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Characterizations of Convex Geometries by Extreme Point Operator*

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Abstract

A closure space is a pair (X, τ) of a finite set X and a closure operator $\tau: 2^X \to 2^X$ with $\tau(\emptyset) = \emptyset$. The extreme point operator $\mathrm{ex}: 2^X \to 2^X$ associated with closure space (X, τ) is defined as $\mathrm{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\}$ $(A \subseteq X)$. We investigate the extreme point operators of closure spaces and convex geometries (or antimatroids), and show that each of the following conditions is necessary and sufficient for a closure space (X, τ) to be a convex geometry: (i) $\mathrm{ex}(A) = \mathrm{ex}(\tau(A))$ for each $A \subseteq X$; (ii) $\mathrm{ex}|_{\mathcal{L}}: \mathcal{L} \to \mathcal{E}$ is bijective; (iii) $\mathrm{ex}(B) \subseteq A \subseteq B$ implies $\mathrm{ex}(A) \subseteq \mathrm{ex}(B)$; (iv) $\mathrm{ex}^{-1}(A)$ is an interval in 2^X for each $A \in \mathcal{E}$, where $\mathcal{L} = \{A \subseteq X \mid \tau(A) = A\}$ and $\mathcal{E} = \{A \subseteq X \mid \mathrm{ex}(A) = A\}$. New characterizations of convex geometry in terms of closure operator are derived as well.

Keywords: convex geometry, closure operator, choice function, antimatroid

1. Introduction

For a finite set X and a closure operator $\tau: 2^X \to 2^X$ with $\tau(\emptyset) = \emptyset$, we call the pair (X, τ) a closure space. A closure space (X, τ) is called a convex geometry if τ satisfies Antiexchange Axiom, which is, in a sense, opposite to the Steinitz-MacLane exchange property of closure operators of matroids. (Precise definitions will be given in the later sections.) Convex geometries (or antimatroids) arise in many situations in combinatorics and combinatorial optimization. See Edelman and Jamison [3], Korte, Lovász and Schrader [8], Dietrich [2] for surveys and examples of convex geometry.

For a closure space (X, τ) the extreme point operator $\mathrm{ex}: 2^X \to 2^X$ is defined as $\mathrm{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\}$ $(A \subseteq X)$. Although convex geometries are studied considerably, less attention has been paid to the behavior of extreme operators of closure spaces and/or convex geometries in the literature of discrete mathematics. In the meanwhile, a remarkable result on the extreme operators of convex geometries was brought from the literature of social choice theory: Koshevoy [9] showed that the path independent choice functions are exactly the extreme point operators of convex geometries. Moreover, Johnson and Dean [6] showed another interesting property of path independent choice functions, which was called the quotient property.

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In this paper, we investigate the extreme point operators of closure spaces and convex geometries, and discuss the results of Koshevoy and Johnson and Dean in the context of the theory of closure operator rather than that of social choice function. As a result, we obtain four characterizations of convex geometries in terms of extreme point operator: Each of the following conditions is necessary and sufficient for a closure space (X, τ) to be a convex geometry.

(i) $\forall A \subseteq X : \mathrm{ex}(\tau(A)) = \mathrm{ex}(A)$.

(ii) $\mathrm{ex}|_{\mathcal{L}} : \mathcal{L} \to \mathcal{E}$ is bijective.

(iii) $\forall A, B \subseteq X : \mathrm{ex}(B) \subseteq A \subseteq B \Longrightarrow \mathrm{ex}(A) \subseteq \mathrm{ex}(B)$.

(iv) $\forall A \in \mathcal{E}$, $\mathrm{ex}^{-1}(A)$ is an interval in 2^X,

where $\mathcal{E} \subseteq 2^X$ is the set of subsets $A \subseteq X$ such that $\mathrm{ex}(A) = A$ and $\mathcal{L}$ is the set of closed subsets of X.

In Section 3, we show that convex geometries can be characterized by Condition (i). It is well known that closure space (X, τ) is a convex geometry if and only if τ and ex satisfy the Minkowski-Krein-Milman property (MKM property): $\forall A \subseteq X : \tau(\mathrm{ex}(A)) = \tau(A)$ (see [3]). The condition (i) is a variant of MKM property.

Kashiwabara and Okamoto [7] observed that if (X, τ) is a convex geometry, then the mapping $\mathrm{ex}|_{\mathcal{L}} : \mathcal{L} \to \mathcal{E}$ is bijective. In Section 4, we prove that this property characterizes convex geometries indeed. We show a similar condition for τ characterizing convex geometries as well.

The condition (iii) is called Aizerman's Axiom (cf. Moulin [10]). As an immediate corollary of Koshevoy's theorem, we can see that a closure space is a convex geometry if and only if its extreme point operator is path independent. Analyzing path independent mappings carefully, we have a slightly stronger result. In Section 5, we show that a closure space is a convex geometry if and only if its extreme point operator satisfies Aizerman's Axiom.

Johnson and Dean [6] called the condition (iv) the quotient property. They showed that for a choice function, path independence is equivalent to Chernoff property and the quotient property. Combining this result with that of Koshevoy, we can see that the extreme point operator of a convex geometry satisfies the quotient property. In Section 6, we will prove that for a closure space the quotient property is also sufficient to be a convex geometry. Moreover, we show the quotient property for closure operators characterizes convex geometries.

2. Closure spaces

In this section, we collect important lemmas concerning extreme point operators of closure spaces, which will be useful in the subsequent sections.

Let X be a finite set. We call mapping $\tau : 2^X \to 2^X$ a *closure operator* if τ satisfies the following conditions.

(C1) $\forall A \subseteq X : A \subseteq \tau(A)$ (Extensionality).

(C2) $\forall A, B \subseteq X : A \subseteq B \Longrightarrow \tau(A) \subseteq \tau(B)$ (Monotonicity).

(C3) $\forall A \subseteq X : \tau(\tau(A)) = \tau(A)$ (Idempotence).

A pair (X, τ) of a finite set X and closure operator τ with $\tau(\emptyset) = \emptyset$ is called a *closure space* (cf. [4]). For a closure space (X, τ), a subset $K \subseteq X$ is called *closed* if $\tau(K) = K$.

Closed subsets of closure spaces are characterized as follows.

Proposition 2.1: *For a closure space* (X, τ) *the family* $\mathcal{L}$ *of closed subsets satisfies the following two conditions:*

(A1) $\emptyset, X \in \mathcal{L}$.

(A2) $A, B \in \mathcal{L} \Longrightarrow A \cap B \in \mathcal{L}$.

Conversely, given a family $\mathcal{L} \subseteq 2^X$ *satisfying* (A1)–(A2), *the pair* (X, τ) *defined by*

$$\tau(A) = \bigcap\{K \mid A \subseteq K \in \mathcal{L}\} \quad (A \subseteq X). \tag{2.1}$$

is a closure space, and $\mathcal{L}$ *is the family of closed subsets of* (X, τ). □

Thus, a closure space is uniquely determined by its family $\mathcal{L}$ of closed subsets. It should be noted here that $\mathcal{L}$ ordered by set-inclusion is a (finite) lattice, and conversely, every finite lattice is isomorphic to the lattice of the closed subsets of a closure space.

Let (X, τ) be a closure space. For $A \subseteq X$ we call $S \subseteq A$ a *spanning set* of A if $\tau(S) = \tau(A)$. A minimal spanning set of A is called a *basis* of A. For $A \subseteq X$ an element $p \in A$ is an *extreme point* of A if $p \notin \tau(A - p)$. We denote by $\mathrm{ex}(A)$ the set of extreme points of A. The mapping $\mathrm{ex}\colon 2^X \to 2^X$ thus defined is called the *extreme point operator* associated with (X, τ).

Throughout the rest of this paper, we always denote by ex the extreme point operator associated with the closure space in context.

Lemma 2.2: *For a closure space* (X, τ) *subset* $A \subseteq X$ *is closed if and only if for each* $p \notin A$ *we have* $p \in \mathrm{ex}(A \cup p)$.

(Proof) Suppose that A is closed. If $p \notin A$, then we have $\tau((A \cup p) - p) = \tau(A) = A \not\ni p$, and hence, $p \in \mathrm{ex}(A \cup p)$.

Conversely, suppose that for each $p \notin A$ we have $p \in \mathrm{ex}(A \cup p)$. If $p \notin A$, we have $p \notin \tau((A \cup p) - p) = \tau(A)$. Therefore, we have $\tau(A) \subseteq A$. □

Extreme point operators of closure spaces can be described as follows.

Lemma 2.3: *Suppose that* (X, τ) *is a closure space. Then, for each* $A \subseteq X$ *we have*

$$\mathrm{ex}(A) = \bigcap\{B \mid B \subseteq A, \tau(B) = \tau(A)\}.$$

(Proof) Let p be an extreme point of A and B be a basis of A. If $p \notin B$, then since $B \subseteq A - p$, we have $\tau(B) \subseteq \tau(A - p) \subsetneq \tau(A)$ contradicting B is a basis of A. We thus have inclusion $\subseteq$.

Conversely, if $p \in A$ is not an extreme point of A, we have $\tau(A - p) = \tau(A)$. Hence, inclusion $\supseteq$ holds. □

Lemma 2.3 is partly due to Edelman and Jamison [3].

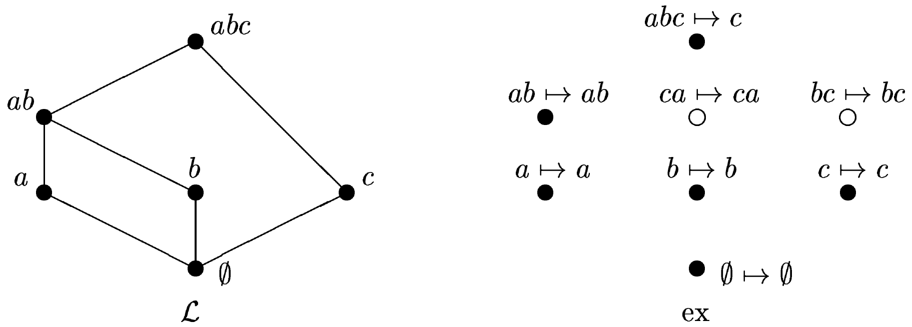
The following proposition shows that the extreme point operator of a closure space has an important property called the *Chernoff property* (cf. Moulin [10]).

Lemma 2.4: *Let* (X, τ) *be a closure space. If* $A \subseteq B \subseteq X$, *we have* $\mathrm{ex}(B) \cap A \subseteq \mathrm{ex}(A)$.

(Proof) If $p \in \mathrm{ex}(B) \cap A$, we have $p \notin \tau(B - p)$. Since $\tau(A - p) \subseteq \tau(B - p)$, we have $p \notin \tau(A - p)$. □

The lemma above was proved for convex geometries by Pfaltz [11].

Example 2.5: Consider a closure space (X, τ), where $X = \{a, b, c\}$ and the associated family of closed subsets is given by $\mathcal{L} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,b,c\}\}$. The left-hand side of the following figure shows the Hasse diagram of $\mathcal{L}$ ordered by set-inclusion and the right-hand side shows the extreme point operator associated with this closure space, where we abbreviate, say, $\{a, b\}$ to ab.



3. A variant of the Minkowski-Krein-Milman property

A closure space (X, τ) is called a *convex geometry* if the following *Antiexchange Axiom* holds.

(AE) $\forall A \subseteq X, \forall p, q \notin \tau(A) \colon q \in \tau(A \cup p) \Longrightarrow p \notin \tau(A \cup q)$.

Convex geometries can be characterized in many ways. Among them are the followings due to Edelman and Jamison [3].

Theorem 3.1 (Edelman and Jamison [3]): *Suppose that (X, τ) is a closure space. Then, the following conditions are equivalent.*

(a) *(X, τ) is a convex geometry.*

(b) *For each closed set $K \neq X$ there exists $p \in X - K$ such that $K \cup p$ is closed.*

(c) *Each $A \subseteq X$ has the unique basis.*

(d) *For each closed set K, we have $K = \tau(\mathrm{ex}(K))$.*

(e) *For each $A \subseteq X$, we have $\tau(A) = \tau(\mathrm{ex}(A))$.*

(f) *For each closed set K and $p \notin K$, we have $p \in \mathrm{ex}(\tau(K \cup p))$.*

□

Each of Conditions (d) and (e) in the above theorem is called the *(Finite) Minkowski-Krein-Milman property.*

Lemma 3.2: *Let (X, τ) be a closure space. For each $A \subseteq X$, we have $\mathrm{ex}(\tau(A)) \subseteq \mathrm{ex}(A)$.*

(Proof) Each spanning set of A is a spanning set of $\tau(A)$, that is,

$$\{B \mid B \subseteq A, \tau(B) = \tau(A)\} \subseteq \{B \mid B \subseteq \tau(A), \tau(B) = \tau(\tau(A))\}.$$

It follows from Lemma 2.3 that $\mathrm{ex}(\tau(A)) \subseteq \mathrm{ex}(A)$. □

We have the following variant of the MKM property, where τ and ex are transposed.

Theorem 3.3: *A closure space* (X, τ) *is a convex geometry if and only if for each* $A \subseteq X$ *we have* $\mathrm{ex}(A) = \mathrm{ex}(\tau(A))$.

(Proof) We have $\mathrm{ex}(A) \supseteq \mathrm{ex}(\tau(A))$ for each $A \subseteq X$ by Lemma 3.2.

If (X, τ) is a convex geometry, then it follows from Theorem 3.1(d) that $\mathrm{ex}(\tau(A))$ is a spanning set of A. Therefore, we have from Lemma 2.3 that $\mathrm{ex}(A) \subseteq \mathrm{ex}(\tau(A))$.

Conversely, suppose that (X, τ) is not a convex geometry. Then, by Theorem 3.1(f), there exists a closed K and $p \notin K$ such that $p \notin \mathrm{ex}(\tau(K \cup p))$. However, since we have $p \in \mathrm{ex}(K \cup p)$ due to Lemma 2.2, it follows that $\mathrm{ex}(K \cup p) \supsetneq \mathrm{ex}(\tau(K \cup p))$. □

4. Bijective properties

The extreme point operator of a closure space is idempotent as is shown in the following proposition.

Proposition 4.1 (Idempotency): *Let* (X, τ) *be a closure space. We have* $\mathrm{ex}(\mathrm{ex}(A)) = \mathrm{ex}(A)$ *for each* $A \subseteq X$.

(Proof) Since we have $\mathrm{ex}(A) \subseteq A$, it follows from Lemma 2.4 that $\mathrm{ex}(A) = \mathrm{ex}(A) \cap A \subseteq \mathrm{ex}(\mathrm{ex}(A))$. □

For a closure space (X, τ) a subset $A \subseteq X$ is called *free* if $\mathrm{ex}(A) = A$. Proposition 4.1 shows that the range of $\mathrm{ex}: 2^X \to 2^X$ is precisely the family of free subsets.

Kashiwabara and Okamoto [7] noticed that if (X, τ) is a convex geometry, then the mapping $\mathrm{ex}|_{\mathcal{L}}: \mathcal{L} \to \mathcal{E}$ is a bijection, where $\mathcal{L} \subseteq 2^X$ and $\mathcal{E} \subseteq 2^X$ are, respectively, the families of closed and free subsets. In fact, this property characterizes convex geometries. To prove it, we need the following lemma.

Lemma 4.2: *Suppose that* (X, τ) *is a closure space. For* $A \subseteq X$, *each basis of* A *is free.*

(Proof) Let B be a basis of A. If $b \in B$, then, since B is a minimal spanning set of A, we have $\tau(B - b) \subsetneq \tau(A) = \tau(B)$, and hence, $b \notin \tau(B - b)$. Therefore, we have $B \subseteq \mathrm{ex}(B)$, i.e., B is free. □

Theorem 4.3: *A closure space* (X, τ) *is a convex geometry if and only if* $\mathrm{ex}|_{\mathcal{L}}: \mathcal{L} \to \mathcal{E}$ *is bijective, where* $\mathcal{L} \subseteq 2^X$ *and* $\mathcal{E} \subseteq 2^X$ *are the families of closed and free subsets, respectively.*

(Proof) ("only if" part:) By Theorem 3.3, we have $\mathrm{ex}(A) = \mathrm{ex}(\tau(A))$ for each $A \subseteq X$. Hence, it is clear that mapping $\mathrm{ex}|_{\mathcal{L}}$ is surjective. If $K, K' \in \mathcal{L}$ and $\mathrm{ex}(K) = \mathrm{ex}(K')$, then it follows from the MKM property (Theorem 3.1(d)) that $K = \tau(\mathrm{ex}(K)) = \tau(\mathrm{ex}(K')) = K'$.

("if" part:) Suppose that $\mathrm{ex}|_{\mathcal{L}}: \mathcal{L} \to \mathcal{E}$ is bijective. If (X, τ) is not a convex geometry, then it follows from Theorem 3.1(d) that there exists a closed set K such that $K \supsetneq \tau(\mathrm{ex}(K))$. Let K be such a closed set with $\mathrm{ex}(K)$ being maximal.

Let B be a basis of K. We have $B \supseteq \mathrm{ex}(K)$ by Lemma 2.3 but since $\mathrm{ex}(K)$ cannot span K, we have $B \supsetneq \mathrm{ex}(K)$. Since $B \in \mathcal{E}$ by Lemma 4.2 and $\mathrm{ex}|_{\mathcal{L}}$ is bijective, there exists $K \neq L \in \mathcal{L}$ such that $B = \mathrm{ex}(L)$. However, the choice of K implies that $L = \tau(\mathrm{ex}(L)) = \tau(B) = K$. This is a contradiction. □

In particular, the extreme point operator of a convex geometry gives rise to a "choice function" (see e.g, Moulin [10]) as the following corollary shows.

Corollary 4.4: *If a closure space* (X, τ) *is a convex geometry, then for each* $\emptyset \neq A \subseteq X$ *we have* $\mathrm{ex}(A) \neq \emptyset$.

(Proof) Let (X, τ) be a convex geometry and $A \subseteq X$. If $\mathrm{ex}(A) = \emptyset$, we have from Theorem 3.3 that $\mathrm{ex}(\tau(A)) = \emptyset$. Furthermore, it follows from Theorem 4.3 that $\tau(A) = \emptyset$. Therefore, we have $A = \emptyset$ due to the extensionality of τ. □

It is very natural to ask whether a similar statement as Theorem 4.3 holds for τ. The following theorem gives the positive answer for this question.

Theorem 4.5: *A closure space (X, τ) is a convex geometry if and only if $\tau|_{\mathcal{E}}: \mathcal{E} \to \mathcal{L}$ is bijective, where $\mathcal{L} \subseteq 2^X$ and $\mathcal{E} \subseteq 2^X$ are the families of closed and free subsets, respectively.*

(Proof) ("only if" part:) Suppose that (X, τ) is a convex geometry. For each $K \in \mathcal{L}$ we have $K = \tau(\mathrm{ex}(K))$ by Theorem 3.1(d). Therefore, $\tau|_{\mathcal{E}}$ is surjective.

Suppose that for $A, B \in \mathcal{E}$ we have $\tau(A) = \tau(B)$. It follows from Theorem 3.3 that

$$A = \mathrm{ex}(A) = \mathrm{ex}(\tau(A)) = \mathrm{ex}(\tau(B)) = \mathrm{ex}(B) = B,$$

and hence, $\tau|_{\mathcal{E}}$ is injective.

("if" part:) Suppose that (X, τ) is not a convex geometry. Then, it follows from Theorem 3.1(c) that there exists a closed K having two distinct bases. Let B_1 and B_2 be such bases. We have $B_1, B_2 \in \mathcal{E}$ by Lemma 4.2 and that $K = \tau(B_1) = \tau(B_2)$ by definition of basis. Therefore, $\tau|_{\mathcal{E}}$ is not bijective. □

The family of free subsets is an independent system for a closure space (X, τ) as the following proposition shows.

Proposition 4.6: *Let (X, τ) be a closure space. If A is free and $B \subseteq A$, then B is free.*

(Proof) Suppose that $A = \mathrm{ex}(A)$. We have from Lemma 2.4 that $B = B \cap A = B \cap \mathrm{ex}(A) \subseteq \mathrm{ex}(B)$. It is clear from definition that $\mathrm{ex}(B) \subseteq B$. □

5. Aizerman's Axiom

First, we summarize properties of extreme operators of closure spaces obtained in Section 2.

Proposition 5.1: *Let (X, τ) be a closure space and let $S = \mathrm{ex}: 2^X \to 2^X$. Then, there hold the following conditions* (Ex1)–(Ex3).

(Ex1) $\forall A \subseteq X: S(A) \subseteq A$ (Intensionality).

(Ex2) $\forall p \in X: S(\{p\}) = \{p\}$ (Singleton Identity).

(Ex3) $A \subseteq B \subseteq X \Longrightarrow S(B) \cap A \subseteq S(A)$ (Chernoff property).

(Proof) Both Conditions (Ex1) and (Ex2) are clear from definitions. Condition (Ex3) follows from Lemma 2.4. □

The following proposition shows implication of Conditions (Ex1)–(Ex3).

Proposition 5.2 (Mounlin [10]): *Condition* (Ex3) *is equivalent to any one of the followings four conditions provided that* (Ex1) *holds.*

(Ex3a) $\forall A, B \subseteq X: S(A \cup B) \subseteq S(S(A) \cup B)$.

(Ex3b) $\forall A, B \subseteq X: S(A \cup B) \subseteq S(S(A) \cup S(B))$.

(Ex3c) $\forall A, B \subseteq X: S(A \cup B) \subseteq S(A) \cup S(B)$.

(Ex3d) $\forall A, B \subseteq X \colon S(A \cup B) \subseteq S(A) \cup B$.

□

The following lemma is due to Koshevoy. We include an alternative proof.

Lemma 5.3 (Koshevoy [9]): *Suppose that a closure space (X, τ) is a convex geometry. Then, $S :=$ ex satisfies following condition.*

(Ex4) $\forall A, B \subseteq X \colon S(A \cup B) = S(S(A) \cup S(B))$ (Path Independence).

(Proof) Let $A, B \subseteq X$. It follows from Theorem 3.3 and Theorem 3.1(e) that we have $\mathrm{ex}(A \cup B) = \mathrm{ex}(\mathrm{ex}(A) \cup \mathrm{ex}(B))$ if and only if $\tau(A \cup B) = \tau(\mathrm{ex}(A) \cup \mathrm{ex}(B))$. Hence, it suffices to prove equation $\tau(A \cup B) = \tau(\mathrm{ex}(A) \cup \mathrm{ex}(B))$.

We have, by the monotonicity of τ, that $\tau(A \cup B) \supseteq \tau(\mathrm{ex}(A) \cup \mathrm{ex}(B))$. Conversely, it follows from Theorem 3.1(e), Proposition 5.2 and the monotonicity of τ that $\tau(A \cup B) = \tau(\mathrm{ex}(A \cup B)) \subseteq \tau(\mathrm{ex}(A) \cup \mathrm{ex}(B))$. □

We consider the following condition (Ex5), which we call *Aizerman's Axiom* (cf. Moulin [10]).

(Ex5) $\forall A, B \subseteq X \colon S(B) \subseteq A \subseteq B \Longrightarrow S(A) \subseteq S(B)$ (Aizerman's Axiom).

Proposition 5.4 (cf. Moulin [10]): *Condition* (Ex5) *is equivalent to the following Condition* (Ex5') *provided that $S(S(A)) = S(A)$ for every $A \subseteq X$.*

(Ex5') $\forall A, B \subseteq X \colon S(B) \subseteq A \subseteq B \Longrightarrow S(A) = S(B)$.

□

It follows from Proposition 4.1 that for the extreme point operator ex of a closure space Condition (Ex5) is equivalent to Condition (Ex5').

Path independent property (Ex4) decomposes into Chernoff (Ex3) and Aizerman's Axiom (Ex5) as the following lemma shows.

Lemma 5.5 (Aizerman and Malishevski [1]; cf. Moulin [10]): *Condition* (Ex4) *is equivalent to Conditions* (Ex3) *and* (Ex5), *provided that* (Ex1) *holds.* □

Theorem 5.6: *A closure space (X, τ) is a convex geometry if and only if* $\mathrm{ex}\colon 2^X \to 2^X$ *satisfies Aizerman's Axiom* (Ex5).

(Proof) Suppose that (X, τ) is a convex geometry. Then, it follows from Lemma 5.3 and Lemma 5.5 that ex satisfies Aizerman's Axiom (Ex5).

Conversely, suppose that ex satisfies Aizerman's Axiom (Ex5). Let $A \subseteq X$ be arbitrary. We have from Lemma 3.2 that $\mathrm{ex}(\tau(A)) \subseteq \mathrm{ex}(A) \subseteq \tau(A)$. Then, by (Ex5') we have $\mathrm{ex}(A) = \mathrm{ex}(\mathrm{ex}(A)) = \mathrm{ex}(\tau(A))$. It thus follows from Theorem 3.3 that (X, τ) is a convex geometry. □

6. Quotient properties

For a closure space (X, τ) and a free subset A, let us consider family

$$\mathrm{ex}^{-1}(A) = \{B \mid B \subseteq X, \mathrm{ex}(B) = A\}.$$

If $B \in \mathrm{ex}^{-1}(A)$, then we have $A = \mathrm{ex}(B) \subseteq B$. Hence, A is the minimum element of $\mathrm{ex}^{-1}(A)$.

The "only if" part of the following theorem was implicitly shown in [6]. For $C \subseteq D \subseteq X$ we denote by $[C, D]$ the family $\{B \mid C \subseteq B \subseteq D\}$. We call such a family an *interval* in 2^X.

Theorem 6.1: *A closure space (X, τ) is a convex geometry if and only if* $\mathrm{ex}^{-1}(A)$ *is an interval in* 2^X *for each free subset* A.

(Proof) ("only if" part:) Suppose that (X, τ) is a convex geometry and let A be free. We claim that $\mathrm{ex}^{-1}(A) = [A, \tau(A)]$. Let $B \in \mathrm{ex}^{-1}(A)$. We know that $A \subseteq B$, and we have from Theorem 3.1(e) that $B \subseteq \tau(B) = \tau(\mathrm{ex}(B)) = \tau(A)$. Conversely, suppose that $A \subseteq B \subseteq \tau(A)$. Since we have from Theorem 3.3 that $A = \mathrm{ex}(\tau(A))$, it follows from Aizerman's Axiom (Ex5') that $\mathrm{ex}(B) = A$.

("if" part:) Suppose that for each free $A \subseteq X$ the family $\mathrm{ex}^{-1}(A)$ is an interval $[A, B]$ for some $B \subseteq X$. We will show that ex satisfies Aizerman's Axiom.

Suppose that $\mathrm{ex}(D) \subseteq C \subseteq D$ for $C, D \subseteq X$. Let $A = \mathrm{ex}(D)$. Since $D \in \mathrm{ex}^{-1}(A)$, we have $C \in \mathrm{ex}^{-1}(A)$, i.e., $\mathrm{ex}(C) = A = \mathrm{ex}(D)$. It follows from Theorem 5.6 that (X, τ) is a convex geometry. □

A parallel statement for Theorem 6.1 holds, where ex is replaced with τ. To prove it, we need the following lemma.

Lemma 6.2: *Let (X, τ) be a closure space. For each closed K we have $K \in \tau^{-1}(K)$ and $\tau^{-1}(K) \subseteq [\mathrm{ex}(K), K]$.*

(Proof) Let K be closed and $A \in \tau^{-1}(K)$. Then, we have $A \subseteq \tau(A) = K$. Also, we have $\tau(K) = K$. Therefore, K is the unique maximal member of $\tau^{-1}(K)$. Furthermore, we have from Lemma 3.2 that

$$\mathrm{ex}(K) = \mathrm{ex}(\tau(A)) \subseteq \mathrm{ex}(A) \subseteq A.$$

□

Theorem 6.3: *A closure space (X, τ) is a convex geometry if and only if $\tau^{-1}(K)$ is an interval in 2^X for each closed K.*

(Proof) ("only if" part:) Suppose that (X, τ) is a convex geometry and K is closed. We will show that $\tau^{-1}(K) = [\mathrm{ex}(K), K]$. Since we have inclusion $\tau^{-1}(K) \subseteq [\mathrm{ex}(K), K]$ by Lemma 6.2, it suffices to show the other inclusion.

Suppose $\mathrm{ex}(K) \subseteq A \subseteq K$. Then, we have by Theorem 3.1(d) and the monotonicity τ that

$$K = \tau(\mathrm{ex}(K)) \subseteq \tau(A) \subseteq K,$$

and hence, $\tau(A) = K$.

("if" part:) Conversely, suppose that $\tau^{-1}(K)$ is an interval for each closed K. This, in particular, means that each closed K has the unique minimal spanning set. It follows from Theorem 3.1(c) that (X, τ) is a convex geometry. □

An seemingly weaker version of Theorem 6.3 is given as follows.

Corollary 6.4 (Edelman and Jamison [3] (cf. Hoffman [5])): *A closure space (X, τ) is a convex geometry if and only if*

$$\forall A, B \subseteq X \colon \tau(A) = \tau(B) \Longrightarrow \tau(A \cap B) = \tau(A) = \tau(B). \tag{6.1}$$

(Proof) The necessity easily follows from Theorem 6.3. To show the sufficiency, suppose that condition (6.1) holds. Let K be closed. We have from Lemma 2.3 that $\tau(\mathrm{ex}(K)) = K$. Then, it follows from Theorem 3.1(d) that (X, τ) is a convex geometry. □

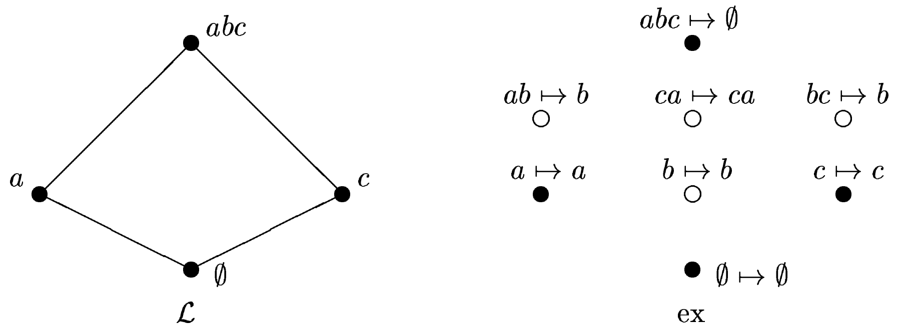
Pfaltz [11] called Condition (6.1) *the unique generation property.*

Remark 6.5: It immediately follows from Theorem 6.1 that if closure space (X, τ) is a convex geometry, then we have

$$\forall A, B \subseteq X\colon \mathrm{ex}(A) = \mathrm{ex}(B) \Longrightarrow \mathrm{ex}(A \cup B) = \mathrm{ex}(A) = \mathrm{ex}(B). \tag{6.2}$$

However, the converse is not true. One can see the closure space in Example 2.5 satisfies (6.2) while it is not a convex geometry.

Also, note that a closure space does not necessarily satisfy condition (6.2). Let us consider the closure space and the associated extreme point operator depicted in the following figure. We have $\mathrm{ex}(\{a, b\}) = \mathrm{ex}(\{b, c\}) = \{b\}$ but $\mathrm{ex}(\{a, b, c\}) = \emptyset$.



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