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Extreme Point Axioms for Closure Spaces*

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Abstract

A pair (X, τ) of a finite set X and a closure operator $\tau: 2^X \rightarrow 2^X$ is called a closure space. Matroids as well as antimatroids are closure spaces. Associated with a closure space (X, τ) , the extreme point operator $\text{ex}: 2^X \rightarrow 2^X$ is defined as $\text{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\}$. We give characterizations of extreme point operators of closure spaces, matroids and antimatroids, respectively.

Keywords: closure operator, matroid, antimatroid, convex geometry, choice function, extreme point

1. Introduction

Let X be a finite set. We call mapping $\tau: 2^X \rightarrow 2^X$ a *closure operator* if τ satisfies the following conditions.

(C1) $\forall A \subseteq X: A \subseteq \tau(A)$. (Extensionality)

(C2) $\forall A, B \subseteq X: A \subseteq B \implies \tau(A) \subseteq \tau(B)$. (Monotonicity)

(C3) $\forall A \subseteq X: \tau(\tau(A)) = \tau(A)$. (Idempotence)

A pair (X, τ) of a finite set X and a closure operator τ is called a *closure space* (see [4]). A closure space (X, τ) is a *matroid* if τ satisfies the following (Steinitz-MacLane) Exchange Axiom:

(EA) $\forall A \subseteq X, \forall q \notin \tau(A): q \in \tau(A \cup p) \implies p \in \tau(A \cup q)$

(see Welsh [10] and Oxley [8]). On the other hand, a closure space (X, τ) is called an *antimatroid* (or *convex geometry*) if τ satisfies $\tau(\emptyset) = \emptyset$ and the following Antiexchange Axiom:

(AE) $\forall A \subseteq X, \forall p, q \notin \tau(A)$ with $p \neq q: q \in \tau(A \cup p) \implies p \notin \tau(A \cup q)$.

See Edelman and Jamison [3] and Korte, Lovász and Schrader [5] for surveys and examples of antimatroids.

The *extreme point operator* $\text{ex}: 2^X \rightarrow 2^X$ of a closure space (X, τ) is defined as $\text{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\}$ ($A \subseteq X$). As the name suggests, the concept of extreme point had first appeared in the context of antimatroid. However, this concept can be applied to general closure spaces. For example, if (X, τ) is a matroid, $\text{ex}(A)$ is the set of isthmuses of A for each $A \subseteq X$ (see Lemma 2.1 below).

We characterize extreme point operators of closure spaces as follows.

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Theorem 1.1: *A mapping $S:2^X \rightarrow 2^X$ is the extreme point operator of a closure space if and only if S satisfies the following (Ex1)–(Ex3).*

$$(Ex1) \quad \forall A \subseteq X: S(A) \subseteq A. \quad (\text{Intensionality})$$

$$(Ex2) \quad A \subseteq B \subseteq X \implies S(B) \cap A \subseteq S(A). \quad (\text{Chernoff property})$$

$$(Ex3) \quad \forall A \subseteq X, \forall p, q \notin A: (p \notin S(A \cup p), q \in S(A \cup q)) \implies q \in S(A \cup p \cup q).$$

□

As corollaries of Theorem 1.1, we have following characterizations of the extreme point operator of matroids and antimatroids, respectively.

Theorem 1.2: *A mapping $S:2^X \rightarrow 2^X$ is the extreme point operator of a matroid if and only if S satisfies (Ex1)–(Ex3) and the following (Ex4).*

$$(Ex4) \quad \forall A \subseteq X, \forall p \in X: p \in S(A \cup p) \implies S(A \cup p) \supseteq S(A) \cup p.$$

□

Theorem 1.3: *A mapping $S:2^X \rightarrow 2^X$ is the extreme point operator of an antimatroid if and only if S satisfies (Ex0)–(Ex2) and (Ex5), where Conditions (Ex0) and (Ex5) are defined as follows.*

$$(Ex0) \quad \forall p \in X: S(\{p\}) = \{p\}. \quad (\text{Singleton Identity})$$

$$(Ex5) \quad \forall A, B \subseteq X: S(B) \subseteq A \subseteq B \implies S(A) \subseteq S(B). \quad (\text{Aizerman's Axiom})$$

□

Theorem 1.3 is equivalent to a result of Koshevoy [6] but seems to be more natural in view of Theorem 1.1. As we shall see, Aizerman's Axiom [1] is a strengthening of Condition (Ex3).

The rest of this paper is organized as follows. In Section 2, we collect previously known results on extreme point operator of closure spaces and antimatroids. In Section 3, we give a proof of Theorem 1.1. In Section 4, we prove Theorems 1.2 and 1.3. In Section 5, we discuss relationship between Theorem 1.3 and the result of Koshevoy [6].

2. Preliminaries

In this section, we collect important lemmas concerning extreme point operators of closure spaces and antimatroids, which will be useful in the subsequent sections.

Extreme point operators of closure spaces can be described as follows.

Lemma 2.1: *Suppose that (X, τ) is a closure space. Then, for each $A \subseteq X$ we have*

$$\text{ex}(A) = \bigcap \{B \mid B \subseteq A, \tau(B) = \tau(A)\}.$$

(Proof) Let p be an extreme point of A . Suppose that $B \subseteq A$ and $\tau(B) = \tau(A)$. If $p \notin B$, then since $B \subseteq A - p$, we have $\tau(B) \subseteq \tau(A - p) \subsetneq \tau(A)$, a contradiction. We thus have inclusion \subseteq .

Conversely, if $p \in A$ is not an extreme point of A , we have $\tau(A - p) = \tau(A)$. Hence, inclusion \supseteq holds. □

Lemma 2.1 is partly due to Edelman and Jamison [3].

The following proposition shows that the extreme point operator of a closure space has an important property called the *Chernoff property* (see Moulin [7]).

Proposition 2.2 (Chernoff property [9]): *Let (X, τ) be a closure space. If $A \subseteq B \subseteq X$, we have $\text{ex}(B) \cap A \subseteq \text{ex}(A)$.*

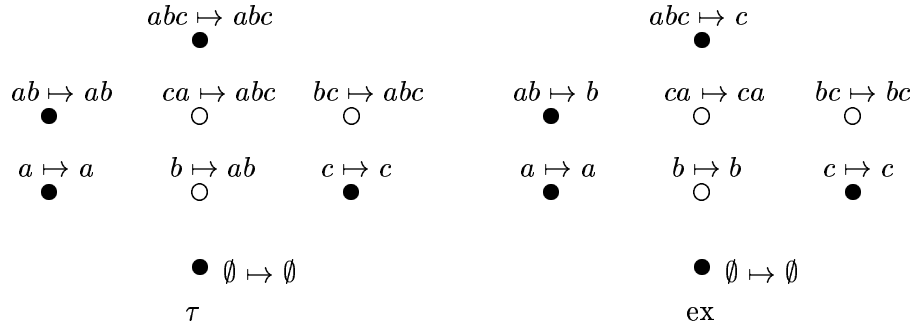
(Proof) If $p \in \text{ex}(B) \cap A$, we have $p \notin \tau(B - p)$. Since $\tau(A - p) \subseteq \tau(B - p)$, we have $p \notin \tau(A - p)$, and hence, we have $p \in \text{ex}(A)$. \square

The extreme point operator of a closure space is idempotent as is shown in the following proposition.

Proposition 2.3 (Idempotency): *Let (X, τ) be a closure space. We have $\text{ex}(\text{ex}(A)) = \text{ex}(A)$ for each $A \subseteq X$.*

(Proof) Since we have $\text{ex}(A) \subseteq A$, it follows from Lemma 2.2 that $\text{ex}(A) = \text{ex}(A) \cap \text{ex}(A) \subseteq \text{ex}(\text{ex}(A))$. \square

Example 2.4: Consider the closure space (X, τ) depicted in the left-hand side of the following figure, where $X = \{a, b, c\}$. The associated extreme point operator is shown in the right-hand side.



Antimatroids can be characterized in many ways. Among them is the following due to Edelman and Jamison [3].

For a closure space (X, τ) , a subset $K \subseteq X$ is called *closed* if $\tau(K) = K$.

Theorem 2.5 (Edelman and Jamison [3]): *Let (X, τ) be a closure space with $\tau(\emptyset) = \emptyset$. The following conditions are equivalent.*

- (a) (X, τ) is an antimatroid.
- (b) $\forall A \subseteq X: \tau(A) = \tau(\text{ex}(A))$.
- (c) For each closed set K and $p \notin K$, we have $p \in \text{ex}(\tau(K \cup p))$.

\square

Condition (b) in the above theorem is called the (*finite*) *Minkowski-Krein-Milman property*.

Lemma 2.6 (Ando [2, Lemma 3.2]): *Let (X, τ) be a closure space. For each $A \subseteq X$, we have $\text{ex}(\tau(A)) \subseteq \text{ex}(A)$.*

(Proof) Let $A \subseteq X$. Since $A \subseteq \tau(A) = \tau(\tau(A))$, we have

$$\{B \mid B \subseteq A, \tau(B) = \tau(A)\} \subseteq \{B \mid B \subseteq \tau(A), \tau(B) = \tau(\tau(A))\}.$$

It follows from Lemma 2.1 that $\text{ex}(\tau(A)) \subseteq \text{ex}(A)$. \square

We have the following variant of the Minkowski-Krein-Milman property, where τ and ex are transposed.

Theorem 2.7 (Ando [2, Theorem 3.3]): *A closure space (X, τ) with $\tau(\emptyset) = \emptyset$ is an antimatroid if and only if for each $A \subseteq X$ we have $\text{ex}(A) = \text{ex}(\tau(A))$.*

(Proof) If (X, τ) is an antimatroid, then it follows from Theorem 2.5(b) that $\tau(\text{ex}(\tau(A))) = \tau(A)$. Also, we have $\text{ex}(\tau(A)) \subseteq \text{ex}(A) \subseteq A$ by Lemma 2.6. Therefore, we have from Lemma 2.1 that $\text{ex}(A) \subseteq \text{ex}(\tau(A))$.

Conversely, suppose that (X, τ) is not an antimatroid. Then, by Theorem 2.5(c), there exists a closed K and $p \notin K$ such that $p \notin \text{ex}(\tau(K \cup p))$. However, since we have $p \in \text{ex}(K \cup p)$ by definition of ex , it follows that $\text{ex}(K \cup p) \not\subseteq \text{ex}(\tau(K \cup p))$. \square

3. Extreme point operator of closure spaces

In this section, we give a proof of Theorem 1.1. The following proposition proves the “only if” part of the theorem.

Proposition 3.1: *Let (X, τ) be a closure space and $S: 2^X \rightarrow 2^X$ be its extreme point operator. Then, there hold Conditions (Ex1)–(Ex3).*

(Proof) (Ex1) is clear from the definition of extreme point operator. Condition (Ex2) follows from Proposition 2.2.

Let us show (Ex3). Suppose that $p, q \notin A, p \notin S(A \cup p)$ and $q \in S(A \cup q)$. Then, by definition of S , we have $p \in \tau(A)$ and $q \notin \tau(A)$. Therefore, we have $\tau(A \cup p) = \tau(A) \not\ni q$, and hence, $q \in S(A \cup p \cup q)$. \square

For a mapping $S: 2^X \rightarrow 2^X$ define $\tau_S: 2^X \rightarrow 2^X$ by

$$\tau_S(A) = A \cup \tilde{A} \quad (A \subseteq X), \quad (3.1)$$

where

$$\tilde{A} = \{q \mid q \notin A, q \notin S(A \cup q)\} \quad (3.2)$$

for each $A \subseteq X$.

Lemma 3.2: *Suppose that $S: 2^X \rightarrow 2^X$ satisfies Conditions (Ex1)–(Ex3). Then, mapping $\tau_S: 2^X \rightarrow 2^X$ defined in (3.1) is a closure operator.*

(Proof) By its definition, τ_S satisfies Extensionality (C1). It remains to show Monotonicity (C2) and Idempotence (C3).

We first show (C2). Suppose $A \subseteq B \subseteq X$. Let $p \in \tau_S(A)$. If $p \in B$, then $p \in \tau_S(B)$ and we are done. Suppose $p \notin B$. Invoking (Ex2) to inclusion $A \cup p \subseteq B \cup p$, we have $S(B \cup p) \cap (A \cup p) \subseteq S(A \cup p)$. Since $p \notin S(A \cup p)$, we have $p \notin \widetilde{S(B \cup p)}$, and hence, $p \in \tau_S(B)$.

Next we show (C3). Let $A \subseteq X$. It suffices to show that $\widetilde{\tau_S(A)} = A \cup \tilde{A} = \emptyset$. Suppose that $q \notin A \cup \tilde{A}$.

We prove by induction on $|B|$ that $q \in S(A \cup B \cup q)$ for each $B \subseteq \tilde{A}$. This is trivially true for $B = \emptyset$ since we have $q \in S(A \cup q)$ by definition of \tilde{A} . Suppose $\emptyset \neq B \subseteq \tilde{A}$ and let $p \in B$. We have $p, q \notin A \cup (B - p)$. By the induction hypothesis, we have $q \in S(A \cup (B - p) \cup q)$. Since $A \cup p \subseteq A \cup B$, we have by (Ex2) that

$$S(A \cup B) \cap (A \cup p) \subseteq S(A \cup p).$$

Since $p \in \tilde{A}$, we have $p \notin S(A \cup p)$, and hence, $p \notin S(A \cup B) = S(A \cup (B - p) \cup p)$. By (Ex3), we have $q \in S(A \cup (B - p) \cup p \cup q) = S(A \cup B \cup q)$.

We have $q \in S(A \cup \tilde{A} \cup q)$ in particular. Since $q \notin A \cup \tilde{A}$ is arbitrary, we have $\widetilde{\tau_S(A)} = \emptyset$. This completes the proof of the present lemma. \square

Note that the set \mathcal{L} of closed subsets of closure space (X, τ_S) is given by

$$\mathcal{L} = \{A \mid A \subseteq X, \forall p \in X - A: p \in S(A \cup p)\}. \quad (3.3)$$

by definition (3.1) of τ_S .

The next theorem proves the “if” part of Theorem 1.1.

Theorem 3.3: *Suppose that a mapping $S: 2^X \rightarrow 2^X$ satisfies (Ex1)–(Ex3). Then, (X, τ_S) defined by (3.1) is a closure space with its extreme point operator being S .*

(Proof) Lemma 3.2 shows that (X, τ_S) is a closure space.

Let $\text{ex}: 2^X \rightarrow 2^X$ be the extreme point operator of (X, τ_S) . We shall show $\text{ex}(A) = S(A)$ for each $A \subseteq X$. Suppose $A \subseteq X$.

Let $p \in \text{ex}(A)$. We have $p \in A$ and $p \notin \tau_S(A - p)$. By definition of τ_S , we have $p \in S((A - p) \cup p) = S(A)$. Conversely, let $p \in S(A)$. Then, by definition of τ_S , we have $p \notin \tau_S(A - p)$. Since $p \in A$ due to (Ex1), we conclude that $p \in \text{ex}(A)$. \square

4. Extreme point operators of matroids and antimatroids

In this section, we prove Theorems 1.2 and 1.3.

We first prove Theorem 1.2 concerning extreme point operators of matroids.

(Proof of Theorem 1.2) Suppose that $S: 2^X \rightarrow 2^X$ is the extreme point operator of a matroid (X, τ) . Let $A \subseteq X$, $p \notin A$ and $p \in S(A \cup p)$. We have to show that $S(A \cup p) \supseteq S(A) \cup p$. Let $q \in S(A)$ and suppose, on the contrary, that $q \notin S(A \cup p)$. Then, by definition of S , we have $q \notin \tau(A - q)$ and $q \in \tau(A - q \cup p)$. It follows from Exchange Axiom that $p \in \tau(A - q \cup p) = \tau(A)$. This means that $p \notin S(A \cup p)$, a contradiction.

Conversely, suppose $S: 2^X \rightarrow 2^X$ satisfies (Ex1)–(Ex4). We know from Theorem 1.1 that S is the extreme point operator of a closure space (X, τ) . Hence, it suffices to show that τ satisfies Exchange Axiom (EA).

Suppose that $p \in \tau(A \cup q) - \tau(A)$. Since $p \notin \tau(A)$, we have $p \notin A$ and $p \in S(A \cup p)$. Then, we have, by (Ex4), that $S(A \cup p) \supseteq S(A) \cup p$. Suppose, on the contrary, that we have $q \in S(A \cup p \cup q)$. Then,

$$S(A \cup p \cup q) \supseteq S(A \cup p) \cup q \supseteq S(A) \cup p \cup q.$$

However, since $p \in \tau(A \cup q)$, we have $p \notin S(A \cup p \cup q)$, a contradiction. Therefore, we have $q \notin S(A \cup p \cup q)$, and hence, $q \in \tau(A \cup p)$. \square

Next, we consider extreme point operators of antimatroids.

Proposition 4.1 (see Mounlin [7]): *Condition (Ex2) is equivalent to any one of the following four conditions, provided that (Ex1) holds.*

$$(Ex2a) \quad \forall A, B \subseteq X: S(A \cup B) \subseteq S(S(A) \cup B).$$

$$(Ex2b) \quad \forall A, B \subseteq X: S(A \cup B) \subseteq S(S(A) \cup S(B)).$$

$$(Ex2c) \quad \forall A, B \subseteq X: S(A \cup B) \subseteq S(A) \cup S(B).$$

$$(Ex2d) \quad \forall A, B \subseteq X: S(A \cup B) \subseteq S(A) \cup B.$$

□

The following lemma shows that Condition (Ex3) is a weakening of Aizerman's Axiom (Ex5).

Lemma 4.2: *Condition (Ex5) implies Condition (Ex3), provided that Conditions (Ex1)–(Ex2) holds.*

(Proof) Suppose that a mapping $S: 2^X \rightarrow 2^X$ satisfies Conditions (Ex1), (Ex2) and (Ex5). Let us consider $A \subseteq X$ and $p, q \notin A$ such that $p \notin S(A \cup p)$ and $q \in S(A \cup q)$. Then, it follows from Proposition 4.1 and (Ex2) that

$$S(A \cup p \cup q) \subseteq S(A \cup p) \cup q \subseteq A \cup q.$$

Applying (Ex5) to the inclusions

$$S(A \cup p \cup q) \subseteq A \cup q \subseteq A \cup p \cup q,$$

we have $q \in S(A \cup q) \subseteq S(A \cup p \cup q)$. □

Theorem 4.3: *Suppose that $S: 2^X \rightarrow 2^X$ satisfies (Ex0)–(Ex2) and (Ex5). Then, (X, τ_S) is an antimatroid with its extreme point operator being S .*

(Proof) We have from Lemma 4.2 and Theorem 3.3 that (X, τ_S) is a closure space and that S is the extreme point operator of (X, τ_S) . Therefore, it suffices to show that (X, τ_S) is an antimatroid. We show that (X, τ_S) satisfies the condition in Theorem 2.7.

Let $A \subseteq X$ be arbitrary. We have from Theorem 3.3 and Lemma 2.6 that

$$S(\tau_S(A)) \subseteq S(A) \subseteq \tau_S(A). \quad (4.1)$$

Applying Aizerman's Axiom (Ex5) to (4.1) yields $S(A) = S(S(A)) \subseteq S(\tau_S(A))$, where the equation follows from Proposition 2.3. Since we have $\tau_S(\emptyset) = \emptyset$ by (Ex0), it follows from Theorem 2.7 that (X, τ_S) is an antimatroid. □

(Proof of Theorem 1.3) The “if” part of the theorem follows from Theorem 4.3.

Let us show the “only if” part. Let S be the extreme point operator of an antimatroid (X, τ) . Since an antimatroid is a closure space, we have (Ex1)–(Ex2) by Proposition 3.1. Also, since $\tau(\emptyset) = \emptyset$, we have $S(\{p\}) = \{p\}$ for each $p \in X$.

To show (Ex5), let us suppose $S(B) \subseteq A \subseteq B$. Then, it follows from the monotonicity of τ and Theorem 2.5(b) that

$$\tau(B) = \tau(S(B)) \subseteq \tau(A) \subseteq \tau(B),$$

and hence, we have $S(A) = S(B)$ by Theorem 2.7. □

5. Koshevoy's theorem

A *choice function* on X is a mapping $S: 2^X \rightarrow 2^X$ satisfying the following two conditions (see Moulin [7]).

(Ex1) $S(A) \subseteq A$ ($A \subseteq X$). (Intensionality)

(NE) $S(A) \neq \emptyset$ ($\emptyset \neq A \subseteq X$). (Nonemptiness)

Koshevoy [6] characterized extreme point operators of antimatroids as path-independent choice functions.

Theorem 5.1 (Koshevoy [6]): *A mapping $S: 2^X \rightarrow 2^X$ is the extreme point operator of an antimatroid if and only if S satisfies (Ex1), (NE) and the following (PI).*

$$(PI) \quad \forall A, B \subseteq X: S(A \cup B) = S(S(A) \cup S(B)). \quad (\text{Path Independence})$$

□

Path independent property (PI) decomposes into Chernoff property (Ex2) and Aizerman's Axiom (Ex5) as the following lemma shows.

Lemma 5.2 (Aizerman and Malishevski [1]; see also Moulin [7]): *Condition (PI) is equivalent to Conditions (Ex2) and (Ex5), provided that (Ex1) holds.* □

The following proposition shows the equivalence of Theorem 1.3 and Theorem 5.1.

Proposition 5.3: *The set of Conditions (Ex0), (Ex1), (Ex2) and (Ex5) is equivalent to that of Conditions (Ex1), (NE) and (PI).*

(Proof) Suppose that $S: 2^X \rightarrow 2^X$ satisfies (Ex1), (NE) and (PI). Then, by Lemma 5.2, S satisfies (Ex2) and (Ex5). Also, Conditions (Ex1) and (NE) implies (Ex0).

Conversely, suppose that S satisfies (Ex0), (Ex1), (Ex2) and (Ex5). Then, by Lemma 5.2, we have (PI). It remains to show that $S: 2^X \rightarrow 2^X$ satisfies (NE). Suppose, on the contrary, that for some $A \neq \emptyset$ we have $S(A) = \emptyset$. Let $p \in A$. Then, we have $S(A) \subseteq \{p\} \subseteq A$. It follows from (Ex5) that $S(\{p\}) \subseteq S(A) = \emptyset$. This contradicts (Ex0). □

Koshevoy proved the “if” part of Theorem 5.1 as follows. He showed that, given a choice function $S: 2^X \rightarrow 2^X$ satisfying (PI), the mapping defined by

$$\bar{S}(A) = \bigcup \{B \mid B \subseteq X, S(B) = S(A)\} \quad (A \subseteq X). \quad (5.1)$$

is a closure operator and that S is the extreme point operator of (X, \bar{S}) . This approach does not work for proving Theorem 1.1 since \bar{S} is not always a closure operator. (Consider the extreme point operator given in Example 2.4. We have $\bar{\text{ex}}(\{c\}) = \{a, b, c\} \not\subseteq \{c, a\} = \bar{\text{ex}}(\{c, a\})$.)

However, if $S: 2^X \rightarrow 2^X$ satisfies the conditions in Theorem 1.3 (or equivalently, those in Theorem 5.1), then we have $\tau_S = \bar{S}$.

Proposition 5.4: *Suppose that mapping $S: 2^X \rightarrow 2^X$ satisfies Conditions (Ex0)–(Ex2) and (Ex5). Then, we have $\tau_S = \bar{S}$, where τ_S and \bar{S} are, respectively, defined by (3.1) and (5.1).*

To show Proposition 5.4, we need the following lemma.

Lemma 5.5: *Let (X, τ_1) and (X, τ_2) be closure spaces with their extreme point operators being ex_1 and ex_2 , respectively. If $\tau_1 \neq \tau_2$, then $\text{ex}_1 \neq \text{ex}_2$.*

(Proof) Let \mathcal{L}_1 and \mathcal{L}_2 be the set of closed subsets of (X, τ_1) and (X, τ_2) , respectively. Since a closure operator is uniquely determined by its closed sets, we have $\mathcal{L}_1 \neq \mathcal{L}_2$. Suppose, say, $\mathcal{L}_1 \not\subseteq \mathcal{L}_2$ and let $A \in \mathcal{L}_1 - \mathcal{L}_2$. Then, we have $\tau_1(A) = A$ and $\tau_2(A) \supsetneq A$. Let $p \in \tau_2(A) - A$. We have $p \in \text{ex}_1(A \cup p)$ and $p \notin \text{ex}_2(A \cup p)$ by definition of extreme point operator. Therefore, we have $\text{ex}_1 \neq \text{ex}_2$. □

(Proof of Proposition 5.4) Suppose that S satisfies (Ex0)–(Ex2) and (Ex5). Then, we have from Theorem 4.3 that (X, τ_S) is an antimatroid with its extreme point operator being S . However, S is also the extreme point operator of (X, \bar{S}) by Theorem 5.1. Then, we must have $\tau_S = \bar{S}$ by Lemma 5.5. □

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