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An Unbiased One-sided Test for the Positional Parameter of the Exponential Distribution

by

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Abstract.

In this paper the underlined distribution is of form

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & \text{for } \theta<x<\infty \\ 0, & \text{otherwise} \end{cases}$$

$(-\infty<\theta<\infty)$ and the author proposes an unbiased one-sided test for testing the hypothesis H_0: $\theta\leqq\theta_0$ versus the alternative hypothesis H_1: $\theta>\theta_0$ with some constant θ_0.

§1. Introduction.

In the paper by Nogami(2000) the author discussed goodness of the two-sided test derived from the Lagrange's method. In this paper we use the same estimate for θ to derive the one-sided test for testing the hypothesis $H_0:\theta\leqq\theta_0$ versus the alternative hypothesis $H_1:\theta>\theta_0$ with some constant θ_0.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & \text{for } \theta<x<\infty \\ 0, & \text{otherwise} \end{cases}$$

$(-\infty<\theta<\infty)$. We use an unbiased estimate $Y=\bar{X}-1$ $(=n^{-1}\sum_{i=1}^{n}X_i-1)$ for θ to get a one-sided test.

§2. Optimal one-sided test.

We shall first find the density of Y and furthermore the density of $T=Y+1-\theta$ to obtain the one-sided test.

Finding the joint density of variables $W=X_1+\ldots+X_n$, $Z_1=X_1, \ldots, Z_{n-1}=X_{n-1}$ and taking the marginal density $g_W(w|\theta)$ of W we obtain

$$g_W(w|\theta)=(\Gamma(n))^{-1}(w-n\theta)^{n-1}e^{-(w-n\theta)}I_{(n\theta,\infty)}(w).$$

Noticing $Y=n^{-1}W-1$ we get the density of Y as follows:

$$h_Y(y|\theta)=g_W(n(y+1)|\theta)n$$

$$=(\Gamma(n))^{-1}n^n(y+1-\theta)^{n-1}e^{-n(y+1-\theta)}I_{[\theta-1,\infty)}(y).$$

Furthermore, letting $t=y+1-\theta$ we have the density of T so that

$$h_T(t)=(\Gamma(n))^{-1}n^n t^{n-1}e^{-nt}I_{[0,\infty)}(t)$$

which is the gamma density with parameters n and n.

Let α be a real number such that $0<\alpha<1$. We propose the one-sided test which rejects H_0 if $\theta_0-1+t_0\leqq Y$ and accepts H_0 if $\theta_0-1+t_0>Y$ where t_0 is given by

$$\int_{t_0}^{\infty} h_T(t) = \alpha.$$

Using the test function we write this test as

$$\phi(y) = \begin{cases} 1, & \text{for } y \geq \theta_0 + t_0 - 1 \\ 0, & \text{for } y < \theta_0 + t_0 - 1. \end{cases}$$

To check unbiasedness of this test we obtain the power function as follows:

$$\pi(\theta) = E_\theta(\phi(Y)) = \int_{\theta_0 + t_0 - 1}^{\infty} h_Y(y|\theta)\, dy$$

$$= \int_{\theta_0 - \theta + t_0}^{\infty} h_T(t)\, dt.$$

Since $d\pi(\theta)/d\theta = h_T(\theta_0 - \theta + t_0)(\geq 0)$, $\forall\theta$ and $\pi(\theta_0) = \alpha$, our one-sided test is unbiased.

REFERENCE:

Nogami, Y. (2000). Optimal two-sided tests for the positional and proportional parameters of the exponential distribution--comparison with the generalized likelihood-ratio tests--. Discussion Paper Series No. 893, Institute of Policy and Planning Sciences, University of Tsukuba, December.