{21}$$

From equations (20) and (21), equation (14) is obtained. Substitution of equation (14) into equation (13) gives

$$\mathrm{Var}(|\hat{s}|_{st})^* = \left(\sum_{i=1}^{m} \sqrt{|s_i|(|S_i| - |s_i|)}\right)^2/n. \tag{22}$$

It can be shown that $\mathrm{Var}(|\hat{s}|_{st})^* \leq \mathrm{Var}(|\hat{s}|_{st})$, i.e., $\mathrm{Var}(|\hat{s}|_{st})^*$ is the minimum variance.

Second, by use of the Cauchy-Schwarz inequality, it is obtained that

$$\left(\sum_{i=1}^{m} \sqrt{|s_i|}\sqrt{|S_i| - |s_i|}\right)^2/n \leq \sum_{i=1}^{m} |s_i| \sum_{i=1}^{m} (|S_i| - |s_i|)/n$$

$$= |s|\ (|S| - |s|)/n. \tag{23}$$

Since the left hand side is the variance of stratified random sampling with equation (14) and the right hand side is the variance of simple random sampling, inequation (15) holds. The equality holds if and only if $\sqrt{|S_i| - |s_i|}/\sqrt{|s_i|}$ is constant for all $i$, which is equivalent to equation (16), proving the theorem.

It is noted that the allocation given by equation (14) may correspond to the Neyman allocation referred to in the literature of sampling with finite population. (See p.97 of Cochran (1954)). Equation (14) says that a large number of sample points should be chosen if the function $s_i$ of $s$ in stratum $S_i$ occupies almost the half of $S_i$ and/or the area of stratum $S_i$ is large.

In Theorem 2, the number of strata is fixed. In a practical situation, however, strata may freely be designed. In this case the following theorem holds.

Theorem 4. For a given stratified random sampling with $n_i \geq 2$, (not necessarily for all $i=1, 2, \ldots, m$), there always exists a more efficient stratified random sampling with $n$ strata having one sample point in each stratum, (i.e., $n_i=1, i=1,2, \ldots, m=n$).

To prove this theorem, the following lemma is first shown.

Lemma 1. If $|S_i| = |S|/n$, $n_i=1$, $i=1, 2, \ldots, n$, then

$$\mathrm{Var}(|\hat{s}|_{st}) \leq \mathrm{Var}(|\hat{s}|_{sm}). \tag{24}$$

The equality holds if and only if $|s_i| = |S|/n$.

Proof. Substitution of $|S_i| = |S|/n$, $n_i=1$ into equation (13) gives

$$\mathrm{Var}(|\hat{s}|_{st}) = \sum_{i=1}^{n} |s_i|(|S|/n - |s_i|)$$

$$= |s||S|/n - \sum_{i=1}^{n} |s_i|^2$$

$$= \mathrm{Var}(|\hat{s}|_{sm}) + |s|^2/n - \sum_{i=1}^{n} |s_i|^2 \tag{25}$$

By use of the Cauchy-Schwarz inequality, it can be shown that

$$n \sum_{i=1}^{n} |s_i|^2 \geq \left(\sum_{i=1}^{n} |s_i|\right)^2. \tag{26}$$

From this inequality and equation (25), inequality (24) is obtained, proving the lemma.

Lemma 1 says that if $S$ is devided into $n$ equal size strata and one sample point is randomly placed in each stratum, then the estimator $|\hat{s}|_{st}$ of stratified random sampling is not less efficient than the estimator $|\hat{s}|_{sm}$ of simple random sampling. It should be noted, however, that Lemma 1 does not say that stratified random sampling with equal size strata is more efficient than that with unequal size strata. A counter example can be given by stratified random sampling with unequal strata which are designed in such a way that $|s_i|$ is close to zero or $|S_i|$.

<u>Proof of Theorem 4.</u> Consider a stratum $S_i$ having $n_i \geq 2$ points and let us devide $S_i$ into $n_i$ equal size strata $S_{i1}$, $S_{i2}$, ...,

and let us devide $S_i$ into $n_i$ equal size strata $S_{i1}, S_{i2}, \ldots, S_{in}$, on each of which one sample point $\underline{x}'_{ij}$ is randomly placed. Let $s_{ij} = s_i \cap S_{ij}$, and

$$y'_{ij} = \begin{cases} 1 & \text{if } \underline{x}'_{ij} \in s_{ij}, \\ 0 & \text{if } \underline{x}'_{ij} \notin s_{ij}. \end{cases} \qquad (27)$$

Then an estimator $|\hat{s}|'_{st}$ of the stratified random sampling with strata $S_{11}, S_{12}, \ldots, S_{1n_1}, S_{21}, \ldots, S_{mn_m}$ is, from equation (10) and (11), given by

$$|\hat{s}|'_{st} = \sum_{i=1}^{m} \sum_{j=1}^{n_i} |S_{ij}|\ y'_{ij}. \qquad (28)$$

By applying Lemma 1 to each strata, it is obtained that

$$\mathrm{Var}\Big(\sum_{j=1}^{n_i} |S_{ij}|\ y'_{ij}\Big) \leq \mathrm{Var}\Big(|S_i| \sum_{j=1}^{n_i} y_{ij}/n_i\Big), \qquad (29)$$

where $y_{ij}$ is given by equation (9). From equation (28) and (29) it follows that

$$\begin{aligned} \mathrm{Var}(|\hat{s}|'_{st}) &= \sum_{i=1}^{m} \mathrm{Var}\Big(\sum_{j=1}^{n_i} |S_{ij}|\ y'_{ij}\Big) \\ &\leq \sum_{i=1}^{m} \mathrm{Var}\Big(|S_i| \sum_{j=1}^{n_i} y_{ij}\Big) \\ &= \mathrm{Var}(|\hat{s}|_{st}). \end{aligned} \qquad (30)$$

Therefore the stratified random sampling with strata $S_{11}, S_{12}, \ldots, S_{mn_m}$ having one sample point in each $S_{ij}$ is not less efficient than the stratified random sampling with strata $S_1$, $S_2$, ..., $S_m$ having $n_i$ sample points in $S_i$. This completes the proof of Theorem 4.

## (4) SYSTEMATIC SAMPLING

In systematic sampling, first an initial sample point is randomly placed on $S$ and next the rest of $n-1$ sample points are placed with equal intervals from the initial point. To be explicit, let $S_1$, $S_2$, ..., $S_n$ be the same shape strata of $S$, and $F_i$ be the Euclid transformation maping $S_1$ onto $S_i$, i.e., $S_i = F_i(S_1)$. Then the initial sample point is randomly placed according to the uniform distribution $f_1(\underline{x})$ given by equation (7) and the rest of $n-1$ sample points are systematically placed according to $\underline{x}_i = F_i(\underline{x}_1)$, $i=2, 3, \ldots, n$. (Note that $F_1(\underline{x}_1) = \underline{x}_1$). Now let $s_i = s \cap S_1$ and

$$y_i = \begin{cases} 1 & \text{if } \underline{x}_i = F_i(\underline{x}_1) \in s_i, \\ 0 & \text{if } \underline{x}_i = F_i(\underline{x}_1) \in s_i, \quad i=1, 2, \ldots, n. \end{cases} \tag{31}$$

Then an estimator $|\hat{s}|_{sy}$ of $|s|$ used in systematic sampling is given by

$$|\hat{s}|_{sy} = |S| \sum_{i=1}^{n} y_i/n. \tag{32}$$

Theorem 5. For systematic sampling,

$$E(|\hat{s}|_{sy}) = |s|, \tag{33}$$

$$\mathrm{Var}(|\hat{s}|_{sy}) = |S||s|/n + 2T_2|S|^2/n^2 - |s|^2, \tag{34}$$

where
$$T_2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} n|s_i' \cap s_j'|/|S|, \tag{35}$$

$$s_i' = F_i^{-1}(s_i), \quad (F_i^{-1} \text{ is the inverse function of } F_i). \tag{36}$$

Proof. (It might be helpful to refer to figure 1). Let $s_{i1}'$, $s_{i2}'$, ..., $s_{ik}'$ be $k$ sets out of $n$ sets $s_1'$, $s_2'$, ..., $s_n'$, where $i1 < i2 < \ldots < ik$. If the initial sample point $\underline{x}_1$ belongs to $s_{i1}' \cap s_{i2}' \cap \ldots \cap s_{ik}'$, $k$ sample points $\underline{x}_{i1} = F_{i1}(\underline{x}_1)$, $\underline{x}_{i2} = F_{i2}(\underline{x}_1)$, ..., $\underline{x}_{ik} = F_{ik}(\underline{x}_1)$ respectively belong to $s_{i1}$, $s_{i2}$, ..., $s_{ik}$, ($s_{ij} \subseteq s$), and hence at least $k$ sample points belong to $s$. This event $\underline{x}_1 \in s_{i1}' \cap s_{i2}' \cap \ldots \cap s_{ik}'$ occurs with probability $|s_{i1}' \cap s_{i2}' \cap \ldots \cap s_{ik}'| / |S_1| = n|s_{i1}' \cap s_{i2}' \cap \ldots \cap s_{ik}'|/|S|$. Let

$$T_k = \sum_{i1=1}^{n-k+1} \sum_{i2=i1+1}^{n-k+2} \cdots \sum_{ik=i(k-1)+1}^{n} n|s_{i1}' \cap s_{i2}' \cap \ldots \cap s_{ik}'|/|S|. \tag{37}$$

By inclusion and exclusion method, the probability $q_k$ of exactly $k$ sample points belonging to $s$ is given by

$$q_k = T_k - \binom{k+1}{k} T_{k+1} + \ldots + (-1)^{n-k}\binom{n}{k} T_n. \tag{38}$$

Since the estimate of $|\hat{s}|_{sy}$ becomes $k|S|/n$ with probability $q_k$, the expected value is obtained from

$$E(|\hat{s}|_{sy}) = \sum_{i=1}^{n} q_k\ k|S|/n = T_1|S|/n$$

$$= \sum_{i=1}^{n} |s_i'| = \sum_{i=1}^{n} |s_i| = |s|, \qquad (39)$$

and the variance is obtained from

$$\mathrm{Var}(|\hat{s}|_{sy}) = E(|\hat{s}|_{sy}^2) - E(|\hat{s}|_{sy})^2$$

$$= \sum_{i=1}^{n} k^2|S|^2\ q_k/n^2 - T_1{}^2|S|^2/n^2$$

$$= (T_1 + 2T_2)|S|^2/n^2 - T_1{}^2|S|^2/n^2, \qquad (40)$$

which is, using $T_1 = n|s|/|S|$, written as equation (34). This completes the proof.

Figure 1

Equation (33) shows that the estimator $|\hat{s}|_{sy}$ is unbiased. Concerning the variance, it may be of interest to notice that equation (34) does not contain $T_k$, $k \geq 3$. This implies that the variance does not depend upon $s_{i1}' \cap s_{i2}' \cap \cdots \cap s_{ik}'$, $k \geq 3$ but it only depends upon $s_{i1}' \cap s_{i2}'$. Stated differently the variance depends upon the overlapped area in $S_1$ with respect to $s_i'$ and $s_j'$, $i \neq j$. With this property in mind, let us rewrite $\mathrm{Var}(|\hat{s}|_{sy})$ in terms of a correlation coefficient. A correlation coefficient $\rho$ between $\underline{x}_i = F_i(\underline{x}_1)$ and $\underline{x}_j = F_j(\underline{x}_1)$ for all $\underline{x}_1 \in S_1$ is defined by

$$\rho = E(y(F_i(\underline{x}_1)) - \bar{y})(y(F_j(\underline{x}_1)) - \bar{y}) / E(y(F_i(\underline{x}_1)) - \bar{y})^2 \quad (41)$$

where $\bar{y} = E(y(F_i(\underline{x}_1))$. Noticing that $y(F_i(\underline{x}_1))^2 = y(F_i(\underline{x}_1))$, $\rho$ is written as

$$\rho = \{ \int_{\underline{x}_1 \in S_1} \sum\sum_{i \neq j} y(F_i(\underline{x}_1)) y(F_j(\underline{x}_1)) d\underline{x}_1 / n(n-1)|S_1| - (|s|/|S|)^2\} / \{|s|/|S| - (|s|/|S|)^2\}. \quad (42)$$

Since $\int_{\underline{x}_1 \in S_1} y(F_i(\underline{x}_1))\, y(F_j(\underline{x}_1))\, d\underline{x}_1 = |s_i' \cap s_j'|$, (43)

the variance given by equation (34) is written as

$$\mathrm{Var}(|\hat{s}|_{sy}) = |s||S|/n + (n-1)|s|(|S| - |s|)\rho/n - |s|^2/n. \quad (44)$$

This result shows that if a positive correlation exists, the variance becomes large, whereas if a negative correlation exists, the variance becomes small.

Concerning relative efficiency, the following theorem is first shown.

Theorem 6. If $\rho \gtreqless 0$, then $\mathrm{Var}(|\hat{s}|_{sy}) \gtreqless \mathrm{Var}(|\hat{s}|_{sm})$. (45)

Proof. Upon substituting equation (6) into (44), equation (44) is written as

$$\mathrm{Var}(|\hat{s}|_{sy}) = \mathrm{Var}(|\hat{s}|_{sm}) + (n-1)|s|(|S| - |s|)\rho/n, \quad (46)$$

from which the result is immediately obtained.

This theorem says that if a positive correlation exists, simple random sampling is more efficient than systematic sampling; if there is no correlation, systematic sampling is as efficient as simple random sampling; if a negative correlation exists, systematic sampling is more efficient than simple random sampling.

Theorem 7. $\mathrm{Var}(|\hat{s}|_{sy}) \gtreqless \mathrm{Var}(|\hat{s}|_{sm})$ if and only if

$$\sum_{i \neq j}\sum |s_i' \cap s_j'|/n(n-1)\ |S| \gtreqless n(|s|/|S|)^2. \qquad (47)$$

$\mathrm{Var}(|\hat{s}|_{sy}) \gtreqless \mathrm{Var}(|\hat{s}|_{st})$ if and only if

$$|S| \sum_{i \neq j}\sum |s_i' \cap s_j'|/n \gtreqless \sum_{i \neq j}\sum |s_i||s_j|, \qquad (48)$$

where $\mathrm{Var}(|\hat{s}|_{st})$ is the variance of $|\hat{s}|_{st}$ in the case of $|S_i| = |S|/n$, $n_i=1$, $i=1, 2, \ldots, n$.

Proof. Noticing that $2T_2|S|^2/n^2 = |S| \sum_{i \neq j}\sum |s_i' \cap s_j'|/n$, the results are obtained, after a few steps of calculation, from comparing equation (32) with equations (6) and (25).

Let us consider implications of this theorem. The left hand side of inequality (47) implies the average ratio of overlapping areas $|s_i' \cap s_j'|/|S_1|$. If this value is smaller than $n$ times the ratio of the area $|s|$ to be estimated to the whole area $|S|$ raised by the second power, then systematic sampling is more efficient than simple random sampling. The implication of inequation (42) is not clear. It should be noted, however, that if overlapped areas $|s_i' \cap s_j'|$ $i \neq j$ is small, systematic sampling will be more efficient than simple random sampling or stratified random sampling. Such a case (i.e., $|s_i' \cap s_j'| \approx 0$) is likely to occur when $|s|$ is small relative to $|S_i|$, (like figure 2), or/and $s$ consists of small areas which are "randomly" distributed in the sense that $s_i'$ and $s_j'$ do not overlap, (like figure 3). If the number of sample points becomes large, $S_i$ becomes small and hence $|s|$ becomes large relative to $|S_i|$. In this case, overlapped areas will increase. (See figure 4 and compare figure 2). It may hence be concluded that if the number of sample points is large, systematic sampling is relatively less efficient than simple random sampling or stratified random sampling, although absolute efficiency increases.

## (5) CONCLUSIONS

Major conclusions of this paper are summarized as follows:

i) If sample points are allocated by equation (14) or strata are the same size and each strata has one sample point, stratified random sampling is not less efficient than simple random sampling. (Theorems 3 and Lemma 1).

ii) If a negative correlation exists, systematic sampling is more efficient than simple random sampling. (Theorem 6).

iii) If an area to be estimated is small relative to each stratum area of the area consists of small areas that are "randomly" distributed, then systematic sampling is likely to be more efficient than stratified random sampling. (Theorem 7).

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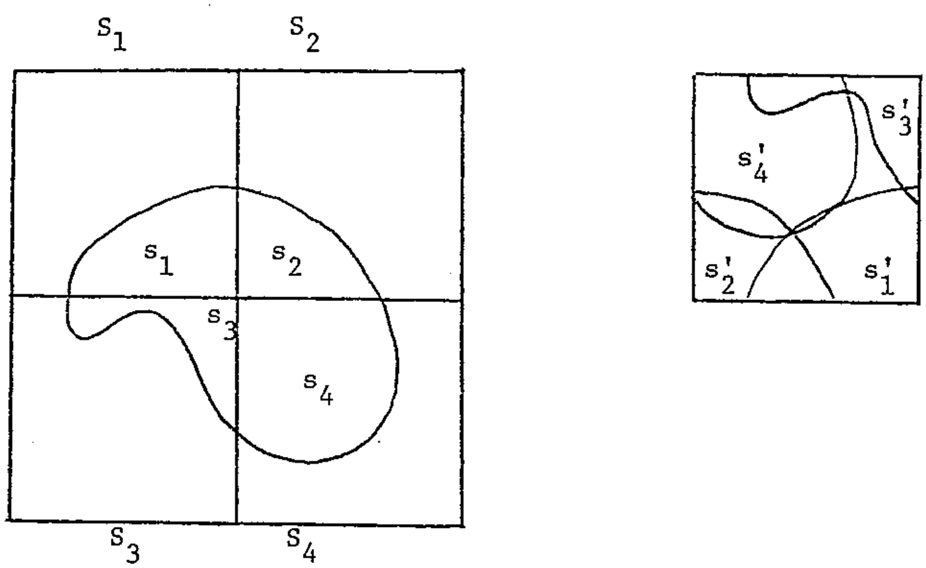


Figure 1

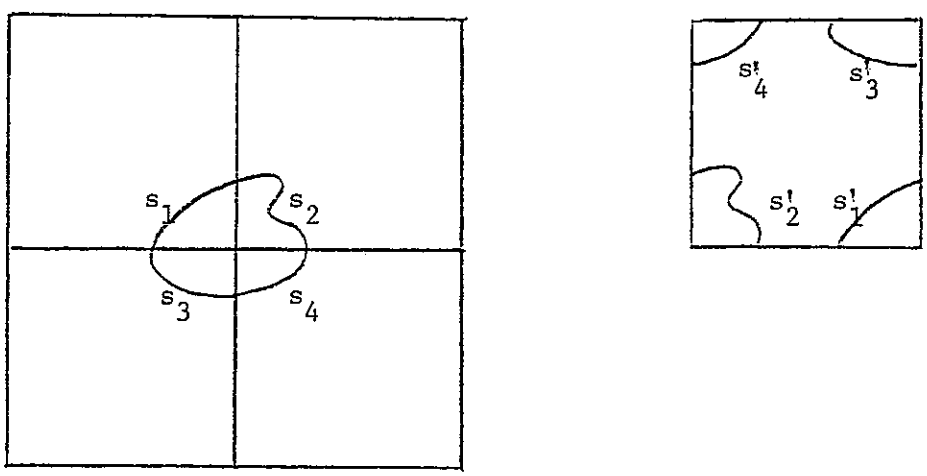


Figure 2

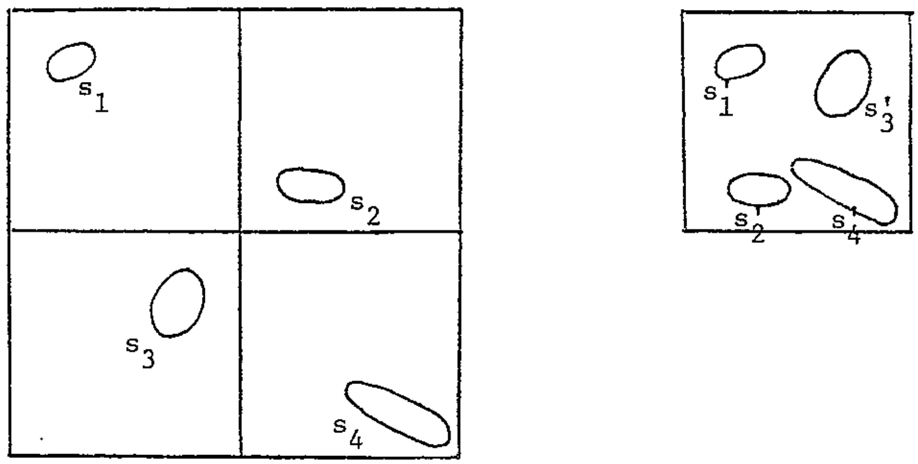


Figure 3

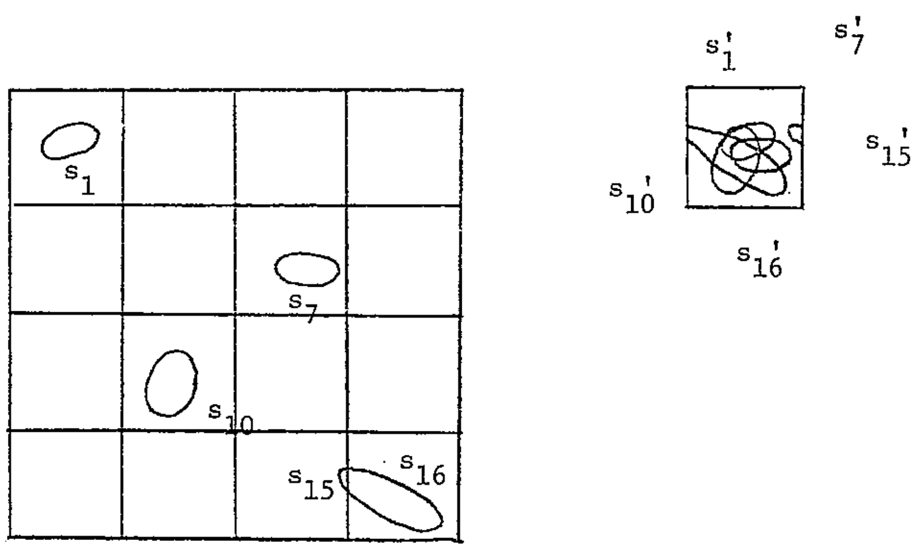


Figure 4