No. 94 (80 - 32)

A generalization of a sequential selection process by introducing an extended shortage function

by

Seizo Ikuta *)

October, 1980

ABSTRACT

A theorem is presented for decision problems such that each action has a different amount of reward, which is defined as the sum of two terms, called an immediate reward and after-effect. The theorem shows that the maximum reward for this kind of decision problems can be always expressed using an extended shortage functions defined in the present paper. Furthermore it will be appreciated that applications of the theorem together with some monotonicty of the extended shortage functions brings about a quite systematic approach to many real-world problems of a certain class, for example, the problem of optimally selling and/or buying within a given period an asset, say stock, with fluctuating price day by day.

*) Institute of Socio-Economic Planning, University of Tsukuba, Sakura-mura, Niiharigun, Ibaraki-ken 305, Japan

CONTENTS

1. INTRODUCTION

This paper develops the method for more generally treating a decision model which has defined in [9] by the auther. The decision model is described as follows: It has a finite action space $\mathbf{A} = \{A^r; r=0,1,\ldots,k\}$, $k \geqq 1$. If an action A^r in $\mathbf{A}$ is chosen after observing an m-vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$, $m \geqq 0$, from a known distribution F, then the reward consisting of two terms, $x^r(\boldsymbol{\theta}) + z^r(\boldsymbol{\theta})$, is gained where both $x^r(\boldsymbol{\theta})$ and $z^r(\boldsymbol{\theta})$ are real-valued functions of $\boldsymbol{\theta}$. For convenience we shall name $x^r(\boldsymbol{\theta})$ and $z^r(\boldsymbol{\theta})$ an immediate reward and after-effect, respectively, for taking action A^r. The objective is here to find the optimal decision rule, attaining the maximum expected reward gained, denoted by v_i. Clearly we have

(1.1) $\quad v = E(v(\boldsymbol{\theta}))$

where E represents an expectation operator of F and where

(1.2) $\quad v(\theta) = \max_{r=0,1,\ldots,k} \{x^r(\theta) + z^r(\theta)\}$

From now on we shall refer to the expression (1.1) or (1.2) as a <u>fundamental expression</u> or <u>FE</u> for brevity. Before proceeding to further discussion, we shall define in advance an extended shortage function by use of which the above FE can be expressed (section 2) and list some properties of the function, which will play an essencial role in applying the above model to many real-world problems as discussed in section 4.

<u>DEFINITION 1</u> For any integrable functions, x, g, h, a, b, and y, of m-vector $\boldsymbol{\theta}$, define the following functions L, $\tilde{L}$, T, and $\tilde{T}$, called a shortage functions,

(1.3) $\quad L(x;g,h) = x(1 - g)I(1 > h)$

(1.4) $\quad L(x;g) = L(x;g,g)$

(1.5) $\quad \tilde{L}(x;g) = xg + L(x;g)$

(1.6) $\quad L(a,b;y) = L(a;y/a) - L(b;y/b) \quad a > 0$ and $b > 0$ for all $\boldsymbol{\theta}$

where I( ) represents an indicator by which we mean that if a given statement S is true, then I(S) = 1, otherwise I(S) = 0. Let $\boldsymbol{\theta}$ be now a random variable from a known distribution F. Then define the expectations of (1.3) to (1.6) as

(1.7) $T(x;g,h) = E(L(x;g,h))$

(1.8) $T(x;g) = T(x;g,g)$

(1.9) $\tilde{T}(x;g) = E(xg) + T(x;g)$

(1.10) $T(a,b;y) = T(a;y/a) - T(b;y/b)$ $a > 0$ and $b > 0$ for all $\boldsymbol{\theta}$

Suppose that $\boldsymbol{\theta}$ is an 1-dimensional vector, that is, $\boldsymbol{\theta} = (\theta)$, where $\theta > 0$ and let a and b be positive real numbers. Then we shall define the functions resulting from replacing x, g, h, a, and b in (1.7) to (1.10) by θ, g/θ, h/θ, $a\theta$, and $b\theta$, respectively, as

(1.13) $T(g,h) = E(L(\theta;g/\theta,h/\theta)) = E((\theta - g)I(\theta > h))$

(1.14) $T(g) = T(g,g) = E((\theta - g)I(\theta > g))$

(1.15) $\tilde{T}(g) = g + T(g)$

(1.16) $T_{a,b}(g) = aT(y/a) - bT(y/b)$ $a > 0$ and $b > 0$

For convenience define

(1.17) $T(x;\infty) = T(\infty) = 0$

Conventionally only T(g) with a constant function g have been called a shortage function. Throughout the paper, for simplicity, that $\{h_n\}$ is an increasing or strictly increasing sequence will be written as, respectively, " $h_n \nearrow$ in n " or " $h_n \nearrow$ in n ". For decreasing or strictly decreasing sequence, symboles $\swarrow$ or $\swarrow$, respectively, is emploied.

LEMMA 1 For any integrable functions, x, g, h, s, a, b, and y, of θ, we have

(a) $L(x;g) \geqq L(x;g,h)$

(b) 1. $L(x;g) \swarrow$ in g, 2. $L(x;0) = x$, 3. $x \geqq L(x;g) \geqq x - xg$,

4. $L(x;g) \geqq 0$

(c) 1. $\tilde{L}(x;g) \nearrow$ in g, 2. $\tilde{L}(x;0) = x$, 3. $x + xg \geqq \tilde{L}(x;g) \geqq x$,

4. $\tilde{L}(x;g) \geqq xg$

(d) If $0 < a \leqq b$ for all θ, we have

1. $L(a,b;y) \leqq 0$, 2. $L(a,b;y) \nearrow$ in y, and 3. $L(a,b;y) \geqq (a - b)$

PROOF (a) Since $\{1 > h\} = \{1 > h \geqq g\} \cup \{1 > g > h\} \cup \{g \geqq 1 > h\}$ and $\{1 > g\} = \{1 > h \geqq g\} \cup \{1 > g > h\} \cup \{h \geqq 1 > g\}$, we get $L(x;g) - L(x;g,h) = x(1 - g)I(h \geqq 1 > g) - x(1 - g)I(g \geqq 1 > h) \geqq 0$. (b1) For $g \geqq s$, we have $L(x;g) - L(x;s) \leqq L(x;g) - L(x;s,g) = x(s - g)I(1 > g) \leqq 0$. (c1) For $g \geqq s$, it follows that $\tilde{L}(x;g) - \tilde{L}(x;s) = x(g - s) + L(x;g) - L(x;s) \geqq x(g - s) + L(x;g,s) - L(x;s) = x(g - s)(1 - I(1 > s)) \geqq 0$. (d1) $L(a,b,;y) \leqq L(a;y/a) - L(b;y/b,y/a) = (a - b)I(a > y) \leqq 0$. (d 2) Using (a), we have, for $y \geqq u \geqq 0$, $L(a,b;y) - L(a,b;u) \geqq \max\{(b - a)(I(a > u) - I(b > y)), (y - u)(I(b > y) - I(a > u))\} \geqq 0$. Q.E.D.

The two lemmas below are clear from lemma 1.

LEMMA 2 We have for any integrable functions, x, g, s, h, a, and b, of θ

(a) $T(x;g) \geqq T(x;g,h)$

(b) 1. $T(x;g) \swarrow$ in g, 2. $T(x;0) = E(x)$, 3. $E(x) \geqq T(x;g) \geqq E(x) - E(xg)$,

4. $T(x;g) \geqq 0$

(c) 1. $\tilde{T}(x;g) \nearrow$ in g, 2. $\tilde{T}(x;0) = E(x)$, 3. $E(x) + E(xg) \geqq \tilde{T}(x;g) \geqq E(x)$,

4. $\tilde{T}(x;g) \geqq E(xg)$

(d) If $0 < a \leqq b$ for all θ, we have

1. $T(a,b;y) \leqq 0$, 2. $T(a,b;y) \nearrow$ in y, and 3. $T(a,b;y) \geqq E(a - b)$

<u>LEMMA 3</u>   We have for any integrable functions g, s, and h, of θ (scalar)

(a) $T(g) \geqq T(g,h)$

(b) 1. $T(g) \swarrow$ in g, 2. $T(0) = E$, 3. $E \geqq T(g) \geqq E - E(g)$, 4. $T(g) \geqq 0$

(c) 1. $\tilde{T}(g) \nearrow$ in g, 2. $\tilde{T}(0) = E$, 3. $E + E(g) \geqq \tilde{T}(g) \geqq E$, 4. $\tilde{T}(g) \geqq E(g)$

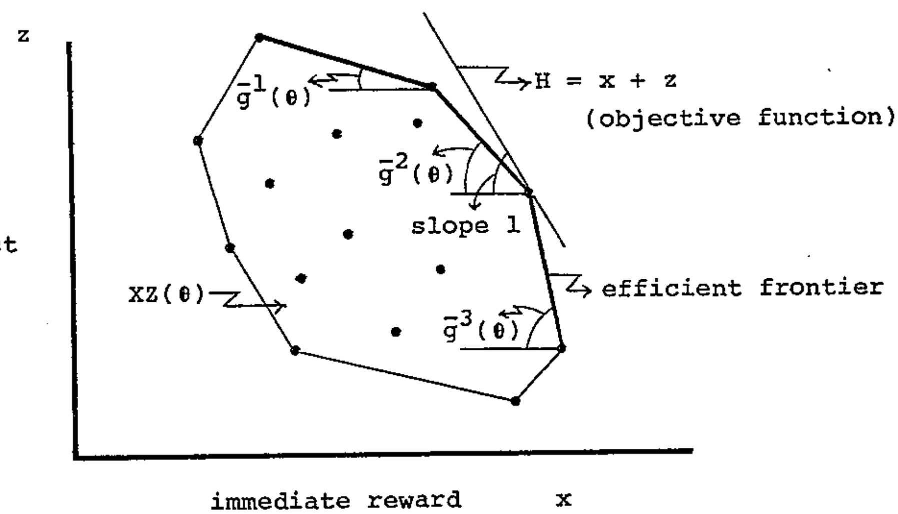
(d) If $0 < a \leqq b$, we have

1. $T_{ab}(y) \leqq 0$, 2. $T_{ab}(y) \nearrow$ in y, 3. $T_{ab}(g) \geqq (a - b)E$.

where E denotes an expectation of θ.

2. DERIVATION OF FUNDAMENTAL EXPRESSION

The maximization problem (1.2) is equivalent to one of maximizing the objective function $H = x + z$ on the set $XZ(\theta) = \{(x^r(\theta), z^r(\theta)): r=0, 1,\ldots,k\}$. This implies that the optimal action, attaining $v(\theta)$, is within the set of efficient actions in $XZ(\theta)$. By term efficient action we shall means an action associated with the point $(\bar{x}^r(\theta), \bar{z}^r(\theta))$ which is on the boundary line of convex set generated from $k + 1$ points in $XZ(\theta)$ and which has not any point $(x^r(\theta), z^r(\theta))$ such that $(x^r(\theta), z^r(\theta)) \geq (\bar{x}^r(\theta), \bar{z}^r(\theta))$ (see Figure below).



Now we shall denote the set of all efficient actions in $\mathbf{A}$ by $\mathbf{A}(\theta) = \{\bar{A}^r: r=0,1,\ldots,k(\theta)\}$ where $\bar{x}^r(\theta) \nearrow$ in r for any θ from the definition of efficient action. Then define

$$(2.1) \quad \bar{g}^r(\theta) = \Delta\bar{z}^r(\theta)/\Delta\bar{x}^r(\theta) \qquad 0 \leqq r \leqq k(\theta)$$

where $\Delta z^r(\theta) = z^{r-1}(\theta) - z^r(\theta)$ and $\Delta x^r(\theta) = x^r(\theta) - x^{r-1}(\theta)$. Here clearly we have $\bar{g}^r(\theta) \nearrow$ in r from the definition of efficient action, and to choose the action $\bar{A}^r$ such that $\bar{g}^{r+1}(\theta) \geqq 1 > \bar{g}^r(\theta)$ is optimal where let $\bar{g}^0(\theta) = -\infty$ and $\bar{g}^{k(\theta)+1}(\theta) = \infty$. From this, we can easily transform (1.2) into

$$(2.2) \qquad v(\theta) = \sum_{r=0}^{k(\theta)} (\bar{x}^r(\theta) + \bar{z}^r(\theta)) I(\bar{g}^{r+1}(\theta) \geqq 1 > \bar{g}^r(\theta))$$

Using the relationship $I(\bar{g}^{r+1}(\theta) \geqq 1 > \bar{g}^r(\theta)) = I(1 > \bar{g}^r(\theta)) - I(1 > \bar{g}^{r+1}(\theta))$, (2.2) can be arranged as follows, using an extended shortage function L,

$$(2.3) \qquad v(\theta) = \bar{x}^0(\theta) + \bar{z}^0(\theta) + \sum_{r=1}^{k(\theta)} L(\Delta\bar{x}^r(\theta);\bar{g}^r(\theta))$$

Then (1.1) becomes

$$(2.4) \qquad v = E(\bar{x}^0(\theta)) + E(\bar{z}^0(\theta)) + E(\sum_{r=1}^{k(\theta)} L(\Delta\bar{x}^r(\theta);\bar{g}^r(\theta)))$$

<u>DEFINITION 2</u> A given action space $\mathbf{A}$ is said to be regular if $A(\theta) = \mathbf{A}$ for all θ.

The next theorem can be easily obtained. (From now on, let an expectation $E(f(\theta))$ be simply written as $E(f)$).

<u>THEOREM 1</u> Suppose that $x^r(\theta) \nearrow$ in r for all θ.

(a) If $g^r(\theta) \nearrow$ in r for all θ, the action space $\mathbf{A}$ is regular. Then the FE (2.4) becomes

$$(2.5) \qquad v = E(x^0) + E(z^0) + \sum_{r=1}^{k} T(\Delta x^r;g^r),$$

and the optimal decision may be described as follows: If an observation θ belongs to a set

$$(2.6) \qquad D^r = \{\theta: g^{r+1}(\theta) \geqq 1 > g^r(\theta)\} \qquad 0 \leqq r \leqq k,$$

then take action A^r, where

$$(2.7) \qquad g^r(\theta) = \Delta z^{r-1}(\theta)/\Delta x^r(\theta) \qquad 0 < r < k,$$

$g^0(\theta) = -\infty$, and $g^{k+1}(\theta) = \infty$.

(b) if $g^r(\theta) \swarrow$ in r for all θ, we have $\mathbf{A}(\theta) = \{A^0, A^k\}$ for all θ. Then we obtain the FE

$$(2.8) \qquad v = E(x^0) + E(z^0)) + T(x^k - x^0;g)$$

where

(2.9) $\quad g(\theta) = (z^0(\theta) - z^k(\theta))/(x^k(\theta) - x^0(\theta)),$

and the optimal decision is that if an observation θ belongs to the set D^0, take the action A^0, and if to D^k, A^k, where

(2.10) $\quad D^0 = \{\theta : g(\theta) \geqq 1\} \qquad D^k = \{\theta : 1 > g(\theta)\}$

<u>REMARK 1</u> When $k = 0$, clearly $= E(x^0) + E(z^0)$. For $r > k$ let us define $\Delta x^r(\theta)$ to be any real number and $g^r(\theta) = \infty$ for all θ. Then since $T(\Delta x^r; g^r) = 0$ from definition (1.17), the FE (2.5) may be rewritten as

(2.11) $\quad v = E(x^0) + E(z^0) + \sum_{r=1}^{\infty} T(\Delta x^r; g^r)$

Here notice that the FE (2.11) is valid even for case of $k = 0$.

<u>REMARK 2</u> In case of $x_i^r(\theta) \nearrow$ in r for all θ, the above theorem holds only if the order of sequence $x^0(\theta), x^1(\theta), \ldots, x^k(\theta)$ is reversed as $x^k(\theta), x^{k-1}(\theta), \ldots, x^0(\theta)$, where the requirements " $g^r(\theta) \nearrow$ in r " in (a) and " $g^r(\theta) \swarrow$ in r " in (b), $g^r(\theta)$ and $g^{r+1}(\theta)$ in (2.6), $x^0(\theta)$ and $x^k(\theta)$, $z^0(\theta)$ and $z^k(\theta)$, D^0 and D^k, $g^0(\theta)$ and $g^{k+1}(\theta)$ must be interchanged in a notational sence and where $\Delta x^r(\theta)$ in (2.5) must be replaced by $-\Delta x^r(\theta)$.

<u>REMARK 3</u> It is clear that some of inequality signs $\geqq$ and $>$ in sets (2.6) may be replaced by, respectively, $>$ and $\geqq$ in such an arbitraly way as not to break the mutual exclusivness of $D^0, D^1, \ldots, D^k$. The simplest but most frequently used one of such replacements is to interchange $\geqq$ and $>$ exhaustively.

Next we shall provide a mathematically strict proof of theorem 1. By the term (k+1)-partition of m-dimensional space R^m let us denote the set B of k+1 mutual exclusive subsets B^r, $r=0,1,\ldots,k$ of R^m, which are all Borel sets such that $\bigcup_{r=0}^{k} B^r = R^m$. Furthermore let us represent the set consisiting of all such partitions by $G^m(k+1)$, called a partition space. Clearly the set $\mathbf{D} = \{D^r; r=0,1,\ldots,k\}$ belongs to $G^m(k+1)$ where let D^r be empty for $0 < r < k$ in (b) of theorem 1. Then we shall call the $\mathbf{D}$ an <u>optimal partition</u> or <u>OP</u> for brevity. Now any (k+1)-partition $\mathbf{B}$ in $G^m(k+1)$ may prescribe a decision rule for the maximization problem (1.1), which implies that if an observation θ belongs to an element B^r in $\mathbf{B}$, choose the action A^r. Then the expected reward from employing the $\mathbf{B}$ as a decision rule becomes

$$V(\mathbf{B}) = \sum_{r=0}^{k} E((x^r + z^r)I(B^r)) \tag{2.12}$$

where $I(B^r) = I(\theta \in B^r)$.

<u>LEMMA 4</u> There exists the maximum expected reward $v = \max_{G^m(k+1)} V(\mathbf{B})$, which is attained by the (k+1)-partition $\mathbf{D} = \{D^r : r=0,1,\ldots,k\}$, where

$$D^r = \{\theta : x^r(\theta) - x^s(\theta) > z^s - z^r \text{ for } 0 \leqq s < r \text{ and } x^r(\theta) - x^s(\theta) \geqq z^s - z^r \text{ for } r < s \leqq k\} \qquad 0 \leqq r \leqq k \tag{2.13}$$

PROOF Let $H^r = x^r + z^r$. Then (2.13) may be written as $D^r = \{H^r > H^s$ for $0 \leqq s < r$ and $H^r \geqq H^s$ for $r < s \leqq k\}$. Two arbitrary sets D^r and D^q with $r > q$ are exclusive because $D^r \subset \{H^r > H^q\}$ and $D^q \subset \{H^q \geqq H^r\}$. Hence $D^0, D^1, \ldots, D^k$ become also mutually exclusive. For any vector θ in R^m let $H^q = H^{q'} = \ldots = H^{q''} > H^u, H^{u'}, \ldots, H^{u''}$ where $(q,q',\ldots,q'',u,u',\ldots,u'')$ is a permutation of integers $0,1,\ldots,k$ such as $q < q' < \ldots < q''$ and $u < u' < \ldots < u''$. Then $q = 0$ leads to $H^0 \geqq H^s$ for $0 < s \leqq k$, $0 < q < k$ to $H^q > H^s$ for $0 \leqq s < q$ and to $H^q \geqq H^s$ for $q < s \leqq k$, and $q = k$ to $H^k > H^s$ for $0 \leqq s < k$. Thus the vector θ must belong to one of $D^0, D^1, \ldots, D^k$. This implies $R^m = \bigcup_{r=0}^{k} D^r$. Hence it follows that $\mathbf{D}$ is a (k+1)-partition of R^m. For any $\mathbf{B}$ in $G^m(k+1)$ we have $V(\mathbf{D}) - V(\mathbf{B}) = \sum_{r,s=0}^{k} E((H^r - H^s)I(D^r \cap B^s))$, using $I(D^r) = \sum_{s=0}^{k} I(D^r \cap B^s)$ and $I(B^s) = \sum_{r=0}^{k} I(D^r \cap B^s)$. Since $H^r \geqq H^s$ on $D^r \cap B^s$, it follows that $V(\mathbf{D}) - V(\mathbf{B}) \geqq 0$, hence $V(\mathbf{D})$ is proved to be the maximum expected reward. Q.E.D.

Then the maximum expected reward $v = V(D)$ is expressed as

(2.14) $v = \sum_{r=0}^{k} E((x^r + z^r)I(D^r))$

LEMMA 5 For any two sequences $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ where x_n are all positive, $y_n/x_n \nearrow (\swarrow)$ in n yields

(a) $y_1/x_1 \leqq (\geqq) (y_1 + y_2 + \ldots + y_n)/(x_1 + x_2 + \ldots + x_n) \leqq (\geqq) y_N/x_N$

(b) $(y_1 + y_2 + \ldots + y_n)/(x_1 + x_2 + \ldots + x_n) \nearrow (\swarrow)$ in n

(c) $(y_n + y_{n+1} + \ldots + y_N)/(x_n + x_{n+1} + \ldots + x_N) \nearrow (\swarrow)$ in n

PROOF Easy

PROOF of THEOREM 1 Define $g^{rs} = g^{sr} = (z^s - z^r)/(x^r - x^s)$ for $r \neq s$. Then we have $g^r = g^{r,r-1} = g^{r-1,r}$, and

$$g^{rs} = (\Delta z^{s+1}+\Delta z^{s+2}+\ldots+\Delta z^r)/(\Delta x^{s+1}+\Delta x^{s+2}+\ldots+\Delta x^r) \quad \text{if } s < r \qquad (*)$$

$$g^{rs} = (\Delta z^{r+1}+\Delta z^{r+2}+\ldots+\Delta z^s)/(\Delta x^{r+1}+\Delta x^{r+2}+\ldots+\Delta x^s) \quad \text{if } r < s \qquad (**)$$

Using g^{rs}, (2.13) can be transformed into

$$\left.\begin{array}{l} D^0 = \{g^{0s} \geqq 1 \text{ for } 0 < s \leqq k\} \\ D^r = \{g^{rs} \geqq 1 \text{ for } r < s \leqq k \text{ and } 1 > g^{rs} \text{ or } 0 \leqq s < r\} \quad 0 < r < k \\ D^k = \{1 > g^{ks} \text{ for } 0 \leqq s < k\} \end{array}\right\} (***)$$

(a) Since $g^r = \Delta z^r/\Delta x^r \nearrow$ from the assumption, we get, from lemma 5(a), $g^{rs} \leqq \Delta z^r/\Delta x^r = g^{r,r-1} = g^r$ for $s < r$ and $g^{rs} \geqq \Delta z^{r+1}/\Delta x^{r+1} = g^{r,r+1} = g^{r+1}$ for $r < s$. Hence the sets (***) can be reduced to the intervals (2.6). Arranging (2.14) by inserting (2.6) in yields (2.5).

(b) If $g^r \swarrow$ in r, then applying lemma 5(b,c) to (**) and (*) yields $g^{0s} \geqq (\Delta z^1 + \Delta z^2 + \ldots + \Delta z^k)/(\Delta x^1 + \Delta x^2 + \ldots + \Delta x^k) = g^{0k} = g$ and $g^{ks} \leqq (\Delta z^1 + \Delta z^2 + \ldots + \Delta z^k)/(\Delta x^1 + \Delta x^2 + \ldots + \Delta x^k) = g^{k0} = g$. Thus we get (2.10). Consequently D^r must be empty for $0 < r < k$. The (2.8) can be obtained by inserting (2.10) in (2.14). Q.E.D.

For the problem with 1-dimensional vector $\boldsymbol{\theta} = (\theta)$, the next corollary offers a necessary condition for D^r in OP, $r=0,1,\ldots,k$, being $k + 1$ successive adjacent intervals on R^1.

<u>COROLLARY 1</u> Suppose that θ is 1-dimensional (that is, $\boldsymbol{\theta} = (\theta)$) and that $x^r(\theta) \nearrow$ in r.

(a) If $g^r(\theta) \nearrow$ in r and $g^r(\theta) \swarrow(\nearrow)$ in θ, define

(2.15) $\quad c^r = \inf\{\theta : 1 > g^r(\theta)\} \;\; (\sup\{\theta : 1 > g^r(\theta)\})$

Then (2.6) is reduced to the intervals

(2.16) $\quad D^r = (c^r,\, c^{r+1}] \quad ((c^{r+1},\, c^r])$

(b) If $g^r(\theta) \swarrow$ in r $g(\theta) \swarrow(\nearrow)$ in θ, define

(2.17) $\quad c = \inf\{\theta : 1 > g(\theta)\} \quad (\sup\{\theta : 1 > g(\theta)\})$

Then (2.10) is reduced to

(2.18) $\quad D^0 = (-\infty,\, c] \;\; ((c, \infty)) \qquad D^k = (c, \infty] \;\; ((-\infty, c])$

<u>COROLLARY 2</u> Let us regard $z^r(\boldsymbol{\theta})$ as a functional of $\boldsymbol{x}(\boldsymbol{\theta}) = (x^0(\boldsymbol{\theta}), x^1(\boldsymbol{\theta}), \ldots, x^k(\boldsymbol{\theta}))$ for all r, that is, $z^r(\boldsymbol{\theta}, x(\boldsymbol{\theta}))$, and assume that

(1) $g^r(\boldsymbol{\theta})$ and $g(\boldsymbol{\theta})$ associated with any model with $x^r(\boldsymbol{\theta}) \nearrow$ in r are independent of $\boldsymbol{x}(.)$ for all r, and

(2) the function $z^r(\boldsymbol{\theta}, \boldsymbol{x})$ of $\boldsymbol{x}$ is right continuous at $\boldsymbol{x} = 0$ for all r.

Then theorem 1 holds also for any model with $x^r(\boldsymbol{\theta}) \nearrow$ in r.

PROOF Let us prove only the case of $g^r(\boldsymbol{\theta}) \nearrow$ in r for all $\boldsymbol{\theta}$. For the case of $g^r(\boldsymbol{\theta}) \swarrow$ in r for all $\boldsymbol{\theta}$ it can be similarly proved. For any model with $x^r(\boldsymbol{\theta}) \nearrow$ in r we can take a positive real number h so that $x^r(\boldsymbol{\theta}) + rh \nearrow$ in r for all $\boldsymbol{\theta}$. Then, from the assumption (1) the $g^r(\boldsymbol{\theta})$ and OP for the model with immediate rewards $x^r(\boldsymbol{\theta}) + rh$ is given by (2.7) and (2.6) (i.e., $\mathbf{D}$), respectively, which are independent of h. Let $V_h(\mathbf{D})$ and $V_h(\mathbf{B})$ be now the expected rewards for the model, associated with the partition $\mathbf{D}$ and any partition $\mathbf{B}$, respectively. Then we have $V_h(\mathbf{D}) = \sum_{r=0}^{k} E((x^r + rh + z^r(\boldsymbol{x} + (0,1,\ldots,k)'h))I(D^r)) =$

$E(x^0) + E(z^0(\mathbf{x} + (0,1,\ldots,k)'h)) + \sum_{r=1}^{k} T(\Delta x^r + h; g^r)$ and $V_h(B) = \sum_{r=0}^{k} E((x^r + h + z^r(\mathbf{x} + (0,1,\ldots,k)'h))I(B^r))$. Since $V_h(B) \leqq V_h(D)$ from the definition of D, we have $\lim_{h\to 0+} V_h(B) \leqq \lim_{h\to 0+} V_h(D)$. Noticing the assumption (2), the two limits provide the expected rewards for the original model with $x^r(\boldsymbol{\theta}) \nearrow$ in r, that is, V(B) and V(D). Therefore we have $V(B) \leqq V(D)$ for any B in $G^m(k+1)$. This means that the D, that is, the OP for any model with $x^r(\boldsymbol{\theta}) \nearrow$ in r, provides the OP for the original model (with $x^r(\boldsymbol{\theta}) \nearrow$ in r) and that we have the FE $v = V(D) = E(x^0) + E(z^0(\mathbf{x})) + \sum_{r=1}^{k} T(\Delta x^r; g^r)$, which is of the identical form with FE (2.5). Q.E.D.

As special cases of our general decision model we may consider the next linear models:

Linear model A that $x^r(\boldsymbol{\theta}) = a^r\theta$, $r=0,1,\ldots,k$, where $a^r \nearrow$ in r and $\boldsymbol{\theta} = (\theta)$ is an 1-vector, assumed $\theta > 0$.

Linear model B that $x^0(\boldsymbol{\theta}) = 0$ and $x^r(\boldsymbol{\theta}) = \theta_1 + \theta_2 + \ldots + \theta_r$ for $1 \leqq r \leqq k$ where $k \leqq m$ and $\theta_1 \geqq \theta_2 \geqq \ldots \geqq \theta_m > 0$. We shall denote the distribution function of θ_r by F_r and its expectation by $E_r \geqq 0$.

Applying theorem 1 to the above two linear models easily produces the next corollaries, provided that an after-effect is independent of $\boldsymbol{\theta}$.

<u>COROLLARY 3</u>(linear model A)

Suppose $k \geqq 1$ and $a^r \nearrow$ in r. Then define

$$c^r = \Delta z^{r-1}/\Delta a^r \qquad 1 \leqq r \leqq k \tag{2.19}$$

where $\Delta z^r = z^{r-1} - z^r$ and $\Delta a^r = a^r - a^{r-1}$, and

$$c = (z^0 - z^k)/(a^k - a^0) \tag{2.20}$$

(a) If $c^r \nearrow$ in r, the action space $\mathbf{A}$ is regular. Then the OP is reduced to a set of k + 1 intervals, r = 0,1,...,k, (2.21) $D^r = (c^r, c^{r+1})$

where $c^0 = -\infty$ and $c^{k+1} = \infty$, and the FE to

(2.22) $v = a^0E + z^0 + \sum_{r=1}^{k} \Delta a^r T(c^r)$

(b) If $c^r \swarrow$ in r, we have $A(\theta) = \{A^0, A^k\}$ for all θ. Then the OP is provided by the intervals

(2.23) $D^0 = (-\infty, c\,]$, $D^k = (c, \infty)$

where D^r is empty for $0 < r < k$,

and the FE by

(2.24) $v = a^0E + z^0 + (a^k - a^0)T(c)$

<u>COROLLARY 4</u>(linear model B)

Suppose $k \geqq 1$ and define

(2.25) $c^r = z^{r-1} - z^r$, $1 \leqq r \leqq k$

Then if $c^r \nearrow$ in r, the action space A is regular. Then the OP becomes

(2.26)
$$\begin{cases} D^0 = \{\theta : \theta_1 \leqq c^1\} \\ D^r = \{\theta : c^r < \theta_r \text{ and } \theta_{r+1} \leqq c^{r+1}\} & 0 < r < k \\ D^k = \{\theta : c^k < \theta_k\} \end{cases}$$

and the FE is given by

(2.27) $v = z^0 + \sum_{r=1}^{k} T_r(c^r)$

where $T_r(g)$ represents a shortage function associated with a distribution function F_r of θ_r.

3. SEQUENTIAL SELECTION PROCESSES

As a dynamic process version of the decision model in the previous sections, the following discrete time stochastic decision process may be considered. It has a finite state space $I = \{i:i=0,1,\ldots,N\}$ and a finite state-dependent action space $\mathbf{A}_i = \{A_i^r:r=0,1,\ldots,k_i\}$. Here it is assumed that m-vectors θ, θ', θ'', ... are observed at successive time points, which are independent random varibles from a given distribution F. If an action A_i^r is taken after observing an m-vector θ when in state i, then an immediate reward $x_i^r(\theta)$ is gained and the state j of the next time is chosen according to a known transition probability p_{ij}^r. The objective is to find the optimal decision strategy, maximizing the total expected reward or expected reward per period over a given time horizon, possibly infinite. From now on we shall refer to the above decision process as a sequential selection process.

3.1 Finite time horizon process

We shall begin with a finite time horizon process. For convenience let times be measured backward from the terminating time point $t = 0$ of the process. For given partitions $\mathcal{B}_i$ in $G^m(k_i+1)$, $i=0,1,\ldots,N$, we shall refer to the set $\bar{B} = \{\mathcal{B}_i:i=0,1,\ldots,N\}$ a decision policy, and denote a time sequence of decision policies, $\bar{B}(t)$, $\bar{B}(t-1)$, ... , $\bar{B}(0)$ by $B(t)$. Here $\bar{B}(0)$ is given as a final(or terminating) condition. Now by $v_i(t)$ we shall denote the maximum total expected reward, starting from time t with state i, where $v_i(0)$ will be assumed to be uniquely determined by a given final condition $\bar{B}(0)$. The decision strategy attaining $v_i(t)$ is referred to as an optimal decision strategy, and furthermore the decision rule and decision policiy associated with the optimal decision strategy as, respectively, an optimal decision rule and optimal decision policy. Now suppose that $v_i(t-1)$, $v_i(t-2)$, ... , $v_i(1)$ exist for all i. Then the after-effect for choosing action A_i^r at time t in state i becomes

$$(3.1) \qquad z_i^r(t) = \beta\sum_{j=0}^{N} p_{ij}^r v_j(t-1)$$

which is independent of an observation θ at time t and where β is a discount factor ($0 < \beta \leqq 1$). From now on, for simplicity, we shall very often suppress a time parameter t and vector θ in $x_i^r(\theta)$, $v_i(t)$, $D_i(t)$, $D_i^r(t)$, $z_i^r(t)$, ... and write them as x_i^r, v_i, D_i, D_i^r, z_i^r, ... Furthermore in general let functions f(t), f(t-1), f(t-2), ... with time parameters t, t-1, t-2, ... be written as f, f', f'', ... Then (3.1) is expressed as

$$(3.2) \qquad z_i^r = \beta \sum_{j=0}^{N} p_{ij}^r v_j' \quad ,$$

and the FE, corresponding to v defined in lemma 4, becomes

$$(3.3) \qquad v_i = \max_{B_i \in G^m(k_i+1)} \sum_{r=0}^{k} E((x_i^r + z_i^r)I(B_i^r))$$

where the maximum of the right hand can be attained by the partition D_i defined by (2.13). The above expression (3.3) can be changed into

$$(3.4) \qquad v_i = R_i(D_i) + \beta\sum_{j=0}^{N} Q_{ij}(D_i)v_j'$$

where

$$(3.5) \qquad R_i(D_i) = \sum_{r=0}^{k_i} E(x_i^r I(D_i^r))$$

$$(3.6) \qquad Q_{ij}(D_i) = \sum_{r=0}^{k_i} p_{ij}^r \; E(I(D_i^r))$$

The above expressions (3.4) to (3.6) reveals the interrelationship between a Markovian decision process with a finite time horizon and our sequential selection process defined in the present paper. It should be noted here that an action space, immediate reward, transition probability, and optimal action which is commonly defined in the former process correspond to, respectively, $G^m(k_i+1)$, $R_i(B_i)$,

$Q_{ij}(B_i)$, and D_i in the latter one.

3.2 Infinite time horizon process

Whenever an infinite time horizon process is discussed in the present paper, a partition space $G^m(k_i+1)$, which provides an action space in Markovian decision process, will be assumed by implication to be finite for all i. For example, if a distribution function F takes positive values only on a given finite domain, then the partition spaces may be defined to be finite. Then we have the next corollary, referring to corollary 6.6 and theorems 6.17 and 6.18 in (15) by Ross.

COROLLARY 5 (a) If a discount factor $\beta < 1$, the limit of $v_i(t)$ as t tends to infinity, denoted by v_i, uniquely exists for all i, and the limiting vector $(v_0, v_1, \ldots, v_N)$ is the unique solution to the system of equations resulting from replacing $v_j^!$ in (3.4) by v_j.

(b) When $\beta < 1$, let $y_i = \beta(v_i - v_{i-1})$ for the limits v_i and suppose $\beta^{-1}y_i$ is bounded uniformly in β for all i. Then there exist u_i and g satisfying

$$(3.7) \qquad g + u_i = R_i(D_i) + \sum_{j=0}^{N} Q_{ij}(D_i)u_j \qquad 0 \leq i \leq N,$$

and for some sequence $\beta_n \rightarrow 1-$ we have $\lim_{n\rightarrow\infty} \beta_n(v_i^n - v_{i-1}^n) = u_i - u_{i-1}$ where v_i^n represents a limit of $v_i(t)$ associated with the discount factor β_n. Then for any i,

$$(3.8) \qquad g = \lim_{n\rightarrow\infty}(1 - \beta_n)v_i^n$$

which provides the maximum expected reward per period for an infinite time horizon process without discounting (i.e., $\beta = 1$).

(c) The optimal decision strategy, attaining limits v_i or g, is given by the stationary strategy $\bar{D}^\infty$ where $\bar{D} = \{D_0, D_1, \ldots, D_N\}$ and where D_i^r are given by (2.6) in which z_i^r is used resulting from replacing $v_j^!$ in (3.2) by v_j or u_j. Such z_i^r and D_i are called a limit z_i^r and limit OP, respectively.

Now the limit OP, that is, D_i in (c) above, can be obtained in a finite number of operations by use of Howard's policy iteration algorithm (PIA) [6]. If $x_i^r(\theta) \nearrow$ in r for all θ and if limits $g_i^r(\theta)$ defined using limits z_i^r are proved to be $\nearrow$ in r, then $G^m(k_i+1)$ may be reduced to the partition space $\tilde{G}^m(k_i+1)$ of only (k_i+1)-partitions of the same form as OP, (2.6), that is,

$$(3.9) \qquad \tilde{G}^m(k_i+1) = \{B : B^r = \{\theta : b_i^{r+1}(\theta) \geqq 1 > b_i^r(\theta)\},\ 0 \leqq r \leqq k_i,$$
$$\text{where for all } \theta$$
$$b_i^0(\theta) = -\infty,\ b_i^{k_i+1}(\theta) = \infty\ ,\ \text{and}$$
$$b_i^r(\theta) \nearrow \text{ in r for } 0 < r < k_i \}$$

When limit $g_i^r(\theta) \swarrow$ in r, it is sufficient to set $b_i(\theta) = b_i^1(\theta) = \ldots = b_i^{k_i}(\theta)$ for all θ. Then if $\beta < 1$, the value determination operation in PIA is to solve for a given partitions B_i, $i=0,1,\ldots,N$, the system of equations (3.4) with B_i and v_j instead of D_i and v_j'. When $\beta = 1$, the operation is perfomed by solving the system of equations (3.7) with B_i instead of D_i, setting $u_0 = 0$. Now arranging the right hand of these equations by inserting $R_i(B_i)$ and $Q_{ij}(B_i)$ associated with a given B_i in $\tilde{G}^m(k_i+1)$ yields the terms $\sum_{r=1}^{k_i} T(\Delta x_i^r; g_i^r, b_i^r)$. If $g_i^r(\theta) \swarrow$ in r for all θ, the same arrangement produces the term $T(x_i^{k_i} - x_i^0; g_i, b_i)$. Then a policy improvement routine in PIA is carried out by finding $b_i^r(\theta)$ maximizing the terms. Applying lemma 2(a), when $g_i^r(\theta) \nearrow$ in r, the maximum of the term is attained by $b_i^r(\theta) = g_i^r(\theta)$, $1 \leqq r \leqq k_i$ and when $g_i^r(\theta) \swarrow$ in r, it is maximized $b_i(\theta) = g_i(\theta)$.

In the next section we will demonstrate to what class of decision problems and how the decision models discussed in the previous sections may be successfully applied, taking the following four examples: target attacking problem, purchasing problem, inventory problem, and customer selection problem. In these applications the following should be noticed:

In a finite time horizon process, the monotonicity of $g_i^r(\theta,t)$ in r for all t will be proved using structural forms of $y_i(t) = \beta(v_i(t) - v_{i-1}(t))$, that is, non-negativity, non-positivity, or monotonicity in i. And the structural forms of $y_i(t)$ for all t will be in turn verified inductively by starting with the terminating conditions $y_i(0) = \beta(v_i(0) - v_{i-1}(0))$, which is inherent in each of the given problems. Furthermore when $l < \beta < 1$, the monotonicity of limits $g_i^r(\theta)$ can be determined as a limiting monotonicity of $g_i^r(\theta,t)$ with the final conditions $v_i(0) = 0$, hence $y_i(0) = 0$, where it should be noted that the $y_i(0)$ as starting points of induction is $\nearrow$ as well as $\swarrow$ in i and bounded uniformly in β for all i. When $\beta = 1$, the monotonicity of limits $g_i^r(\theta)$ can be proved, if requirements stated in (b) of corollary 5 hold, by using the structural form of the limits of " limits y_i^n " as n tends to infinity, noticing $\lim_{n\to\infty} y_i^n = u_i - u_{i-1}$.

4. APPLICATIONS

4.1 Target attacking problem

This section will demonstrate that theorem 1 provides a quite systematic approach to a so-called target attacking problem, which has investigated by Mastran and Thomas in [12], and to a version of it which will be defined below. We shall refer to the model defined in [12] as model 1 and the version as model 2. They can be described, respectively, as follows:

Model 1 : Suppose that a submarine with N torpedos is starting to patrol a sea area over a given mission periods. Assume that one target can be detected every period and that the values of targets detected at successive periods are independent, positive random variables from a known distribution F with expectation $E \geqq 0$. Let $h(r,\theta)$ be the value from firing r torpedos at a target of value θ, where it is assumed that $h(0,\theta) = 0$, $h(r,\theta) \not\nearrow$ in r for all θ, and $\Delta h(r,\theta) = h(r,\theta) - h(r-1,\theta) \not\swarrow$ in r for all θ. A special case of this model is that $h(r,\theta) = (1 - (1 - q(\theta))^r)\theta$ where $q(\theta)$ $(0 < q(\theta) < 1)$ represents a single shot hit probability dependent on θ. It is easy to extend the above model to the case that a target is detected with a probability. The case that $q(\theta)$ is independent of θ is discussed in [12].

Model 2 : It is based on the assumptions that (1) a fleet of m enemy ships may be detected every period (m is a fixed positive integer), and (2) vector of values of m enemy ships detected, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$, is a random variable from a known m-dimensional distribution F, and (3) let the immediate reward from firing r torpedos at a fleet of m enemy ships with value vector $\boldsymbol{\theta}$ be given by $h(r,\boldsymbol{\theta})$ where $h(0,\boldsymbol{\theta}) = 0$, $h(r,\boldsymbol{\theta}) \not\nearrow$ in r, and $\Delta h(r,\boldsymbol{\theta}) = h(r,\boldsymbol{\theta}) - h(r-1,\theta) \not\swarrow$ in r.

The objective in the both models above is to maximize the total expected value from enemy ships hitted by the submarine over the given mission periods.

Analysis of model 1

In the model a state of the system may be described by the number of torpedos remaining at each time in the submarine. Hence the state space is given by $I = \{i:i=0,1,...,N\}$. Let the action of firing r torpedos at a target detected of value θ be denoted by A_i^r, where $0 \leqq r \leqq k_i (= i)$. Then the immediate reward from taking the action A_i^r is given by $x_i^r(\theta) = h(r,\theta)$ for $0 \leqq r \leqq k_i$.

Let $v_i(t)$ be now the maximum total expected value starting from time t with state i. Then the after-effect for action A_i^r is provided by $z_i^r = v'_{i-r}$. Here it is clear that the final values of $v_i(t)$ is given by

(4.1.1) $\quad v_i(0) = E(h(i)),$

and (2.7) becomes

(4.1.2) $\quad g_i^r(\theta) = y'_{i-r+1} / \Delta h(r,\theta) \qquad 1 \leqq r \leqq i$

where $y'_i = v'_i - v'_{i-1}$. Here define the next two statements for all i and all t:

St. 1 $\quad y_i \geqq 0 \qquad$ St. 2 $\quad y_i \swarrow$ in i

Clearly the two statements are true for $t = 0$. Now suppose that they are true for any $t - 1$. Then noticing the assumption $\Delta h(r,\theta) \swarrow$ in r, we have $g_i^r(\theta) \nearrow$ in r, $\swarrow$ in i, and $\geqq 0$. Since $x_i^r(\theta) = h(r,\theta) \nearrow$ in r, the FE (2.5) becomes

(4.1.3) $\quad v_i = v'_i + \sum_{r=1}^{i} T(\Delta h(r); g_i^r)$

from which we obtain

$$(4.1.4) \qquad y_i = y_i' + T(\Delta h(i); g_i^i) + \sum_{r=1}^{i-1}(T(\Delta h(r); g_i^r) - T(\Delta h(r); g_{i-1}^r))$$

$$(4.1.5) \qquad = \tilde{T}(\Delta h(1); g_i^1) + \sum_{r=1}^{i-1} T(\Delta h(r+1), \Delta h(r); y_{i-r}') \qquad 1 \leqq i \leqq N$$

Furthermore from (4.1.5) we have

$$(4.1.6) \qquad y_i - y_{i-1} = \tilde{T}(\Delta h(1); g_i^1) - \tilde{T}(\Delta h(1); g_{i-1}^1) + T(\Delta h(i), \Delta h(i-1); y_1') + \sum_{r=1}^{i-2}(T(\Delta h(r+1), \Delta h(r); y_{i-r}') - T(\Delta h(r+1), \Delta h(r); y_{i-r-1}')) \qquad 2 \leqq i \leqq r$$

Using lemma 2, we easily get $y_i \geqq 0$ (St.2) from (4.1.4) and $y_i - y_{i-1} \leqq 0$ from (4.1.6), hence $y_i \searrow$ in i (St.1). Thus it follows by induction that the above two statements hold for all $t \geqq 1$. Moreover since $y_i \geqq y_i'$ form (4.1.4), it follows that $y_i \nearrow$ in t. From these we have

$$(4.1.7) \begin{cases} g_i^r(\theta) \geqq 0, \nearrow \text{ in } r, \text{ and } \searrow \text{ in } i \text{ for all } t \geqq 1 \\ g_i^r(\theta) \nearrow \text{ in } t. \end{cases}$$

Therefore the FE (4.1.3) holds for all t and the OP is given by (2.6) with $k_i = N - i$ for all t.

Analysis of model 2

The model has exactly the same state space as in model 1. Let A_i^r denote an action of firing r torpedos at a fleet of m enemy ships of value vectoe θ, detected when in state i. Then clearly $0 \leqq r \leqq k_i (= \min\{i,m\})$. An immediate reward and after-effect for action A_i^r becomes $x_i^r(\theta) = h(r,\theta)$ and $z_i^r = v_{i-r}'$, respectively. The final value of $v_i(t)$ becomes

$$(4.1.7) \qquad v_i(0) = E(h(k_i)) \qquad 1 \leqq i \leqq N$$

Then (2.7) becomes

$$(4.1.8) \qquad g_i^r(\theta) = y_{i-r+1}' / \Delta h(r,\theta) \qquad 0 < r \leqq k_i$$

Define the next two statements for all i and all t:

St. 1 $\quad y_i \geqq 0 \qquad$ St. 2 $\quad y_i \swarrow$ in i

Now since $y_i(0) = E(\Delta h(i)) \geqq 0$ for $i \leqq m$ and $y_i(0) = 0$ for $m < i \leqq N$, it follows that $y_i(0) - y_{i-1}(0) = E(\Delta h(i) - \Delta h(i-1)) \leqq 0$ for $i \leqq m$, $y_{m+1}(0) - y_m(0) = -E(\Delta h(m)) \leqq 0$, and $y_i(0) - y_{i-1}(0) = 0$ for $m+1 < i \leqq N$. Hence the two statements hold for t = 0. Suppose the two statements are true for any t - 1. Then we have $g_i^r(\theta) \geqq 0$, $\nearrow$ in r, and $\swarrow$ in i, and the FE (2.5) becomes

$$v_i = v_i' + \sum_{r=1}^{k_i} T(\Delta h(r); g_i^r) \qquad 0 \leqq i \leqq N \tag{4.1.9}$$

from which we get

for $1 \leqq i \leqq m$

$$y_i = y_i' + \sum_{r=i}^{i} T(\Delta h(r); g_i^r) - \sum_{r=1}^{i-1} T(\Delta h(r); g_{i-1}^r) \tag{4.1.10}$$

$$= y_i' + T(\Delta h(i); g_i^i) + \sum_{r=1}^{i-1} (T(\Delta h(r); g_i^r) - T(\Delta h(r); g_{i-1}^r)) \tag{4.1.11}$$

$$= \tilde{T}(\Delta h(1); y_i') + \sum_{r=1}^{i-1} T(\Delta h(r+1), \Delta h(r); y_{i-r}') \tag{4.1.12}$$

for $m < i \leqq N$

$$y_i = y_i' + \sum_{r=1}^{m} (T(\Delta h(r); g_i^r) - T(\Delta h(r); g_{i-1}^r)) \tag{4.1.13}$$

$$= \tilde{T}(\Delta h(1); g_i^1) - T(\Delta h(m); g_{i-1}^m) + \sum_{r=1}^{m-1} T(\Delta h(r+1), \Delta h(r); y_{i-r}') \tag{4.1.14}$$

It is clear from (4.1.11,13) that St. 1 is true. Now the next expressions can be obtained from (4.1.12,14)

$$(4.1.15)\quad y_i - y_{i-1} = \tilde{T}(\Delta h(1);g_i^1) - \tilde{T}(\Delta h(1);g_{i-1}^r)$$

$$+ \sum_{r=1}^{k_{i-1}-1} (T(\Delta h(r+1),\Delta h(r);y'_{i-r}) - T(\Delta h(r+1),\Delta h(r);y'_{i-r-1}))$$

$$+ \begin{cases} T(\Delta h(r+1),\Delta h(r);y'_{i-r}) & 2 \leqq i \leqq m \\ -T(\Delta h(m);g_m^m) & i = m+1 \\ T(\Delta h(m);g_{i-2}^m) - T(\Delta h(m);g_{i-1}^m) & m+1 < i \leqq N \end{cases}$$

Applying lemma 2 to the right hand of (4.1.15) yields $y_i - y_{i-1} \leqq 0$ for all i. Hence St. 2 becomes true. Furhtermore from (4.1.10,13) we obtain $y_i \geqq y'_i$, hence $y_i \nearrow$ in t. Thus it follows by induction that the two statements hold for all t, which produces

$$(4.1.16)\quad \begin{cases} g_i^r(\theta) \geqq 0,\ \nearrow \text{ in } r, \text{ and } \swarrow \text{ in } i \text{ for all } t \\ g_i^r(\theta) \nearrow \text{ in } t \end{cases}$$

and that for all t the FE and OP are provided by (4.1.3) and (2.6), respectively.

4.2 Purchasing problem

Now let N units of a certain kind of material have to be purchased within a given number of days so as to minimize the total expected purchase price. Per-unit prices of the material at subsequent days, θ, θ', θ'', ... are assumed to be positive and to be independent identically distributed random variables from a known distribution F with an expectation $E \geqq 0$. If r units are bought at the day of price θ, then the immediate total purchase price $h(r,\theta)$ is incurred, assumed that $h(0,\theta) = 0$ and $h(r,\theta) \nearrow$ in r. Let a discount factor $0 < \beta < 1$. We shall here discuss the next two cases: quantity-discount case, i.e., $\Delta h(r,\theta) = h(r,\theta) - h(r-1,\theta) \swarrow$ in r, and quantity-premium one, i.e., $\Delta h(r,\theta) \nearrow$ in r. It has been already revealed by Morris [13] that if

$h(r,\theta) = \theta$ for $r > 0$, optimal is a single procurement policy, that is, a policy of buying nothing or the entire quantity required if the purchase decision is made. The section will prove that the optimal purchasing policies are given in quantity-discount case by a single procurement policy but in quantity-premium one by a multiple procurement policy in which the amount to be purchased depends on the price quoted each day.

The above procurement problem has different versions and many related real-world topics of interest, which have been studied under the next different names : purchasing problem by Morris [13], hause selling problem by Simon [17], asset disposing problem by Karlin [10], divestiure problem by Hayes [4], persistence problem by MacQueen and Miller [11], stopping problem by Hockman [5], sampling design problem by Sakaguchi [16], and so forth. In [13] and [17], a differential calculus are used as an optimization thechnique. In other many literatures, a direct application of optimality principle in dynamic programming is emploied. In the section it will be appreciated that the application of theorem 1 brings about a very systematic as well as general approach to all of them.

Now in the problem defined above the state space is given by $I = \{0,1,\ldots,N\}$ where state i means the number of units having been already purchased, and the action space by $\mathbf{A}_i = \{A_i^r : r=0,1,\ldots,k_i\}$, $k_i = N-i$, in which A_i^r denotes the action of purchasing r units at the day when in state i. Then for an action A_i^r we have the immediate reward $x_i^r(\theta) = -h(r,\theta)$ and after-effect $z_i^r = v'_{i+r}$ where v_i represents -1 time the mimimum total expected purchase price starting from day t in state i. Since clearly $v'_N = 0$ for all t, (2.7) and (2.9) become, noticing remark 2 throughout the subsection, respectively, (4.2.1) $\qquad g_i^r(\theta) = y'_{i+r}/\Delta h(r,\theta)$

(4.2.2) $\qquad g_i(\theta) = -v'_i/h(N-i,\theta)$

Clearly the final values of $v_i(t)$ becomes from the assumption above,

(4.2.3) $\qquad v_i(0) = -E(h(N-i))$

hence $y_i(0) = E(\Delta h(N-i+1))$. Because in the problem $x_i^r(\theta) \not\swarrow$ in r for all θ, notice remark 2 throughout the section.

Case of quantity-discount

Here define the following statements for all i and all t:

St. 1 $\quad y_i \nearrow$ in i $\qquad$ St. 2 $\quad y_i \geqq 0$ (that is, $v_i \nearrow$ in i)

Clearly the two statements above hold for t = 0. Suppose that the two statements hold for any t − 1. Then since we have $g_i^r(\theta) \nearrow$ in r, the FE (2.8) becomes

(4.2.4) $\qquad v_i = -E(h(N-i)) + T(h(N-i);g_i) \qquad\qquad 0 \leqq i \leqq N$

(4.2.5) $\qquad\quad = v'_i - E(h(n-i)) + \tilde{T}(h(N-i);g_i),$

From (4.2.4) we obtain

(4.2.6) $\qquad \beta^{-1}y_i = E(\Delta h(N-i+1)) + T(h(N-i;g_i) - T(h(N-i+1);g_{i-1})$

(4.2.7) $\qquad\qquad \geqq E(\Delta h(N-i+1)(1 - I(1 > g_{i-1})) + y'_i I(1 > g_{i-1}))$

From (4.2.6), the next expression can be derived, letting $\Delta^2 h(r,\theta) = \Delta h(r,\theta) - \Delta h(r-1,\theta)$

(4.2.8) $\qquad \beta^{-1}(y_i - y_{i-1}) = -E(\Delta^2 h(N-i+2))$

$$+ T(h(N-i);g_i) - T(h(N-i+1);g_{i-1})$$

$$- T(h(N-i+1);g_{i-1}) + T(h(N-i+2);g_{i-2})$$

Now noticing the lemma 2(a), we obtain from (4.2.8)

$$(4.2.9)\quad \beta^{-1}(y_i - y_{i-1}) \geqq -E(\Delta^2 h(N-i+1))$$

$$- E(\Delta h(N-i+1)I(1 > g_{i-1}) + y_i I(1 > g_{i-1}))$$

$$+ E(\Delta h(N-i+2)I(1 > g_{i-1}) - y_{i-1}I(1 > g_{i-1}))$$

$$= E(\Delta^2 h(N-i+1)(I(1 > g_{i-1}) - 1)$$

$$+ (y_i' - y_{i-1}')I(1 > g_{i-1}))$$

Hence since $\Delta^2 h(N-i+1) \leqq 0$ from the assumption, we have $y_i - y_{i-1} \geqq 0$. Hence St. 1 becomes true. Thus it follows by induction that the two statements above hold for all t. Moreover we get $v_i \geqq v_i'$ from (4.2.5), that is, $v_i \nearrow$ in t. Therefore for all t we have the FE (4.2.4) and the OP (2.10) with k_i = N-i where

$$(4.2.10)\quad \begin{cases} g_i(\theta) \swarrow \text{ both in t and in i} \\ E(\Delta h(N-i))/h(N-i,\theta) \geqq 0 \end{cases}$$

notucing that $E(\Delta h(N-i)) \geqq -v_i \geqq 0$ from (4.2.4,5). That the OP is given by (2.10) means that a single procurement policy is optimal.

Case of quantity-premium

First define the following two statements for all i and all t:

St.1 $y_i \swarrow$ in i St. 2 $y_i \geqq 0$

Evidently the two statements hold for t = 0. Suppose the two statements are true for any t - 1. Then we have $g_i^r(\theta) \geqq 0$ and $\swarrow$ both in r and in i. Hence (2.5) becomes

$$(4.2.11)\quad v_i = -E(h(N-i)) + \sum_{r=1}^{N-i} T(\Delta h(r); g_i^r),$$

from which we obtain

$$(4.2.12)\quad y_i = E(\Delta h(N-i+1)) - T(h(N-i+1); g_{i-1}^{N-i+1})$$

$$+ \sum_{r=1}^{N-i} (T(\Delta h(r); g_i^r) - T(\Delta h(r); g_{i-1}^r))$$

$$(4.2.13) \qquad = E(\Delta h(N-i+1)) - T(\Delta h(1);g_{i-1}^{1}) + \sum_{r=1}^{N-i} T(\Delta h(r),\Delta h(r+1);y'_{i+r})$$

$$(4.2.14) \qquad = y'_i + E(\Delta h(N-i+1)) - T(\Delta h(1);g_{i-1}^{1}) + \sum_{r=1}^{N-i} T(\Delta h(r),\Delta h(r+1);y'_{i+r})$$

It is clear from (4.2.12) that St. 2 holds. From (4.2.13) we have

$$(4.2.15) \quad y_i - y_{i-1} = (T(\Delta h(1);g_{i-2}^{1}) - T(\Delta h(1);g_{i-1}^{1})) + (E(\Delta h(N-i+1) - \Delta h(N-i+2)) - T(\Delta h(N-i+1),\Delta h(N-i+2);y'_N)) + \sum_{r=1}^{N-i} (T(\Delta h(r),\Delta h(r+1);y'_{i+r}) - T(\Delta h(r),\Delta h(r+1);y'_{i+r-1}))$$

All of the three terms in the right-hand of (4.2.15) become $\leqq 0$, applying lemma 2(b1), (d3), and (d2), respectively. Hence St. 1 becomes true. Thus it follows by induction that the two statements above are true for all t. Consequently, for all t, the FE is given by (4.2.11) and the OP by (2.6), which means that a multiple purchasing policy is optimal for all t. Furhtermore we have $y_i \leqq y'_i$ from (4.2.14). Thus the following hold

$$(4.2.16) \begin{cases} g_i^r(\theta) \geqq 0, \swarrow \text{ both in } r \text{ and in } i \text{ for all } t \\ g_i^r(\theta) \swarrow \text{ in } t. \end{cases}$$

4.3 Inventory problem

As an example in application of an infinite time horizon process, the section will discuss the problem of purchasing a material cousumed at a constant rate of w units every day, where w is a fixed positive integer. We shall assume that (1) per-unit prices at successive days, θ, θ', θ'', ... , are positive random variables from a given distribution F with an expectation $E \geqq 0$, (2) the total purchase price is given as the function $h(r,\theta)$ if r units are bought at the day of price θ where $h(0,\theta) = 0$ and $\nearrow$ in r, (3) any shortage cannot be permitted, in other words, a shortage cost is infinite, (4) the amount placed an order is delivered instantaneously, (5) the capacity of storage is limited to N(integer $\geqq$ w) units, (6) a discount factor $\beta <$ 1. The objective is to minimize the total expected purchasing price over an infinite number of days.

Now the state of the purchasing process can be described by the number of units in stock at the morning of each day. Hence the state space is given by $I = \{0,1,\ldots,N\}$. Let A_i^r represent an action of buying r units when in state i. Clearly the domain of r must be $(w-i) \leqq r \leqq N-i$ from the assumptions (3) and (5) above. Then for an action A_i^r the immediate reward becomes $x_i^r(\theta) = -h(r,\theta)$ and the after-effect is given by $z_i^r = \beta v'_{i+r-w}$, where v_i $(= v_i(t))$ represents -1 time the minimum total expected purchasing price starting from time t in state i, given $v_i(0) = 0$ for all i. Then (2.7) and (2.9) become, respectively,

$$(4.3.1) \qquad g_i^r(\theta) = y'_{i+r-w}/\Delta h(r,\theta) \qquad (w - i)^+ < r \leqq N - i.$$

$$(4.3.2) \qquad g_i(\theta) = \bar{y}_i/h(N-i,\theta)$$

where $\bar{y}_i = \beta(v_{N-w} - v_{i-w})$. Throughout the section the readers should keep their mind on remark 2 because $x_i^r(\theta) \swarrow$ in r for all θ. Let $\Delta h(r,\theta) = h(r,\theta) - h(r-1,\theta)$.

<u>Case of quantity-discount, i.e., $\Delta h(r,\theta) \swarrow$ in r</u>

First define the next two statements for all i and all t:

St. 1 $\beta E(\Delta h(N-i+1)) \geqq y_i \geqq 0$ St. 2 $y_i \nearrow$ in i

Suppose that the both statements are true for any t - 1. Then we have $g_i^r(\theta) \nearrow$ in r. Hence (2.8) becomes

$$(4.3.3) \qquad v_i = -E(h(N-i)) + \beta v'_{N-w} + T(h(N-i);g_i)$$

from which we get, using lemma 2(a),

$$(4.3.4)\quad \beta^{-1}y_i = E(\Delta h(N-i+1)) + T(h(N-i);g_i) - T(h(N-i+1);g_{i-1})$$

$$\geqq E(\Delta h(N-i+1))(1 - I(1 > g_{i-1})) + y'_{i-w}I(1 > g_{i-1})) \geqq 0$$

Hence we have $y_i \geqq 0$. Furhtermore we can obtain through the similar way above

$$(4.3.5)\quad \beta^{-1}y_i \leqq E(\Delta h(N-i+1)) - (E(\Delta h(N-i+1)) - y'_{i-w})I(1 > g_i))$$

from which we have $\beta^{-1}y_i \leqq E(\Delta h(N-i+1))$. Consequently it follows that St. 1 is true. From (4.3.4) we can easily obtain, letting $\Delta^2 h(r,\theta) = h(r,\theta) - h(r-1,\theta)$ ($\leqq 0$),

$$(4.3.6)\quad \beta^{-1}(y_i - y_{i-1}) \geqq -E(\Delta^2 h(N-i+2))$$

$$+ T(h(N-i);g_i,g_{i-1}) - T(h(N-i+1);g_{i-1})$$

$$- T(h(N-i+1);g_{i-1}) + T(h(N-i+2);g_{i-2},g_{i-1})$$

$$= E(\Delta^2 h(N-i+2)(I(1 > g_{i-1}) - 1)$$

$$+ (y'_{i-w} - y'_{i-w-1})I(1 > g_{i-1})) \geqq 0$$

Hence we have $y_i - y_{i-1} \geqq 0$. Thus St. 2 becomes true. Consequently it follows by induction that Sts. 1 and 2 hold for all t, hence also for limit y_i. Then the FE (4.3.3) holds for all t as well as for limit v_i, and the limit OP is given by (2.10), where for limit $g_i(\theta)$ we have, noticing $\bar{y}_i = \sum_{s=i-w+1}^{N-w} y_s$

$$(4.3.7)\quad \beta E(h(N-i+w) - h(w-1))/h(N-i) \geqq g_i(\theta) \geqq 0$$

Furthermore since $\beta^{-1}y_i$ is bounded uniformly in β from St. 1, the minimum expected purchase price per day over an infinit time horizon without discount from corollary 5, noticing $v_N = \beta v_{N-i}$ for limits v_N and v_{N-i}

$$(4.3.8)\quad \tilde{g} = -g = y_N + y_{N-1} + \cdots + y_{N-w+1},$$

using limit y_i with $\beta = 1$.

Case of quantity-premium, i.e., $\Delta h(r,\theta) \nearrow$ in r

First define the two statements for all i and all t:

St. 1 $y_i \geqq 0$ St. 2 $y_i \swarrow$ in i

Suppose the both statements are true for any t - 1. Then since $g_i^r(\theta) \swarrow$ in r, (2.5) may be written, letting $J(i) = (w-i)^+ + 1$,

$$(4.3.10) \quad v_i = -E(h(N-i)) + \beta v'_{N-w} + \sum_{r=J(i)}^{N-i} T(\Delta h(r,\theta); g_i^r) \qquad 0 \leqq i \leqq N$$

from which we have

$$(4.3.11) \quad \beta^{-1} y_i = E(\Delta h(N-i+1)) + \sum_{r=J(i)}^{N-i} T(\Delta h(r), \Delta h(r+1); y'_{i+r-w}) + \begin{cases} 0 & \text{for } 1 \leqq i \leqq w \\ -T(\Delta h(1); g_{i-1}^1) & \text{for } w < i \leqq N \end{cases}$$

$$(4.3.12) \quad = E(\Delta h(N-i+1)) - T(\Delta h(N-i+1); g_{i-1}^{N-i+1}) + \sum_{r=J(i)}^{N-i} (T(\Delta h(r); g_i^r) - T(\Delta h(r); g_{i-1}^r)) + \begin{cases} T(\Delta h(N-i+1); g_i^{w-i+1}) & \text{for } 1 \leqq i \leqq w \\ 0 & \text{for } w < i \leqq N \end{cases}$$

From (4.3.11) we have St. 2. Furthermore from (4.3.11) we have, letting $K(i) = (w - i)^+ + 2$ for $2 \leqq i \leqq w$ and $K(i) = 1$ for $w < i \leqq N$,

$$(4.3.13) \quad \beta^{-1}(y_i - y_{i-1}) = E(\Delta h(N-i+1) - \Delta h(N-i+2)) - T(\Delta h(N-i+1), \Delta h(N-i+2); y'_{N-w}) + \sum_{r=K(i)}^{N-i} (T(\Delta h(r), \Delta h(r+1); y'_{i+r-w}) - T(\Delta h(r), \Delta h(r+1); y'_{i+r-w-1})) + \begin{cases} T(\Delta h(w-i+1), \Delta h(w-i+2); y'_1) & \text{for } 2 \leqq i \leqq w \\ -T(\Delta h(1); g_w^1) & \text{for } i = w+1 \\ T(\Delta h(1); g_{i-2}^1) - T(\Delta h(1); g_{i-1}^1) & \text{for } w+2 \leqq i \leqq N \end{cases}$$

All of the three terms of the right hand of (4.3.13) becomes $\leqq 0$ by applying lemma 2. Hence we have $y_i - y_{i-1} \leqq 0$, that is, St. 2

holds. Thus it follows by induction that Sts. 1 and 2 become true for all t. Furthermore we have $\beta(\Delta h(N-i+1)) \geqq y_i$ for all t from (4.3.11). Since the these properties of y_i hold also for limit y_i, we have for limit $g_i^r(\theta)$

(4.3.14) $\quad \beta E(\Delta h(N-i-r+w+1))/\Delta h(r,\theta) \geqq g_i^r(\theta) \geqq 0$

(4.3.15) $\quad g_i^r(\theta) \not\swarrow$ both in i and in r .

Then the FE (4.3.10) holds for limits v_i and the limit OP is given by (2.6). Now we have $E(\Delta h(N-i+1)) \geqq \beta^{-1} y_i \geqq 0$ for limit y_i, in other words, $\beta^{-1} y_i$ is bounded uniformly in β for all i. Therefore the minimum expected purchase price per day for an infinite time horizon process without discounting exists and is given by the same expressions as (4.3.8), because in this case we have in the quantity-premium also $v_N = \beta v_{N-w}$ from (4.3.10) for limits v_N and v_{N-w}.

4.4 Customer selection problem

This section will deal with the following discrete time bulk queuing system. Let time points be taken at an equally spaced interval and assume that every arriving customer has the same fixed service time ω(periods, time lengths between successive two time points), where ω is a positive integer. We shall refer to the total time periods required for completion of all customers now in the system as backlog. Assume that the backlog must be equal to or less than N(integer $\geqq$ 1) periods at any instant, that m customers arrive at the beginning of each period where m is a positive fixed integer, that each arriving customer has an amount of profit, and that a per-period discount factor for the profit, β, is less than 1, that is, $\beta < 1$. Now let the vector of profits of m customers arriving be denoted by $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$, and assume that if $r(\leqq m)$ customers are accepted out of the m ones, then the total amount of profit $h(r,\theta)$ can be gained where $h(0,\theta) = 0$, $h(r,\theta) \nearrow$ in r for all θ, and $\Delta h(r,\theta) = h(r,\theta) - h(r-1,\theta) \swarrow$ in r for all θ. Furhtermore assume that the profit vector θ is a random variable from a known distribution F. The decision maker must decide whether or not to accept each of them on arrival on the basis of its profit. The objective is here to maximize the total expected amount of profit over an infinit time horizon.

Now in the model a state of the system can be described by the amount of backlog at the beginning of each period, hence the state space becomes $I = \{0,1,\ldots,N\}$. The above definition of a state implies that accepting r customers in state i leads to the next state $[i+r\omega-1]^+$, $r = 0,1,\ldots$ where $[x] = \max\{0,x\}$. Since it must be $\leqq N$ from the assumption above, the maximum permissible number of customers to be accepted in state i is given by

$$(4.4.1) \qquad k_i = \min\{m,[(N-i+1)/\omega]\}$$

where [ ] represents Gauss' symbol. Let us denote an action of accepting r customers by A_i^r , $0 \leqq r \leqq k_i$. Then the immediate reward and after-effect for taking an action A_i^r is given by $x_i^r(\theta) = h(r,\theta)$ and $z_i^r = v'_{[i+r\omega-1]^+}$, respectively, where $v_i = (v_i(t))$ denote the maximum total expected amount of profit starting from time t with state i, given $v_i(0) = 0$ for all i. Now define

$$(4.4.2) \qquad y_i = \beta(v_i - v_{[i-w]^+}) \qquad 1 \leqq i \leqq N$$

$$(4.4.3) \qquad \Delta\tilde{y}_i = \tilde{y}_i - \tilde{y}_{i-1} \qquad 2 \leqq i \leqq N.$$

Then (2.7) becomes

$$(4.4.4) \qquad g_i^r(\theta) = -\tilde{y}'_{i+r\omega-1}/\Delta h(r,\theta) \qquad 1 \leqq r \leqq k_i$$

For convenience define $y_i^r = \tilde{y}'_{i+r\omega-1}$ for $1 \leqq r \leqq k_i$ and $y_i^r = \infty$ for $r > k_i$ and the set consisting of only the finite y_i^r's by $Y = \{ y_i^r : 1 \leqq r \leqq k_i,\ 0 \leqq i \leqq N\}$. Then on Y we have

(4.4.5) $\quad g_i^r(\theta) = -y_i^r/\Delta h(r,\theta)$

(4.4.6) $\quad y_i^r = y_{i-\omega}^{r+1} = y_{i+\omega}^{r-1},$

(4.4.7) $\quad y_i^r - y_{i-1}^r = \Delta\tilde{y}'_{i+r\omega-1}$, and

(4.4.8) $\quad y_i^r - y_i^{r-1} = \Delta\tilde{y}'_{i+r\omega-1} + \Delta\tilde{y}'_{i+r\omega-2} + \dots + \Delta\tilde{y}'_{i+r\omega-\omega}$

Now define the two statements for all i and all t:

St. 1 $\quad \tilde{y}_i \leqq 0 \qquad$ St. 2 $\quad \Delta\tilde{y}_i \leqq 0$ (that is, $\tilde{y}_i \swarrow$ in i)

Suppose that the two statements above are true for any t − 1. Then it follows that $g_i^r(\theta) \geqq 0$ and $\nearrow$ both in i and in r and that $y_i^r \leqq 0$ and $\swarrow$ both in i and in r. Hence (2.5) becomes

(4.4.9) $\quad v_i = \beta v'_{[i-1]^+} + \sum_{r=1}^{k_i} T(\Delta h(r);g_i^r) \qquad 0 \leqq i \leqq N$

From this we get, noticing remark 1,

(4.4.10) $\quad \beta^{-1}\tilde{y}_1 = \sum_{r=1}^{\infty}(T(\Delta h(r);g_1^r) - T(\Delta h(r);g_0^r))$

(4.4.11) $\quad \beta^{-1}\tilde{y}_i = \tilde{y}'_{i-1} + \sum_{r=1}^{\infty}(T(\Delta h(r);g_i^r) - T(\Delta h(r);g_0^r)) \quad 2 \leqq i \leqq \omega$

(4.4.12) $\quad \beta^{-1}\tilde{y}_i = \tilde{y}'_{i-1} + \sum_{r=1}^{\infty}(T(\Delta h(r);g_i^r) - T(\Delta h(r);g_{i-\omega}^r)) \quad \omega < i \leqq N$

From three expressions above we have

(4.4.13) $\quad \beta^{-1}\Delta\tilde{y}_2 = \tilde{y}'_1 + \sum_{r=1}^{\infty}(T(\Delta h(r);g_2^r) - T(\Delta h(r);g_1^r))$

(4.4.14) $\quad \beta^{-1}\Delta\tilde{y}_i = \Delta\tilde{y}'_{i-1} + \sum_{r=1}^{\infty}(T(\Delta h(r);g_i^r) - T(\Delta h(r);g_{i-1}^r)) \quad 3 < i \leqq \omega$

Noticing that since $1 \leqq k_{i-w}$ and $1 \leqq k_{i-w-1}$, both $y_{i-\omega}^1$ and $y_{i-\omega-1}^1$ are finite, that is, in Y, we have from (4.4.14)

$$(4.4.15)\quad \beta^{-1}\Delta\tilde{y}_i = (\tilde{T}(\Delta h(1);g^1_{i-\omega-1}) - \tilde{T}(\Delta h(1);g^1_{i-\omega}))^*$$

$$-\sum_{r=2}^{\infty}((T(\Delta h(r-1);\underbrace{g^{r-1}_{i-1}}_{A}) - T(\Delta h(r);\underbrace{g^{r}_{i-\omega-1}}_{B}))$$

$$-(T(\Delta h(r-1);\underbrace{g^{r-1}_{i}}_{C}) - T(\Delta h(r);\underbrace{g^{r}_{i-\omega}}_{D})))^{**}\quad \omega < i \leqq N$$

From (4.4.10,11,12) we have St. 1 for all i and from (4.4.13,14) St. 2 for $2 \leqq i \leqq \omega$. It is also clear that the term $(\ )^* \leqq 0$ in (4.4.15). Next we shall shown that the term $(\ )^{**} \geqq 0$ in (4.4.15).

First notice that

(I) $k_i \not\swarrow$ in i

(II) $k_{i-\omega}$ is greater by at most 1 than k_i

(III) $[x - 1]^+ + 1 > [x]$ is impossible for any real number x

Now arguments A, B, C, and D in the right hand of (4.4.15) are not always finite, in other words, not always in Y. For this reason a somewhat cumbersome treatment as stated below will be needed in inspecting non-negativity or non-positivity of the terms $(\ )^{**}$ of (4.4.15). Noticing (I) and (II) above we have the next relationships: (1) $r-1 > k_{i-1} \rightarrow r > k_{i-\omega-1}$, $r-1 > k_i$ and $r > k_{i-\omega}$, (2) $r > k_{i-\omega-1} \rightarrow r > k_{i-\omega}$, and (3) $r-1 > k_i \rightarrow r > k_{i-\omega}$. These imply, respectively,

(1') $A = \infty \rightarrow B = \infty$, $C = \infty$, $D = \infty$,

(2') $B = \infty \rightarrow D = \infty$,

(3') $C = \infty \rightarrow D = \infty$

Next let us show (4) the impossibility of the joint event " $r \leqq k_{i-\omega-1}$, $r-1 \leqq k_i$, $k_{i-\omega} < r$ ". If the event may occur, then we have $k_{i-\omega} < k_{i-\omega-1}$. This leads to $\min\{m,[A]\} < \min\{m,[A+1/\omega]\}$ where $A = (N-i+1+\omega)/\omega$. This inequality means that m must be at least greater that [A], that is, $m > [A]$. Then we have $k_{i-\omega} = [A]$ and $k_i = [A-1]$. Now since $k_{i-\omega} < k_i - 1$, it follows that $[A] < [A-1]+1$ which leads to a contradiction from above (III). Hence the above joint event, in other words,

(4') " $B < \infty$ and $C < \infty$ and $D = \infty$ " can not occur at all

It is easily seen from above (1') to (4') that the possible combinations as to finiteness(F) or infiniteness(∞) of each of A, B, C, and D are the following five:

Table I

	A	B	C	D
(a)	∞	∞	∞	∞
(b)	F	∞	∞	∞
(c)	F	∞	F	∞
(d)	F	F	∞	∞
(e)	F	F	F	F

For each of (a) to (e) in the above table, the term $(\ )^{**}$ of (4.3.15) becomes, respectively,

(a) 0 ,

(b) $T(\Delta h(r-1);g_{i-1}^{r-1}) \geqq 0$,

(c) $T(\Delta h(r-1);g_{i-1}^{r-1}) - T(\Delta h(r-1);g_i^{r-1}) \geqq 0$,

(d) $-T(\Delta h(r);\Delta h(r-1);y_{i-1}^{r-1}) \geqq 0$,

(e) $-T(\Delta h(r),\Delta h(r-1);-y_{i-1}^{r-1}) + T(\Delta h(r),\Delta h(r-1);-y_i^{r-1}) \geqq 0$

Consequently we have the term $(\ )^{**} \geqq 0$ of (4.4.15), hence St. 1 holds for $\omega < i \leqq N$. Thus by induction Sts. 1 and 2 become true for all t. Therefore it follows that both statements hold also for limit $\tilde{y}_i$. Hence we have for limit $g_i^r(\theta)$

(4.4.16) $\quad g_i^r(\theta) \geqq 0$ and $\nearrow$ both in i and in r

Thus FE (4.4.9) holds for all t as well as for limits v_i and the limit OP is given by (2.6). Now let $h_i = \sum_{r=1}^{k_i} E(h(r))$ ($\geqq 0$). Then from (4.4.10,11,12) we get, for limits $\tilde{y}_i$, $0 \geqq \tilde{y}_1 \geqq -\beta h_1 \geqq -h_1$ and $0 \geqq \tilde{y}_i \geqq \beta(\tilde{y}_{i-1} - h_i) \geqq \tilde{y}_{i-1} - h_i$ for $2 \leqq i \leqq N$. From these inequalities it follows that $0 \geqq \tilde{y}_i \geqq -\beta\sum_{r=1}^{N} h_r$ for $1 \leqq i \leqq N$. Hence $\beta^{-1}\tilde{y}_i$ becomes bounded uniformly in β for all i. Thus the maximum expected amount of profit gained per period for an infinite time horizon problem without discounting becomes from corollary 5, noticing $v_0 = \beta v_0 + \sum_{r=1}^{k_0} T(\Delta h(r); g_0^r)$ for limits v_i

(4.4.16) $$g = \sum_{r=1}^{k_0} T(\Delta h(r); g_0^r)$$

in which $g_0^r(\theta)$ represents a limit $g_0^r(\theta)$ with $\beta = 1$.

REFERENCES

(1) Bellman, R., Dynamic programming, Princeton university press, (1957)

(2) Fabian, T., Fisher. J.L., Sasieni, M.W. and Yardeni, A. Purchasing raw material on a fluctuating market, Operations Research, Vol.7, No.1, 107-122 (1959)

(3) Gilbert, J.P. and Mosteller, F., Recognizing the maximum of a sequence, J. Amer. Statist. Assoc. Vol.16, 35-73 (1966)

(4) Hayes, R.H., Optimal strategies for divestiture, Operations Research, Vol.17, No.2, 292-310 (1969)

(5) Hockman. E, An optimal stopping problem of a growing inventory, Management Science, Vol.19, No.11, 1289-1291 (1973)

(6) Howard, R.A., Dynamic programming and Markov processes, MIT press, (1960)

(7) Ikuta, S., A basic theory of optimal order-selection problem (Japanese), Journal of Japan industrial management association, Vol.46, 17-26 (1971)

(8) Ikuta, S., A method of optimal order-selection (Japanese), Dissertation, Keio University, (1975)

(9) Ikuta, S., A sequential selection process and its applications, Discussion paper series, No. 74(80-12), University of Tsukuba, Institute of Socio-Economic Planning (1980)

(10) Karlin, S., Stochastic models and optimal policy for selling an asset, Arrow, Karlin and Scarf (eds.) " Studies in applied probability and management science" Stanford university press, chapter 9, 148-158 (1962)

(11) MacQueen, J. and Miller, R.G. jr., Optimal persistence policies, Operations Research Vol.8, No.3, 362-380 (1960)

(12) Mastran, D.V. and Thomas, C.J., Decison rules for attacking targets of opportunity, Naval Research Logistic Quarterly, Vol.20, 661-672, (1973)

(13) Morris, W.T., Some analysis of purchasing policy, Management Science, Vol.5, NO.4, 443-452 (1959)

(14) Nishimura, S., Monotone optimal control of arrivals distinguished by reward and service time, Discussion paper series, No.66 (80-4), University of Tsukuba, Institute of Socio-Economic plannning (1980)

(15) Ross, S.M, Applied probability models with optimization applications, Holden-Day, San Francisco (1970)

(16) Sakaguchi, M., Dynamic programming of some sequential sampling design, Journal of Mathematical Analysis and Applications, Vol.2, 446-466, (1961)

(17) Simon, H.A., Models of man, John Wily & Sons, Inc., 1957

(18) Wagner, H.M., Principles of operations research with applications to managerial decisions, Prentice-Hall, (1975)