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An Optimal Hostage Rescue Problem
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by

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Abstract

We propose the following mathematical models for an optimal rescue problem concerning hostages. Suppose persons are taken hostage at a given point in time. We have to make a decision to attempt either rescue or no rescue. In this paper we mainly consider two different objectives: Minimization of the expected number of hostages being killed and maximization of the probability of no hostage being killed. Several properties of an optimal rescuing rule are revealed.

Keywords: Dynamic programming; Stochastic processes; Hostage rescue

1 Introduction(001.033)

Acts involving hostage taking occur for different reasons, e.g., social inequality, poverty, religious problems, racial problems, political problems, and so on. The problem has become an urgent issue to be tackled worldwide. Typical examples in recent years include:

1 (001.074)A 17-year-old youth wielding a knife hijacked a bus on the Sanyo Expressway in Japan and killed a 68-year-old hostage. After 15 hours, the police stormed the bus, the other hostages were rescued, and the hijacker was arrested (May 4, 2000).

2 (001.075)An armed man took a Finance Ministry official hostage in the Tokyo Stock Exchange building and demanded a meeting with the Finance Minister. He surrendered to the police after a tense, five and half hour standoff (January 12, 1998).

3 (001.076)Fourteen guerrillas stormed the home of the Japanese Ambassador to Peru and took about three hundred people hostage, including diplomats and government officials attending a party to celebrate the Emperor of Japan's birthday. All but one of the hostages were rescued while all the rebels were killed when special forces stormed the building (December 17, 1996).

4 (001.077)A man with a knife broke into a house and took a 2-year-old boy hostage. The police finally rushed into the house, set the uninjured boy free, and arrested the criminal (December 1, 1995).

Although not all the information is available for accurate statistics, it could be said that different hostage scenarios continue to occur all over the world. The most important decision for the person in charge of a crisis settlement is the timing to enact rescue of the hostages, especially after all possible negotiations have broken down. Wrestling with the problem, needless to say, involves many factors, political, economical, sociological, psychological, and so on, and all must be taken into acount, together with the safety of hostages, the demands of criminals, the repercussions of success or failure in a rescue attempt, and so on. The purpose of this paper is to propose two types of mathematical models of an optimal hostage rescue problem by using the concept of a sequential stochastic decision processes and examine the properties of optimal rescuing rules. Unfortunately, to assist in our problem we were unable to find any reference material based on a mathematical approach. Accordingly, we can not list any references to be directly cited except [1].

2 Models

Consider the following sequential stochastic decision process with a finite planning horizon. Here, for convenience, let points in time be numbered backward from the final point in time of the planning horizon, time 0, as 0, 1, $\cdots$, and so on. Let the time interval between two successive points, say times t and $t-1$, be called the period t. Here, assume that time 0 is the deadline at which a rescue attempt is considered as the only course of action for some reason, say, the hostage's health condition, the degree of criminal desperation, and so on.

Suppose $i \geq 1$ persons are taken as hostages at a given point in time t, and we have to make a decision on attempting either rescue or no rescue. Let x denote a decision variable of a certain point in time t where $x = 0$ if no rescue is attempted and $x = 1$ if a rescue is attempted, and X_t denote the set of possible decisions of time t, i.e., $X_t = \{0, 1\}$ for $t \geq 1$ and $X_0 = \{1\}$.

Let p $(0 < p < 1)$ be the probability of a hostage being killed if $x = 1$, and let s $(0 \leq s < 1)$ be the probability of criminal(s) surrendering up to the next point in time if $x = 0$, hence $1-s$ is the probability of criminal(s) not surrendering. Further, let q and r $(0 < q < 1,\ 0 \leq r < 1$, and $0 < q + r < 1)$ be the probabilities of a hostage being, respectively, killed and set free up to the next point in time if $x = 0$ and criminal(s) not surrendering; accordingly, $1 - q - r$ is the probability of the hostage being neither killed nor set free. Here, the case of $p = 0$, $p = 1$, $s = 1$, $q = 0$, $q = 1$, $r = 1$, and $q + r = 1$ makes the problem trivial; accordingly, these are all excluded in the definition of the model.

In this paper we mainly consider the following two different objective functions:

1. The expected number of hostages killed, which is to be minimized.
2. The probability of no hostage being killed, which is to be maximized.

For convenience, let us call the model with the former objective function the *expectation model*, and the one with the later objective function the *probability model.*

Now, in the model defined above, the p, s, q and r are all implicitly assumed to be *deterministic.* In Section 7 we also consider a *stochastic* case where they are all random variables. However, we only examine the case of $i = 1$ because the mathematical treatment for the general case of $i \geq 1$ is expected to be quite intractable. It goes without saying that the expectation model and probability model also can be defined for the stochastic case.

3 Structure of Decisions

For simplicity, by A and W let us denote the decisions of, respectively, "attempting a rescue" and "waiting up to the next point in time, i.e., not attempting a rescue". If the decisions are optimal at a given point in time t, let us employ the symbols $\mathtt{A}_t$ and $\mathtt{W}_t$, and if $\mathtt{A}_t$ and $\mathtt{W}_t$ are indifferent, let us use the symbol $\mathtt{A}_t \sim \mathtt{W}_t$. Further, if the decisions are optimal for all times $t \geq 1$, we use $\mathtt{A}_{t\geq 1}$ and $\mathtt{W}_{t\geq 1}$. In our paper the decision rule for both models can be depicted as in Figure 1.

Here, for convenience in later discussions, let us in advance make an examination of the optmal decision rule for the simplest case of $i = 1$ and $t = 1$, i.e., only the hostage remains at time 1.

1 Suppose a rescue attempt is made. Then, the *expected* number of hostages being killed is $p \times 1 + (1 - p) \times 0 = p$ and the *probability* of hostages being killed is also p by definition.

2 Suppose a rescue attempt is not made.

  i Assume that the hostage taker(s) does not surender up to time 0 with probability $1-s$. Then, if the hostage is killed with probability q, the number of hostages being killed is 1 and the probability of

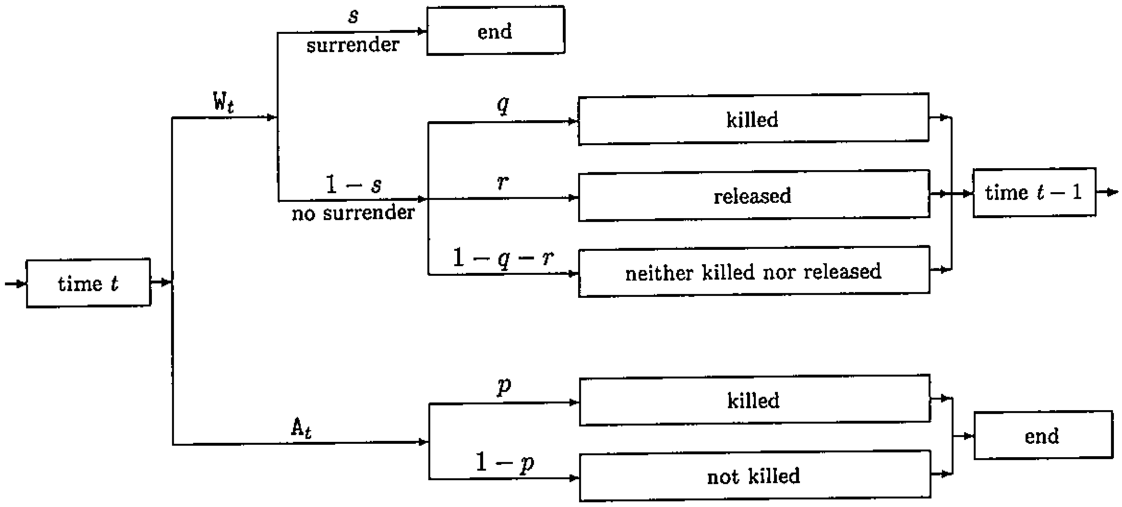

Figure 1: Decision Tree

hostages being killed is also 1. If the hostage is released with probability r, the number of hostages being killed is 0 and the probability of hostages being killed is also 0, and if the hostage is neither killed nor released with probability $1 - q - r$, with a rescue attempt to be necessarily made at the next time 0, the expected number of hostages being killed is p and the probability of hostages being killed is also p at time 0. Accordingly, the *expected* number of hostages being killed and the *probability* of hostages being killed are both given by

$$z = q \times 1 + r \times 0 + (1 - q - r)p = q + (1 - q - r)p, \tag{3.1}$$

where

$$0 < z < 1 \tag{3.2}$$

due to the assumptions of p, q and r.

ii Assume that the hostage taker(s) surenders up to time 0 with probability s Then, the number of hostages being killed is 0 and the probability of hostages being killed is 0.

Consequently, the *expected* number of hostages being killed and the *probability* of hostages being killed are both given by $(1 - s)z + s \times 0 = (1 - s)z$.

From the above we can eventually obtain the following lemma.

Lemma 3.1 *Suppose a hostage is taken at time* 1*, i.e.,* $i = 1$ *and* $t = 1$*. Then, the optimal decision rule at time* 1 *can be prescribed as follows.*

1 *If* $p < (1 - s)z$*, then* A_1.

2 *If* $p = (1 - s)z$*, then* $A_1 \sim W_1$.

3 *If* $p > (1 - s)z$*, then* W_1.

Now, first stating a conclusion, which is strictly verified in the subsequent sections, it follows that the above optimal decision rule at time 1 holds also for any time t. In other words, the optimal decision rule is t-independent. The practical implication of this fact, which must be said to possess a remarkably singular property, will be stated in Section 8.

4 Preliminaries

Let $f_p(m|i)$ be the probability of m hostages being killed among i hostages if a rescue attempt is made ($x = 1$), given by

$$f_p(m|i) = \binom{i}{m} p^m (1-p)^{i-m}, \quad i \geq 1, \quad 0 \leq m \leq i. \tag{4.1}$$

Further, let $f_{qr}(k, \ell|i)$ be the probability of k hostages being killed and ℓ hostages being set free among i hostages if a rescue attempt is not made ($x = 0$) and criminal(s) does not surrender up to the next point in time with probability $1 - s$, given by

$$f_{qr}(k, \ell|i) = \frac{i!}{k!\ell!(i-k-\ell)!} q^k r^\ell (1-q-r)^{i-k-\ell}, \quad i \geq 1, \quad 0 \leq k + \ell \leq i. \tag{4.2}$$

Then, we have

$$\sum_{k=0}^{i} \sum_{\ell=0}^{i-k} k f_{qr}(k, \ell|i) = iq, \tag{4.3}$$

$$\sum_{k=0}^{i} \sum_{\ell=0}^{i-k} (i-k-\ell) f_{qr}(k, \ell|i) = i(1-q-r). \tag{4.4}$$

See Appendix A for the Eqs. (4.3) and (4.4). The Lemma below will be used in the subsequent sections.

Lemma 4.1 *For $i \geq 1$ we have*

$$\lim_{i\to\infty} (1-p)^i = \lim_{i\to\infty} (1-z)^i = 0, \tag{4.5}$$

$$\lim_{i\to\infty} \sum_{\ell=0}^{i-1} f_{qr}(k, \ell|i) = 0. \tag{4.6}$$

Proof. See Appendix B and Appendix C. ∎

For convenience in later discussions let us define

$$\alpha = s + (1-s)r, \quad \beta = (1-s)(1-q-r), \quad \lambda = 1 - q - r \tag{4.7}$$

where

$$0 \leq \alpha < 1, \quad 0 < \beta < 1, \quad 0 < \lambda < 1, \quad 0 < \alpha + \beta < 1 \tag{4.8}$$

due to the assumption of s, q and r. We use the following two inequalities in Sections 6.3.2 and 6.3.3.

$$0 < (r + q\lambda^t)/(q + r) \leq 1, \qquad t \geq 0, \tag{4.9}$$

$$0 < \left(r + (z-p)\lambda^t\right)/(q + r) < 1, \quad t \geq 0. \tag{4.10}$$

Eq. (4.9) is immediate. See Appendix D for Eq. (4.10).

5 Expectation Model

5.1 Optimal Equation

By $A(i)$ let us denote the expected number of hostages being killed when i hostages are taken, provided that a rescue attempt is made ($x = 1$) at any time t. Then, clearly

$$A(i) = \sum_{m=0}^{i} m f_p(m|i) = ip, \quad i \geq 1. \tag{5.1}$$

Now, let $v_t(i)$ be the minimum expected number of hostages being killed, starting from time $t \geq 0$ with i hostages, expressed as

$$v_0(i) = A(i), \qquad i \ge 1, \tag{5.2}$$
$$v_t(i) = \min\{A(i), W_t(i)\}, \quad i \ge 1, \quad t \ge 1, \tag{5.3}$$

where $W_t(i)$ is the minimum expected number of hostages being killed over the period from time t to 0 (the deadline) if no rescue attempt is made ($x = 0$). Noting $k + \ell \le i$, we can express $W_t(i)$ as

$$\begin{aligned} W_t(i) &= s \times 0 + (1-s) \sum_{k+\ell \le i} \big(k + v_{t-1}(i-k-\ell)\big) f_{qr}(k,\ell|i) \\ &= (1-s) \sum_{k=0}^{i} \sum_{\ell=0}^{i-k} \big(k + v_{t-1}(i-k-\ell)\big) f_{qr}(k,\ell|i), \quad i \ge 1, \quad t \ge 1. \end{aligned} \tag{5.4}$$

Here, noting Eqs. (5.1), (4.3), (4.4) and (3.1), we can rewrite $W_1(i)$ as follows.

$$\begin{aligned} W_1(i) &= (1-s) \sum_{k=0}^{i} \sum_{\ell=0}^{i-k} \big(k + (i-k-\ell)p\big) f_{qr}(k,\ell|i) \\ &= (1-s)\big(iq + i(1-q-r)p\big) = i(1-s)z, \quad i \ge 1. \end{aligned} \tag{5.5}$$

Now, let

$$V_t(i) = W_t(i) - A(i), \quad i \ge 1, \quad t \ge 1. \tag{5.6}$$

Using Eqs. (5.5) and (5.1), we have

$$V_1(i) = W_1(i) - A(i) = i\big((1-s)z - p\big), \quad i \ge 1. \tag{5.7}$$

Further, we can rewrite Eq. (5.3) as follows.

$$v_t(i) = \min\{0, V_t(i)\} + A(i), \quad i \ge 1, \quad t \ge 1. \tag{5.8}$$

Accordingly, using Eqs. (5.4), (5.8), (5.1), (4.3), (4.4), (3.1) and (5.7), we can rewrite Eq. (5.6) for $t \ge 2$ as follows.

$$\begin{aligned} V_t(i) &= (1-s) \sum_{k=0}^{i} \sum_{\ell=0}^{i-k} \Big(k + \min\{0, V_{t-1}(i-k-\ell)\} + (i-k-\ell)p\Big) f_{qr}(k,\ell|i) - ip \\ &= (1-s)\big(iq + i(1-q-r)p\big) + (1-s) \sum_{k=0}^{i} \sum_{\ell=0}^{i-k} \min\{0, V_{t-1}(i-k-\ell)\} f_{qr}(k,\ell|i) - ip \\ &= i\big((1-s)z - p\big) + (1-s) \sum_{k=0}^{i} \sum_{\ell=0}^{i-k} \min\{0, V_{t-1}(i-k-\ell)\} f_{qr}(k,\ell|i) \\ &= V_1(i) + (1-s) \sum_{k=0}^{i} \sum_{\ell=0}^{i-k} \min\{0, V_{t-1}(i-k-\ell)\} f_{qr}(k,\ell|i), \quad i \ge 1, \quad t \ge 2. \end{aligned} \tag{5.9}$$

5.2 Optimal Decision Rule

From Eq. (5.6) the optimal decision rule can be stated as follows:

(a) If $V_t(i) > 0$, attempting a rescue is optimal, i.e., $\mathtt{A}_t$.
(b) If $V_t(i) = 0$, attempting a rescue and waiting up to the next time are indifferent, i.e., $\mathtt{A}_t \sim \mathtt{W}_t$.
(c) If $V_t(i) < 0$, waiting up to the next time is optimal, i.e., $\mathtt{W}_t$.

5.3 Analysis

Lemma 5.1

(a) *$V_t(i)$ is nonincreasing in t for $i \ge 1$.*
(b) *If $V_t(i) < 0$ for $i \ge 1$ and $t \ge 1$, then $v_t(i)$ is nonincreasing in t for $i \ge 1$.*

Proof.

(a) Since $\min\{0, V_{t-1}(i-k-\ell)\} \le 0$ for $t \ge 2$, we have $V_2(i) \le V_1(i)$ for $i \ge 1$ from Eq. (5.9) with $t = 2$. Hence, the assertion can be immediately proven by induction starting with this.

(b) Immediate from Eq. (5.8) and (a). ∎

Lemma 5.2 *If $V_1(i) > (=) 0$ for $i \ge 1$, then $V_t(i) = V_1(i) > (=) 0$ for $i \ge 1$ and $t \ge 1$.*

Proof. The assertion is evident for $t = 1$. Suppose $V_{t-1}(i) = V_1(i) > (=) 0$ for $i \ge 1$. Then, since $\min\{0, V_{t-1}(i-k-\ell)\} = 0$ for $i > k+\ell$, we immediately get $V_t(i) = V_1(i) > (=) 0$ for $i \ge 1$ from Eq. (5.9). This completes the induction. ∎

Lemma 5.3 *For $i \ge 1$ and $t \ge 1$ we have*

(a) *If $p < (1-s)z$, then $V_t(i) > 0$, hence $\mathtt{A}_{t\ge1}$.*

(b) *If $p = (1-s)z$, then $V_t(i) = 0$, hence $\mathtt{A}_{t\ge1} \sim \mathtt{W}_{t\ge1}$.*

(c) *If $p > (1-s)z$, then $V_t(i) < 0$, hence $\mathtt{W}_{t\ge1}$.*

Proof.

(a) Let $p < (1-s)z$. Then, from Eq. (5.7) we have $V_1(i) > 0$ for $i \ge 1$, hence from Lemma 5.2 we have $V_t(i) = V_1(i) > 0$ for $i \ge 1$ and $t \ge 1$, thus $\mathtt{A}_{t\ge1}$ due to (a) in Section 5.2.

(b) Let $p = (1-s)z$. Then, from Eq. (5.7) we have $V_1(i) = 0$ for $i \ge 1$, hence from Lemma 5.2 we have $V_t(i) = V_1(i) = 0$ for $i \ge 1$ and $t \ge 1$, thus $\mathtt{A}_{t\ge1} \sim \mathtt{W}_{t\ge1}$ due to (b) in Section 5.2.

(c) Let $p > (1-s)z$. Then, from (a) of Lemma 5.1 and Eq. (5.7) we have $V_t(i) \le V_1(i) < 0$ for $i \ge 1$ and $t \ge 1$, thus $\mathtt{W}_{t\ge1}$ due to (c) in Section 5.2. ∎

Lemma 5.4 *For $i \ge 1$ and $t \ge 1$ we have*

(a) *If $\mathtt{A}_{t\ge1}$, then $v_t(i) = ip$.*

(b) *If $\mathtt{A}_{t\ge1} \sim \mathtt{W}_{t\ge1}$, then $v_t(i) = ip$.*

(c) *If $\mathtt{W}_{t\ge1}$, then $v_t(i)$ is nonincreasing in t and*

$$v_t(i) = i\big(p + \big((1-s)z - p\big)(1-\beta)^{-1}(1-\beta^t)\big), \tag{5.10}$$

which is linear in i.

(d) *$v_t(i)$ converges to*

$$v(i) = i\big(p + \big((1-s)z - p\big)(1-\beta)^{-1}\big) \tag{5.11}$$

as $t \to \infty$, which converges to ∞ as $i \to \infty$.

Proof.

(a) Let $\mathtt{A}_{t\ge1}$, i.e., $V_t(i) > 0$ for $i \ge 1$ and $t \ge 1$ due to (a) in Section 5.2. Then, from Eq. (5.8) we have $v_t(i) = A(i) = ip$ for $i \ge 1$ and $t \ge 1$.

(b) Almost the same as the proof of (a).

(c) Let $\mathtt{W}_{t\ge1}$, i.e., $V_t(i) < 0$ for $i \ge 1$ and $t \ge 1$ due to (c) in Section 5.2. Then, from (b) of Lemma 5.1 we have $v_t(i)$ is nonincreasing t for $i \ge 1$, and from Eq. (5.8) we have $v_t(i) = V_t(i) + A(i) = W_t(i)$ for $i \ge 1$ and $t \ge 1$. Now, from this and Eq. (5.5) we have $v_1(i) = W_1(i) = i(1-s)z$, which is identical with Eq. (5.10) for $t = 1$. Hence, the assertion is true for $t = 1$. Suppose

$$v_{t-1}(i) = i\big(p + \big((1-s)z - p\big)(1-\beta)^{-1}(1-\beta^{t-1})\big), \quad i \ge 1. \tag{5.12}$$

Then, using Eqs. (5.4) and (5.12), we get

$$\begin{aligned}
v_t(i) &= W_t(i) = (1-s)\sum_{k=0}^{i}\sum_{\ell=0}^{i-k}(k+v_{t-1}(i-k-\ell))f_{qr}(k,\ell|i) \\
&= (1-s)\sum_{k=0}^{i}\sum_{\ell=0}^{i-k}\Big(k+(i-k-\ell)p+(i-k-\ell)\big((1-s)z-p\big)(1-\beta)^{-1}(1-\beta^{t-1})\Big)f_{qr}(k,\ell|i) \\
&= i(p+\big((1-s)z-p\big)(1-\beta)^{-1}(1-\beta^t)) \quad \text{(See Appendix E).} \qquad (5.13)
\end{aligned}$$

This completes the induction. Further, from this we get $v_t(i)$ is linear in i for $t \geq 1$.

(d) The assertion holds from Eqs. (5.10) and (4.8). ∎

6 Probability Model

6.1 Optimal Equation

By $A(i)$ let us denote the probability of no hostage being killed if a rescue attempt is made ($x = 1$) at any time t. Then

$$A(i) = f_p(0|i) = (1-p)^i, \quad i \geq 1. \qquad (6.1)$$

Now, let $v_t(i)$ be the maximum probability of no hostage being killed, starting from time $t \geq 0$ with i hostages, expressed as

$$v_0(i) = A(i), \qquad i \geq 1, \qquad (6.2)$$

$$v_t(i) = \max\{A(i), W_t(i)\}, \quad i \geq 1, \quad t \geq 1, \qquad (6.3)$$

where $W_t(i)$ is the maximun probability of no hostage being killed over the period from time t to 0 (the deadline) if no rescue attempt is made ($x = 0$). Noting $k + \ell \leq i$, we can express $W_t(i)$ as

$$W_t(i) = s \times 1 + (1-s)\Big(f_{qr}(0,i|i) \times 1 + \sum_{\ell \leq i-1} f_{qr}(0,\ell|i)v_{t-1}(i-\ell)\Big), \quad i \geq 1, \quad t \geq 1, \qquad (6.4)$$

the right hand side of which implies the following: At any given point in time t,

1 Suppose the hostage taker(s) surrenders with probability s. Then, the hostages are released; in other words, no hostage is killed; accordingly, the probability of no hostage being killed is equal to 1.

2 Suppose the hostage taker(s) does not surrender with probability $1 - s$. Then, the probability of no hostage being killed is $f_{qr}(0, \ell|i)$ if ℓ hostages are released.

  i If all the hostages, i.e., i hostages are released ($\ell = i$), the probability of no hostage being killed is equal to 1.

  ii If $i - 1$ or less hostages are released ($\ell \leq i - 1$), the number of remaining hostages becomes $i - \ell$ at time $t - 1$, implying that the probability of no hostage being killed over the period from time $t - 1$ to 0 is equal to $v_{t-1}(i - \ell)$ by definition.

Further, Eq. (6.4) can be easily rewritten as follows.

$$W_t(i) = s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)v_{t-1}(i-\ell)\Big), \quad i \geq 1, \quad t \geq 1, \qquad (6.5)$$

where

$$\begin{aligned}
W_1(i) &= s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)(1-p)^i\Big) \qquad (6.6) \\
&= s + (1-s)(1-z)^i \quad \text{(See Appendix F)}, \quad i \geq 1. \qquad (6.7)
\end{aligned}$$

Now, define

$$V_t(i) = W_t(i) - A(i), \quad i \geq 1, \quad t \geq 1 \qquad (6.8)$$

where $-1 \le V_t(i) \le 1$ for i and t. Using Eqs. (6.7) and (6.1), we can rewrite $V_1(i)$ and $V_1(1)$ as follows, respectively,

$$V_1(i) = = s + (1-s)(1-z)^i - (1-p)^i, \quad i \ge 1, \tag{6.9}$$

$$V_1(1) = p - (1-s)z. \tag{6.10}$$

Let us rewrite Eq. (6.3) as follows.

$$v_t(i) = \max\{0, V_t(i)\} + A(i), \quad i \ge 1, \quad t \ge 1. \tag{6.11}$$

Accordingly, using Eqs. (6.5), (6.11), (6.1) and (6.9), we can rewrite Eq. (6.8) for $t \ge 2$ as follows.

$$\begin{aligned}
V_t(i) &= s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(\max\{0, V_{t-1}(i-\ell)\} + (1-p)^{i-\ell}\Big)\Big) - (1-p)^i \\
&= s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)(1-p)^{i-\ell}\Big) - (1-p)^i + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\} \\
&= s + (1-s)(1-z)^i - (1-p)^i + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\} \quad \text{(See Appendix F)} \\
&= V_1(i) + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\}, \quad i \ge 1, \quad t \ge 2.
\end{aligned} \tag{6.12}$$

6.2 Optimal Decision Rule

From Eq. (6.8) the optimal decision rule can be stated as follows:

(a) If $V_t(i) < 0$, attempting a rescue is optimal, i.e., $\mathtt{A}_t$.
(b) If $V_t(i) = 0$, attempting a rescue and waiting up to the next time are indifferent, i.e., $\mathtt{A}_t \sim \mathtt{W}_t$.
(c) If $V_t(i) > 0$, waiting up to the next time is optimal, i.e., $\mathtt{W}_t$.

6.3 Analysis

6.3.1 Properties of $V_t(i)$

Lemma 6.1 *For any given $i_2 > i_1 \ge 1$ we have*

(a) *If $p \ge z$, the following two inequalities can not coincide.*

$$V_1(i_1) > V_1(i_1+1), \tag{6.13}$$

$$V_1(i_2) < V_1(i_2+1). \tag{6.14}$$

(b) *If $p \le z$, the following two inequalities can not coincide.*

$$V_1(i_1) < V_1(i_1+1), \tag{6.15}$$

$$V_1(i_2) > V_1(i_2+1). \tag{6.16}$$

Proof. For convenience let, for $i \ge 1$,

$$g(i) = V_1(i+1) - V_1(i). \tag{6.17}$$

Then, from Eq. (6.9) we have

$$\begin{aligned}
g(i) &= s + (1-s)(1-z)^{i+1} - (1-p)^{i+1} - s - (1-s)(1-z)^i + (1-p)^i \\
&= (1-s)(1-z)^i(1-z-1) - (1-p)^i(1-p-1) \\
&= p(1-p)^i - (1-s)z(1-z)^i.
\end{aligned} \tag{6.18}$$

(a) From Eq. (6.17) we have $V_1(i+1) = V_1(i) + g(i)$. If Eqs. (6.13) and (6.14) are both satisfied, then

$$V_1(i_1) > V_1(i_1) + g(i_1), \quad V_1(i_2) < V_1(i_2) + g(i_2),$$

from which $g(i_1) < 0$ and $g(i_2) > 0$; equivalently,

$$p(1-p)^{i_1} < (1-s)z(1-z)^{i_1}, \quad p(1-p)^{i_2} > (1-s)z(1-z)^{i_2}.$$

Consquently, we obtain

$$(1-s)z\Big(\frac{1-z}{1-p}\Big)^{i_2} < p < (1-s)z\Big(\frac{1-z}{1-p}\Big)^{i_1}.$$

Thus, we get

$$\Big(\frac{1-z}{1-p}\Big)^{i_2} < \Big(\frac{1-z}{1-p}\Big)^{i_1}.$$

Since $i_2 > i_1 \geq 1$ by assumption, it must be that $(1-z)/(1-p) < 1$, i.e., $p < z$, which is a contradiction to the assumption, i.e., $p \geq z$. Accoordingly, the assertion holds.

(b) Almost the same as the proof of (a). ∎

From Lemma 6.1 we immediately get the following corollary.

Corollary 6.1 *For any given $i_2 > i_1 \geq 1$ we have*

(a) *If $p \geq z$, the following two inequalities can not coincide.*

$$V_1(i_1) > 0 \geq V_1(i_1+1), \tag{6.19}$$

$$V_1(i_2) \leq 0 < V_1(i_2+1). \tag{6.20}$$

(b) *If $p \leq z$, the following two inequalities can not coincide.*

$$V_1(i_1) < 0 \leq V_1(i_1+1), \tag{6.21}$$

$$V_1(i_2) \geq 0 > V_1(i_2+1). \tag{6.22}$$

Lemma 6.2 *For any given $i' \geq 1$, if $V_1(i) < (=) 0$ for $1 \leq i \leq i'$, then $V_t(i) = V_1(i) < (=) 0$ for $1 \leq i \leq i'$ and $t \geq 1$.*

Proof. The assertion is evident for $t = 1$. Suppose $V_{t-1}(i) = V_1(i) < (=) 0$ for $1 \leq i \leq i'$. Then, since $\max\{0, V_{t-1}(i-\ell)\} = 0$ for $1 \leq i-\ell \leq i'$, we get $V_t(i) = V_1(i) < (=) 0$ from Eq. (6.12) for $1 \leq i \leq i'$ and $t \geq 1$. This completes the induction. ∎

Lemma 6.3

(a) *$V_t(i)$ is nondecreasing in t for $i \geq 1$.*

(b) *If $V_t(i) > 0$ for $i \geq 1$ and $t \geq 1$, then $v_t(i)$ is nondecreasing in t for $i \geq 1$.*

Proof.

(a) Since $\max\{0, V_{t-1}(i-\ell)\} \geq 0$ for $t \geq 2$, we have $V_2(i) \geq V_1(i)$ for $i \geq 1$ from Eq. (6.12) with $t = 2$. The assertion can be immediately proven by induction starting with this.

(b) Immediate from Eq. (6.11) and (a). ∎

Lemma 6.4 $\lim_{i\to\infty} V_t(i) = s$ *for $t \geq 1$.*

Proof. Let

$$B_t(i) = \sum_{\ell=0}^{i-1} f_{qr}(0, \ell | i) \max\{0, V_{t-1}(i-\ell)\}, \quad i \geq 1, \quad t \geq 2.$$

Then, Eq. (6.12) can be rewritten as $V_t(i) = V_1(i) + (1-s)B_t(i)$ for $i \geq 1$ and $t \geq 2$. Now, since clearly $|\max\{0, V_{t-1}(i-\ell)\}| \leq 1$ for $i - \ell \geq 1$ and $t \geq 2$, we obtain

$$0 \le |B_t(i)| \le \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)|\max\{0, V_{t-1}(i-\ell)\}| \le \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i).$$

Since $\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)$ converges to 0 as $i \to \infty$ from Eq. (4.6), it follows that $\lim_{i\to\infty} |B_t(i)| = 0$, hence $\lim_{i\to\infty} B_t(i) = 0$ for $t \ge 2$. From Eqs. (6.9) and (4.5) we immediately obtain $\lim_{i\to\infty} V_1(i) = s$. Accordingly, it follows that $\lim_{i\to\infty} V_t(i) = \lim_{i\to\infty} V_1(i) + (1-s)\lim_{i\to\infty} B_t(i) = s$ for $t \ge 1$. ∎

6.3.2 Case of $s = 0$

In hostage events perpetrated by a person who is determined to go through with it no matter what, and not surrender on any terms, he knows that, if arrested, he will be condemned to death or life imprisonment. This can be regarded as the case of $s = 0$.

Lemma 6.5 *Let $s = 0$. For $i \ge 1$ and $t \ge 1$ we have*
(a) *If $p < (1-s)z$, i.e., $p < z$, then $V_t(i) < 0$, hence $\mathtt{A}_{t\ge1}$.*
(b) *If $p = (1-s)z$, i.e., $p = z$, then $V_t(i) = 0$, hence $\mathtt{A}_{t\ge1} \sim \mathtt{W}_{t\ge1}$.*
(c) *If $p > (1-s)z$, i.e., $p > z$, then $V_t(i) > 0$, hence $\mathtt{W}_{t\ge1}$.*

Proof. Let $s = 0$. Then Eq. (6.9) becomes

$$V_1(i) = (1-z)^i - (1-p)^i = (p-z)\sum_{h=0}^{i-1}(1-z)^{i-1-h}(1-p)^h. \tag{6.23}$$

(a) If $p < (1-s)z$, i.e., $p < z$, from Eq. (6.23) we have $V_1(i) < 0$ for $i \ge 1$, hence $V_t(i) < 0$ for $i \ge 1$ and $t \ge 1$ due to Lemma 6.2, thus $\mathtt{A}_{t\ge1}$ due to (a) in Section 6.2.
(b) If $p = (1-s)z$, i.e., $p = z$, from Eq. (6.23) we have $V_1(i) = 0$ for $i \ge 1$, hence $V_t(i) = 0$ for $i \ge 1$ and $t \ge 1$ due to Lemma 6.2, thus $\mathtt{A}_{t\ge1} \sim \mathtt{W}_{t\ge1}$ due to (b) in Section 6.2.
(c) If $p > z$, from Eq. (6.23) we have $V_1(i) > 0$ for $i \ge 1$, hence $V_t(i) > 0$ for $i \ge 1$ and $t \ge 1$ from (a) of Lemma 6.3, thus $\mathtt{W}_{t\ge1}$ due to (c) in Section 6.2. ∎

Lemma 6.6 *Let $s = 0$. For $i \ge 1$ and $t \ge 1$ we have*
(a) *If $\mathtt{A}_{t\ge1}$, then $v_t(i) = (1-p)^i$.*
(b) *If $\mathtt{A}_{t\ge1} \sim \mathtt{W}_{t\ge1}$, then $v_t(i) = (1-p)^i$.*
(c) *If $\mathtt{W}_{t\ge1}$, then $v_t(i)$ is nondecreasing t and*

$$v_t(i) = \left(\frac{r + (z-p)\lambda^t}{q+r}\right)^i, \tag{6.24}$$

which is strictly decreasing in i.
(d) *$v_t(i)$ converges to*

$$v(i) = \big(r/(q+r)\big)^i \tag{6.25}$$

as $t \to \infty$, which converges to 0 as $i \to \infty$.

Proof.
(a) Let $\mathtt{A}_{t\ge1}$, i.e., $V_t(i) < 0$ for $i \ge 1$ and $t \ge 1$ due to (a) in Section 6.2. Then, from Eq. (6.11) we have $v_t(i) = A(i) = (1-p)^i$ for $i \ge 1$ and $t \ge 1$.
(b) Almost the same as the proof of (a).
(c) Let $\mathtt{W}_{t\ge1}$, i.e., $V_t(i) > 0$ for $i \ge 1$ and $t \ge 1$ due to (c) in Section 6.2. Then, from (b) of Lemma 6.3 we have $v_t(i)$ is nondecreasing in t for $i \ge 1$, and from Eq. (6.11) we get $v_t(i) = V_t(i) + A(i) = W_t(i)$ for $i \ge 1$ and $t \ge 1$. Now, from this and Eq. (6.7) with $s = 0$ we have $v_1(i) = W_1(i) = (1-z)^i$ for $i \ge 1$, which is identical with Eq. (6.24) for $t = 1$. Hence, the assertion is true for $t = 1$. Suppose the assertion

is true for $t-1$, i.e., $v_{t-1}(i) = \big(r + (z-p)\lambda^{t-1})/(q+r)\big)^i$ for $i \geq 1$. Then, using Eq. (6.5) with $s = 0$, we have

$$v_t(i) = W_t(i) = r^i + \sum_{\ell=0}^{i-1} \binom{i}{\ell} r^\ell \lambda^{i-\ell} \Big(\frac{r + (z-p)\lambda^{t-1}}{q+r}\Big)^{i-\ell} = \Big(r + \lambda \frac{r + (z-p)\lambda^{t-1}}{q+r}\Big)^i = \Big(\frac{r + (z-p)\lambda^t}{q+r}\Big)^i.$$

From this and Eq. (4.10) it immediately follows that $v_t(i)$ is strictly decreasing in i for $t \geq 1$.

(d) The assertion holds from Eqs. (6.24) and (4.7). ∎

6.3.3 Case of $s > 0$

By i_t let us define a i satisfying

$$V_t(i-1) < 0 \leq V_t(i), \quad i \geq 2, \quad t \geq 1, \tag{6.26}$$

if it exists.

Lemma 6.7 *Let $s > 0$. For $t \geq 1$ we have*

(a) *Suppose $p > (1-s)z$, then $V_t(i) > 0$ for $i \geq 1$, hence $\mathtt{W}_{t\geq 1}$ for $i \geq 1$.*

(b) *Suppose $p = (1-s)z$, then, $V_t(1) = 0$ and $V_t(i) > 0$ for $i \geq 2$, hence $\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$ for $i = 1$ and $\mathtt{W}_{t\geq 1}$ for $i \geq 2$.*

(c) *Suppose $p < (1-s)z$. Then,*

1. *If $V_1(i) < 0$ and $V_1(i+1) = 0$ for a certain $i \geq 1$, then $V_1(i+2) > 0$.*
2. *There exists a unique $i_t \geq 2$, which is independent of t, hence let $i^* = i_t \geq 2$, thus $V_t(i) < 0$ for $1 \leq i < i^*$, $V_t(i^*) \geq 0$ and $V_t(i) > 0$ for $i > i^*$; accordingly, $\mathtt{A}_{t\geq 1}$ for $1 \leq i < i^*$ and $\mathtt{W}_{t\geq 1}$ for $i \geq i^*$.*

Proof. Assume $s > 0$.

(a) Let $p > (1-s)z$. We have $V_1(1) > 0$ from Eq. (6.10). If $p < z$, then $1-p > 1-z$. Now, from Eqs. (6.17) and (6.18) we have

$$V_1(i+1) - V_1(i) = g(i) > (1-s)z(1-z)^i - (1-s)z(1-z)^i = 0$$

for $i \geq 1$, hence it follows that $V_1(i)$ is strictly increasing in i, thus $V_1(i) > V_1(1) > 0$ for $i \geq 1$. If $p \geq z$. Suppose Eq. (6.19) is satisfied. Then, Eq. (6.20) is not satisfied from (a) of Corollary 6.1. This implies that once $V_1(i)$ becomes less than or equal to 0 for a certain $i'' \geq 1$, it follows that $V_1(i) \leq 0$ for $i \geq i''$, hence the limit of $V_1(i)$ as $i \to \infty$ becomes less than or equal to 0, which is a contradiction due to Lemma 6.4, thus Eq. (6.19) does not occur; in other words, it follows that $V_1(i) > 0$ for $i \geq 1$. Accordingly, whether $p < z$ or $p \geq z$, we have $V_1(i) > 0$ for $i \geq 1$. From (a) of Lemma 6.3 we get $V_t(i) \geq V_1(i) > 0$ for $i \geq 1$ and $t \geq 1$, hence $\mathtt{W}_{t\geq 1}$ for $i \geq 1$ due to (c) in Section 6.2.

(b) Let $p = (1-s)z$. We have $V_1(1) = 0$ from Eq. (6.10), hence $V_t(1) = 0$ for $t \geq 1$ due to Lemma 6.2 with $i' = 1$, thus $\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$ for $i = 1$ due to (b) in Section 6.2. Now, since $z > 0$ and $s > 0$, clearly $z > (1-s)z = p$, hence $1-p > 1-z$. Then, from Eqs. (6.17) and (6.18) we get

$$V_1(i+1) - V_1(i) = g(i) = p(1-p)^i - p(1-z)^i = p\big((1-p)^i - (1-z)^i\big) > 0$$

for $i \geq 1$, hence it follows that $V_1(i)$ is strictly increasing in i, thus $V_1(i) > V_1(1) = 0$ for $i \geq 2$. Accordingly, from (a) of Lemma 6.3 we get $V_t(i) \geq V_1(i) > 0$ for $i \geq 2$ and $t \geq 1$, thus $\mathtt{W}_{t\geq 1}$ for $i \geq 2$ due to (c) in Section 6.2.

(c) Let $p < (1-s)z$. We have $V_1(1) < 0$ from Eq. (6.10). Now since $z > (1-s)z$ due to $z > 0$ and $s > 0$, we get $p < z$.

(c1) Let $V_1(i) < 0$ and $V_1(i+1) = 0$ for a certain $i \geq 1$. Then, from Eq. (6.17) we get $g(i) = -V_1(i) > 0$ and $V_1(i+2) = g(i+1)$. Accordingly, from $g(i) > 0$ and Eq. (6.18) we have $p(1-p)^i > (1-s)z(1-z)^i$. Hence, using $V_1(i+2) = g(i+1)$ and Eq. (6.18), we obtain

$$\begin{aligned} V_1(i+2) &= p(1-p)^{i+1} - (1-s)z(1-z)^{i+1} > (1-s)z(1-z)^i(1-p) - (1-s)z(1-z)^{i+1} \\ &= (1-s)z(1-z)^i(1-p-1+z) = (1-s)z(1-z)^i(z-p) > 0. \end{aligned}$$

(c2) Since $V_1(1) < 0$, we have $V_t(1) < 0$ for $t \geq 1$ due to Lemma 6.2 with $i' = 1$, and since $V_t(i) > 0$ for a sufficiently large i due to Lemma 6.4, it follows that there exists at least one $i_t \geq 2$ for $t \geq 1$. Now, for $t = 1$, let Eq. (6.21) be satisfied. Then Eq. (6.22) does not occur due to (b) of Lemma 6.1. Noting this and (c1), it must be that i_1 is unique, hence let $i^* = i_1$, implying that $V_1(i) < 0$ for $1 \leq i < i^*$, $V_1(i^*) \geq 0$ and $V_1(i) > 0$ for $i > i^*$ due to Eq. (6.26). Further, from $V_1(i) < 0$ for $1 \leq i < i^*$ we have $V_t(i) < 0$ for $1 \leq i < i^*$ and $t \geq 1$ due to Lemma 6.2 with $i' = i^* - 1$, and $V_t(i^*) \geq V_1(i^*) \geq 0$ and $V_t(i) \geq V_1(i) > 0$ for $i > i^*$ and $t \geq 1$ from (a) of Lemma 6.3. Accordingly, by the definition of i_t it follows that $i_t = i^*$, unique for $t \geq 1$ and independent of t. Hence $\mathtt{A}_{t\geq 1}$ for $1 \leq i < i^*$ due to (a) in Section 6.2 and $\mathtt{W}_{t\geq 1}$ for $i \geq i^*$ due to (c) in Section 6.2. ∎

Now, let us define

$$I = \frac{\log\big(sp/(1-s)(z-p)\big)}{\log(1-z)}, \tag{6.27}$$

if it exists.

Lemma 6.8 *Let $s > 0$ and $p < (1-s)z$.*
(a) *The i^* in* (c2) *of* Lemma 6.7 *is given by a i such that*

$$(1-p)\big(s + (1-s)(1-z)^{i-1}\big) < (1-p)^i \leq s + (1-s)(1-z)^i. \tag{6.28}$$

(b) *There exists $I > 0$, and $i^* > I + 1$.*

Proof.
(a) Since the i^* is t-independent from (c2) of Lemma 6.7, it is given by a i such that $V_1(i-1) < 0 \leq V_1(i)$ by the definition Eq. (6.26), which can be rearranged into Eq. (6.28) by using Eq. (6.9).
(b) Since $p < (1-s)z$, we have $z > p$ due to $s > 0$ and $z > 0$, hence I exixts from Eq. (6.27). Noting $(1-s)(z-p) - sp = (1-s)z - p > 0$, we get $\log\big(sp/(1-s)(z-p)\big) < 0$, hence $I > 0$. Further, from Eq. (6.28) we have $(1-p)\big(s + (1-s)(1-z)^{i^*-1}\big) < s + (1-s)(1-z)^{i^*}$, which can be easily rearranged into $i^* > I + 1$. ∎

Lemma 6.9 *Let $s > 0$. For $i \geq 1$ and $t \geq 1$ we have*
(a) *If $\mathtt{A}_{t\geq 1}$, then $v_t(i) = (1-p)^i$.*
(b) *If $\mathtt{A}_{t\geq 1} \sim \mathtt{W}_{t\geq 1}$, then $v_t(i) = (1-p)^i$.*
(c) *If $\mathtt{W}_{t\geq 1}$, then $v_t(i)$ is nondecreasing t and*

$$v_t(i) = s\sum_{\tau=0}^{t-1}(1-s)^\tau\Big(\frac{r + q\lambda^\tau}{q+r}\Big)^i + (1-s)^t\Big(\frac{r + (z-p)\lambda^t}{q+r}\Big)^i, \tag{6.29}$$

which is strictly decreasing in i.
(d) *$v_t(i)$ converges to*

$$v(i) = s(q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k q^{i-k}\Big(1 - (1-s)\lambda^{i-k}\Big)^{-1} \tag{6.30}$$

as $t \to \infty$, which converges to s as $i \to \infty$.

Proof.

(a) Let $A_{t\geq 1}$, i.e., $V_t(i) < 0$ for $i \geq 1$ and $t \geq 1$ due to (a) in Section 6.2. Then, from Eq. (6.11) we have $v_t(i) = A(i) = (1-p)^i$ for $i \geq 1$ and $t \geq 1$.

(b) Almost the same as the proof of (a).

(c) Let $W_{t\geq 1}$, i.e., $V_t(i) > 0$ for $i \geq 1$ and $t \geq 1$ due to (c) in Section 6.2. Then, from (b) of Lemma 6.3 we have $v_t(i)$ is nondecreasing in t for $i \geq 1$, and from Eq. (6.11) we get $v_t(i) = V_t(i) + A(i) = W_t(i)$ for $i \geq 1$ and $t \geq 1$. Now, from this and Eq. (6.7) we get $v_1(i) = W_1(i) = s + (1-s)(1-z)^i$ for $i \geq 1$, which is identical with Eq. (6.29) for $t = 1$. Hence, the assertion is true for $t = 1$. Suppose the assertion is true for $t-1$, i.e.,

$$v_{t-1} = s\sum_{\tau=0}^{t-2}(1-s)^\tau\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^i + (1-s)^{t-1}\Big(\frac{r+(z-p)\lambda^{t-1}}{q+r}\Big)^i, \quad i \geq 1. \tag{6.31}$$

Then, using Eqs. (6.5) and (6.31), for $i \geq 1$ and $t \geq 1$ we get

$$\begin{aligned} v_t(i) &= W_t(i) = s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)v_{t-1}(i-\ell)\Big) \\ &= s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(s\sum_{\tau=0}^{t-2}(1-s)^\tau\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^{i-\ell} + (1-s)^{t-1}\Big(\frac{r+(z-p)\lambda^{t-1}}{q+r}\Big)^{i-\ell}\Big)\Big) \\ &= s\sum_{\tau=0}^{t-1}(1-s)^\tau\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^i + (1-s)^t\Big(\frac{r+(z-p)\lambda^t}{q+r}\Big)^i \quad \text{(See Appendix G).} \end{aligned} \tag{6.32}$$

Form Eqs. (4.9) and (4.10) it follows that $v_t(i)$ is strictly decreasing in i for $t \geq 1$.

(d) Noting

$$\begin{aligned} \sum_{\tau=0}^{t-1}(1-s)^\tau\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^i &= (q+r)^{-i}\sum_{\tau=0}^{t-1}(1-s)^\tau\sum_{k=0}^{i}\binom{i}{k}r^k(q\lambda^\tau)^{i-k} \\ &= (q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^kq^{i-k}\sum_{\tau=0}^{t-1}((1-s)\lambda^{i-k})^\tau = (q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^kq^{i-k}\frac{1-((1-s)\lambda^{i-k})^t}{1-(1-s)\lambda^{i-k}}, \end{aligned}$$

we have Eq. (6.30) from Eq. (6.29). Further, we can rewrite Eq. (6.30) as follows.

$$v(i) = s + s(1-s)(q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k(q\lambda)^{i-k}\Big(1-(1-s)\lambda^{i-k}\Big)^{-1} \quad \text{(See Appendix H).} \tag{6.33}$$

Now, since $\big(1-(1-s)\lambda^{i-k}\big)^{-1} \leq s^{-1}$, we have

$$0 \leq (q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k(q\lambda)^{i-k}\Big(1-(1-s)\lambda^{i-k}\Big)^{-1} \leq s^{-1}\Big(\frac{r+q\lambda}{q+r}\Big)^i,$$

which converges to 0 as $i \to \infty$. Accordingly, $v(i)$ converges to s as $i \to \infty$ from Eq. (6.33). ∎

7 Case where p, s, q and r are Random Variables

7.1 Assumptions

In many real cases, perpetrator operate with confused motives, which causes the probabilities p, s, q and r to change randomly from one minute to the next. This consideration leads us to the case in which p, s, q and r are random variables with a distribution function. For convenice, let $\xi = (p, s, q, r)$ and $\xi' = (p', s', q', r')$. By $F(\xi)$ let us denote the distribution function of ξ. Define $\mu = E(p)$, the expectation of p. Unfortunately, a general analysis for case of $i \geq 1$ is very intractable, so that in our paper we devote ourselves solely to the simplest case of $i = 1$.

7.2 Optimal Equation of the Expectation Model

Let $\tilde{u}_t(\xi)$ be the minimum expectation of the hostage being killed, starting from time t with ξ. Then,

$$\begin{aligned}\tilde{u}_0(\xi) &= p \times 1 + (1-p) \times 0 = p, && (7.1)\\ \tilde{u}_t(\xi) &= \min\{p \times 1 + (1-p) \times 0, s \times 0 + (1-s)\big(q \times 1 + r \times 0 + (1-q-r)\int_{\xi'} \tilde{u}_{t-1}(\xi')dF(\xi')\big)\}\\ &= \min\{p, (1-s)\Big(q + (1-q-r)\int_{\xi'} \tilde{u}_{t-1}(\xi')dF(\xi')\Big)\}, \quad t \geq 1. && (7.2)\end{aligned}$$

Let

$$\tilde{\nu}_t = \int_{\xi} \tilde{u}_t(\xi)dF(\xi), \quad t \geq 0. \tag{7.3}$$

Then,

$$\tilde{\nu}_t = \int_{\xi} \min\{p, (1-s)\big(q + (1-q-r)\tilde{\nu}_{t-1}\big)\}dF(\xi), \quad t \geq 1, \quad \tilde{\nu}_0 = \mu. \tag{7.4}$$

Further, from Eq. (4.7) we can rewrite Eq. (7.4) as follows.

$$\tilde{\nu}_t = \int_{\xi} \min\{p, 1 - \alpha - \beta(1 - \tilde{\nu}_{t-1})\}dF(\xi), \quad t \geq 1, \quad \tilde{\nu}_0 = \mu. \tag{7.5}$$

7.3 Optimal Equation of the Probability Model

Let $u_t(\xi)$ be the maximum probability of the hostage not being killed, starting from time t with ξ. Then,

$$\begin{aligned}u_0(\xi) &= 1 - p, && (7.6)\\ u_t(\xi) &= \max\{1 - p, s + (1-s)\big(r + (1-q-r)\int_{\xi'} u_{t-1}(\xi')dF(\xi')\big)\}, \quad t \geq 1. && (7.7)\end{aligned}$$

Let

$$\nu_t = \int_{\xi} u_t(\xi)dF(\xi), \quad t \geq 0. \tag{7.8}$$

Then,

$$\nu_t = \int_{\xi} \max\{1 - p, s + (1-s)\big(r + (1-q-r)\nu_{t-1}\big)\}dF(\xi), \quad t \geq 1, \quad \nu_0 = 1 - \mu. \tag{7.9}$$

Now, noting Eq. (4.7), we can rewrite Eq. (7.9) as follows.

$$\nu_t = \int_{\xi} \max\{1 - p, \alpha + \beta\nu_{t-1}\}dF(\xi), \quad t \geq 1, \quad \nu_0 = 1 - \mu. \tag{7.10}$$

7.4 Equivalence of Both Models

From Eq. (7.5) we get

$$1 - \tilde{\nu}_t = \int_{\xi} \max\{1 - p, \alpha + \beta(1 - \tilde{\nu}_{t-1})\}dF(\xi), \quad t \geq 1, \quad 1 - \tilde{\nu}_0 = 1 - \mu. \tag{7.11}$$

Now, let $\nu_t = 1 - \tilde{\nu}_t$, implying the maximum expectation of hostages not being killed for one hostage. Then, we can rewrite Eq. (7.11) as follows.

$$\nu_t = \int_{\xi} \max\{1 - p, \alpha + \beta\nu_{t-1}\}dF(\xi), \quad t \geq 1, \quad \nu_0 = 1 - \mu, \tag{7.12}$$

which is identical with Eq. (7.10). Therefore, it follows that the optimal equation of both models can be said to be substantially equivalent. Accordingly, we only consider the probability model in the later analysis.

7.5 Analysis

Lemma 7.1 *ν_t is nondecreasing in t and converges to a finite ν as $t \to \infty$, which is provided by a unique solution of*

$$\nu = \int_\xi \max\{1-p, \alpha+\beta\nu\}dF(\xi) \tag{7.13}$$

where $0 < \nu < 1$.

Proof. The monotonicity of ν_t in t can be proved by induction starting from the fact that $\nu_1 \geq \int_\xi (1-p)dF(\xi) = 1-\mu = \nu_0$. Accordingly, it converges to a finite ν as $t \to \infty$ since ν_t is bounded in t. Let $H(\nu_{t-1}) = \int_\xi \max\{1-p, \alpha+\beta\nu_{t-1}\}dF(\xi)$, hence $\nu_t = H(\nu_{t-1})$ from Eq. (7.10). Then, noting a general formula $|\max_i a_i - \max_i b_i| \leq \max_i |a_i - b_i|$, we obtain

$$|H(\nu_{t-1}) - H(\nu'_{t-1})| \leq |\nu_{t-1} - \nu'_{t-1}| \int_\xi \beta dF(\xi)$$

where $0 < \int_\xi \beta dF(\xi) < 1$ due to Eq. (4.8). Thus, $H(\nu_{t-1})$ is a contration mapping, implying that Eq. (7.13) holds and has a unique solution. Since $0 < p < 1$ by the assumption, we have $0 < \mu < 1$, hence $\nu_0 > 0$; accordingly, $\nu > \nu_0 > 0$. Suppose $\nu = 1$. Then, from Eq. (7.13) we get $1 = \int_\xi \max\{1-p, \alpha+\beta\}dF(\xi)$, hence $0 = \int_\xi \max\{-p, \alpha+\beta-1\}dF(\xi) < 0$ due to $0 < p < 1$ and Eq. (4.8), which is a contradiction. Hence, it must be that $\nu < 1$. ∎

For convenience, let

$$z_\mu = q + (1-q-r)\mu, \tag{7.14}$$

$$U_t(\xi) = \big(\alpha + \beta\nu_{t-1}\big) - (1-p), \quad t \geq 1. \tag{7.15}$$

Then, using Eq. (4.7), we have

$$U_1(\xi) = \alpha + \beta(1-\mu) - 1 + p = p - (1-s)z_\mu. \tag{7.16}$$

From Lemma 7.1 and Eqs. (7.15) and Eq. (4.7) we have the following corollary.

Corollary 7.1 *$U_t(\xi)$ is nondecreasing in t and converges to*

$$U(\xi) = \alpha + \beta\nu - 1 + p = p - (1-s)\big(1 - r - (1-q-r)\nu\big) \tag{7.17}$$

as $t \to \infty$ for all ξ.

Now, from Eq. (7.15) the optimal decision rule can be stated as follows:

(a) If $U_t(\xi) < 0$, attempting a rescue is optimal, i.e., $\mathtt{A}_t$.
(b) If $U_t(\xi) = 0$, attempting a rescue and waiting up to the next time are indifferent, i.e., $\mathtt{A}_t \sim \mathtt{W}_t$.
(c) If $U_t(\xi) > 0$, waiting up to the next time is optimal, i.e., $\mathtt{W}_t$.

Define

$$t'(\xi) = \min\{t \mid U_t(\xi) \geq 0\}, \tag{7.18}$$

$$t''(\xi) = \min\{t \mid U_t(\xi) > 0\}, \tag{7.19}$$

if they exist.

Lemma 7.2 *The assertions below hold for all ξ.*

(a) *Suppose $p > (1-s)z_\mu$. Then $U_t(\xi) > 0$ for $t \geq 1$, hence, $\mathtt{W}_{t\geq 1}$.*
(b) *Suppose $p = (1-s)z_\mu$.*

1 *If $t''(\xi)$ does not exist, then $U_t(\xi) = 0$ for $t \geq 1$, hence $\mathtt{W}_{t\geq 1} \sim \mathtt{A}_{t\geq 1}$.*

2 *If $t''(\xi)$ exists, then $U_t(\xi) = 0$ for $t < t''(\xi)$ and $U_t(\xi) > 0$ for $t \geq t''(\xi)$ with $t''(\xi) > 1$, hence,* $\mathtt{W}_{t<t''(\xi)} \sim \mathtt{A}_{t<t''(\xi)}$ *and* $\mathtt{W}_{t\geq t''(\xi)}$.

(c) *Suppose* $p < (1-s)z_\mu$.

1 *If* $p < (1-s)\big(1-r-(1-q-r)\nu\big)$, *then* $U_t(\xi) < 0$ *for* $t \geq 1$, *hence,* $\mathtt{A}_{t\geq 1}$.

2 *If* $p = (1-s)\big(1-r-(1-q-r)\nu\big)$, *then*

i *If $t'(\xi)$ does not exist, then $U_t(\xi) < 0$ for $t \geq 1$, hence* $\mathtt{A}_{t\geq 1}$.

ii *If $t'(\xi)$ exists, then $U_t(\xi) < 0$ for $t < t'(\xi)$ and $U_t(\xi) = 0$ for $t \geq t'(\xi)$ with $t'(\xi) > 1$, hence,* $\mathtt{A}_{t<t'(\xi)}$ *and* $\mathtt{A}_{t\geq t'(\xi)} \sim \mathtt{W}_{t\geq t'(\xi)}$.

3 *If* $p > (1-s)\big(1-r-(1-q-r)\nu\big)$, *there exist $t'(\xi)$ and $t''(\xi)$ such that $t''(\xi) \geq t'(\xi) > 1$, hence $U_t(\xi) < 0$ for $t < t'(\xi)$, $U_t(\xi) = 0$ for $t'(\xi) \leq t < t''(\xi)$ and $U_t(\xi) > 0$ for $t \geq t''(\xi)$, thus* $\mathtt{A}_{t<t'(\xi)}$, $\mathtt{A}_{t'(\xi)\leq t<t''(\xi)} \sim \mathtt{W}_{t'(\xi)\leq t<t''(\xi)}$ *and* $\mathtt{W}_{t\geq t''(\xi)}$.

Proof.

(a) Suppose $p > (1-s)z_\mu$. Then $U_1(\xi) > 0$ due to Eq. (7.16), hence $U_t(\xi) > 0$ for $t \geq 1$ due to Corollary 7.1, thus $\mathtt{W}_{t\geq 1}$.

(b) Suppose $p = (1-s)z_\mu$. Then $U_1(\xi) = 0$ due to Eq. (7.16).

(b1) If $t''(\xi)$ does not exist, then $U_t(\xi) = 0$ for $t \geq 1$ due to Corollary 7.1, hence $\mathtt{W}_{t\geq 1} \sim \mathtt{A}_{t\geq 1}$.

(b2) If $t''(\xi)$ exists, then $t''(\xi) > 1$ due to Eq. (7.19), hence $U_t(\xi) = 0$ for $t < t''(\xi)$ and $U_t(\xi) > 0$ for $t \geq t''(\xi)$ due to Corollary 7.1, thus $\mathtt{W}_{t<t''(\xi)} \sim \mathtt{A}_{t<t''(\xi)}$ and $\mathtt{W}_{t\geq t''(\xi)}$.

(c) Suppose $p < (1-s)z_\mu$. Then $U_1(\xi) < 0$ due to Eq. (7.16).

(c1) If $p < (1-s)\big(1-r-(1-q-r)\nu\big)$. Then $U(\xi) < 0$ due to Eq. (7.17), hence $U_t(\xi) < 0$ for $t \geq 1$ due to Corollary 7.1, thus $\mathtt{A}_{t\geq 1}$.

(c2) If $p = (1-s)\big(1-r-(1-q-r)\nu\big)$. Then $U(\xi) = 0$ due to Eq. (7.17).

(c2i) If $t'(\xi)$ does not exist, then $U_t(\xi) < 0$ for $t \geq 1$ due to Corollary 7.1, hence $\mathtt{A}_{t\geq 1}$.

(c2ii) If $t'(\xi)$ exists, then $t'(\xi) > 1$ due to Eq. (7.18), hence $U_t(\xi) < 0$ for $t < t'(\xi)$ and $U_t(\xi) = 0$ for $t \geq t'(\xi)$ due to Corollary 7.1, thus $\mathtt{A}_{t<t'(\xi)}$ and $\mathtt{A}_{t\geq t'(\xi)} \sim \mathtt{W}_{t\geq t'(\xi)}$.

(c3) If $p > (1-s)\big(1-r-(1-q-r)\nu\big)$. Then $U(\xi) > 0$ due to Eq. (7.17). Since $U_1(\xi) < 0$, there exist $t'(\xi)$ and $t''(\xi)$ such that $t''(\xi) \geq t'(\xi) > 1$ for any ξ, hence $U_t(\xi) < 0$ for $t < t'(\xi)$, $U_t(\xi) = 0$ for $t'(\xi) \leq t < t''(\xi)$ and $U_t(\xi) > 0$ for $t \geq t''(\xi)$ due to Corollary 7.1, thus $\mathtt{A}_{t<t'(\xi)}$, $\mathtt{A}_{t'(\xi)\leq t<t''(\xi)} \sim \mathtt{W}_{t'(\xi)\leq t<t''(\xi)}$ and $\mathtt{W}_{t\geq t''(\xi)}$. ∎

Now, since $0 < (1-s)z_\mu < 1$, which is independent of p, clearly each of the three conditions in (a), (b) and (c) of Lemma 7.2, i.e., $p > (1-s)z_\mu$, $p = (1-s)z_\mu$ and $p < (1-s)z_\mu$, is possible. Further, in (c), by numerial examples we can show that the conditions of (c1) and (c3) are possible. Let $Y = (1-s)(1-r-(1-q-r)\nu$. Then, if $p = 0.2$ and 0.5 with probabilities 0.6 and 0.4, respectively, $s = 0.2$, $q = 0.5$ and $r = 0.1$, we get $\nu \simeq 0.800$ and $Y \simeq 0.464$ by numerial calculation. Accordingly, $p = 0.2 < Y$ and $p = 0.5 > Y$, impling that any of both conditions is possible.

8 Summary of Conclusions

A. From Lemmas 5.3, 6.5 and 6.7 we can eventually summarize the optimal decision rules for both models as in Table 8.1.

It should be noted in Table 8.1 that

Table 8.1: Summary of Optimal Decision Rules

	Expectation Model	Probability Model		
	Lemma 5.3	Lemma 6.5	Lemma 6.7	Lemma 6.7
	$s \geq 0,\ i \geq 1$	$s = 0$	$i = 1$	$s > 0,\ i \geq 2$
$p < (1-s)z$	$A_{t\geq 1}$	$A_{t\geq 1}$	$A_{t\geq 1}$	$i < i^* \Rightarrow A_{t\geq 1}$ $i \geq i^* \Rightarrow W_{t\geq 1}$
$p = (1-s)z$	$A_{t\geq 1} \sim W_{t\geq 1}$	$A_{t\geq 1} \sim W_{t\geq 1}$	$A_{t\geq 1} \sim W_{t\geq 1}$	$W_{t\geq 1}$
$p > (1-s)z$	$W_{t\geq 1}$	$W_{t\geq 1}$	$W_{t\geq 1}$	$W_{t\geq 1}$

1 $A_{t\geq 1}$ means that it is optimal to attempt a rescue at the time the hostage event occurs and is detected, and $W_{t\geq 1}$ means that it is optimal to wait up to the deadline and attempt a rescue at that time.

2 Let $p < (1-s)z$. Then $A_{t\geq 1}$ except the probability model with $s > 0$ and $i \geq 2$. In this case, there exists a t-independent $i^* \geq 2$ at which $A_{t\geq 1}$ if $i < i^*$ and $W_{t\geq 1}$ if $i \geq i^*$.

3 Let $p = (1-s)z$. Then $A_{t\geq 1} \sim W_{t\geq 1}$ except the probability model with $s \geq 0$ and $i \geq 2$.

4 Let $p > (1-s)z$. Then always $W_{t\geq 1}$ for both models.

5 If empolying $W_{t\geq 1}$ when $A_{t\geq 1} \sim W_{t\geq 1}$, it follows that if $p \leq (1-s)z$, then $W_{t\geq 1}$ for both models.

6 In the probability model with $p < (1-s)z$, $s > 0$ and $i \geq 2$, the decision is made as in the following scenario: If the number of hostages $i < i^*$ are taken at the time when the hostage event occurs, immediately attempt a rescue, and if $i \geq i^*$, wait up to the time when the number of hostages decreases by i^* (i.e., $i < i^*$) either due to the fact that they are killed or released with time, and attempt a rescue. Here, the i^* is given by a i satisfying Eq. (6.28).

B. In general, an optimal decision rule of a sequential decision process depends on time t. However, althought quite rare, it can be that it becomes independent of time t; in other words, the optimal decision rule is the same as that of time 1. This must be said to be quite a singular property. As seen in Table 8.1, one of the most major conclusions of this paper is that the property holds for two models in the paper except for the case in Section 7. This implies that it is optimal to behave always *as if* only a single period of planning horizon remains, i.e., as if the next point in time is a deadline. Usually, this property is called a *myopic property*.

C. The properties of $v_t(i)$.

1 If $A_{t\geq 1}$, then $v_t(i) = ip$ in the expectation model and $v_t(i) = (1-p)^i$ in the probability model.

2 If $W_{t\geq 1}$, then in the expectation model, $v_t(i)$ can be explicitly expressed by Eq. (5.10), which is linear in i, nonincreasing in t, and converges to a finite $v(i)$ as $t \to \infty$, given by Eq. (5.11), further, the $v(i)$ converges to ∞ as $i \to \infty$; and in the probability model, it can be explicitly expressed by Eq. (6.24) for $s = 0$ and Eq. (6.29) for $s > 0$, both of which is strictly decreasing in i, nondecreasing in t, and converges to a finite $v(i)$ as $t \to \infty$, given by Eq. (6.25) for $s = 0$ and Eq. (6.30) for $s > 0$, further, the $v(i)$ converge to 0 for $s = 0$ and s for $s > 0$ as $i \to \infty$.

D. If $\xi = (p, s, q, r)$ is a random variables having a distrubution function $F(\xi)$, there may exist any case of $t'(\xi) > 1$, $t''(\xi) > 1$ and $t''(\xi) \geq t'(\xi) > 1$ for any ξ. However, if ξ is deterministic, there does not exist such $t'(\xi)$ and $t''(\xi)$.

9 Suggested Future Studies

In our paper we have proposed a basic mathematical model for an optimal rescuing problem involving hostages. To take different real hostage situations into account, we feel a need to modify the model from the following viewpoints:

A. In Section 7, we obtained the conclusion that if $p < (1-s)z_\mu$ and $p > (1-s)(1-r-(1-q-r)\nu)$, there exist $t''(\xi)$ and $t'(\xi)$ such that $t''(\xi) \geq t'(\xi) > 1$. However, in the case, the question may be raised as to which of the following three points occurs: (1) always $t''(\xi) > t'(\xi) > 1$, (2) always $t''(\xi) = t'(\xi) > 1$, and (3) both of $t'(\xi) > t'(\xi) > 1$ and $t''(\xi) = t'(\xi) > 1$ are possible. In addition, the existence of $t''(\xi)$ if $p = (1-s)z_\mu$, and $t'(\xi)$ if $p < (1-s)z_\mu$ and $p = (1-s)(1-r-(1-q-r)\nu)$ must be examined. Unfortunately, it is quite difficult to mathematically answer these questions. They must remain for future study.

B. For the case in Section 7 where $i = 1$ is assumed, a clear conclusions could be relatively easily derived. However, as a future study, the mathematicl treatment leading to the conclusions should be developed into a general case of $i \geq 1$.

C. In real hostage events, some courses of action can be considered: Whether or not to submit to the demands to be airlifted to another country, to provide a means of escape, to pay the ransom, to release comrades in prison, and so on. Choosing such a course of action will influence the probabilities p, s, q and r to a greater or lesser degree. A problem arises as to when to act and what course of action to enact.

D. In the present paper, all the hostages are implicitly assumed to be homogenous. As seen in many hostages crises, however, special considerations are given for females, the aged, the sick, children, and so on. Models in which such nonhomogenous classes of hostages are taken into consideration should also be looked into.

E. In many real cases, the deadline is not always definite. In other words, it should be regarded as a random variable. A model with this assumption should be examined in the future.

F. In order for our models to be more realistically effective, the probabilities p, s, q and r must be measured and known in advance for each hostage crisis. Although such a measurement would be a very difficult task, it should be tackled through the united efforts of researchers in different fields, say, psychologists, sociologists, political scientists, engineers, and so on.

Appendices

—Proofs of Equations—

A. Eqs. (4.3) and (4.4)

$$\begin{aligned}
\sum_{k=0}^{i}\sum_{\ell=0}^{i-k} k f_{qr}(k,\ell|i) &= \sum_{k=1}^{i}\sum_{\ell=0}^{i-k} k\frac{i(i-1)!}{k(k-1)!\ell!(i-k-\ell)!}q^k r^\ell (1-q-r)^{i-k-\ell} \\
&= iq\sum_{k=1}^{i}\sum_{\ell=0}^{i-k} \frac{(i-1)!}{(k-1)!\ell!(i-k-\ell)!}q^{k-1} r^\ell (1-q-r)^{i-k-\ell} \\
&= iq\sum_{k=0}^{i-1}\sum_{\ell=0}^{i-1-k} \frac{(i-1)!}{k!\ell!((i-1)-k-\ell)!}q^{k} r^\ell (1-q-r)^{i-1-k-\ell} = iq.
\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{i}\sum_{\ell=0}^{i-k}(i-k-\ell)f_{qr}(k,\ell|i) &= \sum_{k=0}^{i-1}\sum_{\ell=0}^{i-1-k}(i-k-\ell)f_{qr}(k,\ell|i)\\ &= \sum_{k=0}^{i-1}\sum_{\ell=0}^{i-1-k}(i-k-\ell)\frac{i(i-1)!}{k!\ell!(i-k-\ell)(i-k-\ell-1)!}q^k r^\ell(1-q-r)^{i-k-\ell}\\ &= i(1-q-r)\sum_{k=0}^{i-1}\sum_{\ell=0}^{i-1-k}\frac{(i-1)!}{k!\ell!(i-1-k-\ell)!}q^k r^\ell(1-q-r)^{i-1-k-\ell}\\ &= i(1-q-r). \quad \blacksquare\end{aligned}$$

B. Eq. (4.5)

For convenience, let $\rho = 1-p$. Then $0 < \rho < 1$ due to the assumptions of p. Now, consider $\delta > 0$ such that $\rho = 1/(1+\delta)$, then for any sufficient $i \geq 1$ we have $(1+\delta)^i = 1 + i\delta + \cdots + \delta^i > i\delta$. Hence $(1-p)^i = \rho^i = 1/(1+\delta)^i < 1/i\delta$. Accordingly, $(1-p)^i$ converge to 0 as $i \to \infty$. In quite a similar way we can show $\lim_{i\to\infty}(1-z)^i = 0$. $\blacksquare$

C. Eq. (4.6)

Using the Stirling asymptotic formula $i! \sim \sqrt{2\pi}\, i^{i+0.5}e^{-i}$, we obtain

$$\begin{aligned} i f_{qr}(k,\ell|i) &= i\frac{i!}{k!\ell!(i-k-\ell)!}q^k r^\ell (1-q-r)^{i-k-\ell}\\ &\sim \frac{q^k r^\ell}{k!\ell!(1-q-r)^{k+\ell}}\frac{\sqrt{2\pi}\, i i^{i+0.5}e^{-i}}{\sqrt{2\pi}(i-k-\ell)^{i-k-\ell+0.5}e^{-(i-k-\ell)}}(1-q-r)^i\\ &= \frac{q^k r^\ell e^{-(k+\ell)}}{k!\ell!(1-q-r)^{k+\ell}}\left(\frac{i}{i-k-\ell}\right)^{0.5}\left(\frac{i}{i-k-\ell}\right)^{i-k-\ell} i^{k+\ell+1}(1-q-r)^i.\end{aligned}$$

For convenience, let $\rho = 1-q-r$. Then $0 < \rho < 1$ due to the assumptions of q and r. Now, consider $\delta > 0$ such that $\rho = 1/(1+\delta)$, then for any sufficient $i \geq 1$ we have

$$\begin{aligned}(1+\delta)^i &= 1 + i\delta + \cdots + \binom{i}{k+\ell+2}\delta^{k+\ell+2} + \cdots + \delta^i\\ &> \binom{i}{k+\ell+2}\delta^{k+\ell+2} = \frac{i!}{(k+\ell+2)!(i-k-\ell-2)!}\delta^{k+\ell+2},\end{aligned}$$

from which we get

$$\rho^i = 1/(1+\delta)^i < \frac{(k+\ell+2)!(i-k-\ell-2)!}{i!\delta^{k+\ell+2}} = \frac{(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\delta^{k+\ell+2}}.$$

Now,

$$\begin{aligned} i^{k+\ell+1}(1-q-r)^i = i^{k+\ell+1}\rho^i &< \frac{i^{k+\ell+1}(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\delta^{k+\ell+2}}\\ &= \frac{(k+\ell+2)!}{\delta^{k+\ell+2}}\frac{1}{i(1-\frac{1}{i})(1-\frac{2}{i})\cdots(1-\frac{k+\ell}{i})(1-\frac{k+\ell+1}{i})},\end{aligned}$$

which converge to 0 as $i \to \infty$. Further, we have

$$\begin{aligned}\lim_{i\to\infty}\left(\frac{i}{i-k-\ell}\right)^{0.5} &= \lim_{i\to\infty}\left(\frac{1}{1-\frac{k+\ell}{i}}\right)^{0.5} = 1,\\ \lim_{i\to\infty}\left(\frac{i}{i-k-\ell}\right)^{i-k-\ell} &= \lim_{i\to\infty}\left(1+\frac{k+\ell}{i-k-\ell}\right)^{i-k-\ell} = e^{k+\ell}.\end{aligned}$$

Accordingly, it follows that $if_{qr}(k,\ell|i)$ converge to 0 as $i \to \infty$, implying that for any infinitesimal $\varepsilon > 0$ there exists a certain $I(\varepsilon)$ such that $if_{qr}(k,\ell|i) < \varepsilon$, i.e., $f_{qr}(k,\ell|i) < \varepsilon/i$ for $i > I(\varepsilon)$. Hence, $\sum_{\ell=0}^{i-1} f_{qr}(k,\ell|i) < \sum_{\ell=0}^{i-1} \varepsilon/i = \varepsilon$ for $i > I(\varepsilon)$. Thus, it follows that $\sum_{\ell=0}^{i-1} f_{qr}(k,\ell|i)$ converges to 0 as $i \to \infty$. ∎

D. Eq. (4.10)

From Eq. (3.1) we have $z-p = q-p(q+r)$. Now, since $r+(z-p)\lambda^t-(q+r) = r+\big(q-p(q+r)\big)\lambda^t-q-r = -q(1-\lambda^t)-p(q+r)\lambda^t < 0$ for $t \geq 0$, then $\big(r+(z-p)\lambda^t\big)/(q+r) < 1$ for $t \geq 0$. Further, since $r+(z-p) = r+q-p(q+r) = (q+r)(1-p) > 0$, we have the assertion holds for $t=0$. Suppose $r+(z-p)\lambda^{t-1} > 0$. Then, $r+(z-p)\lambda^t = r+\lambda(z-p)\lambda^{t-1}+r\lambda-r\lambda = r(1-\lambda)+\lambda\big(r+(z-p)\lambda^{t-1}\big) > 0$ by induction. Accordingly, $0 < \big(r+(z-p)\lambda^t\big)/(q+r) < 1$ for $t \geq 0$. ∎

E. Eq. (5.13)

Using Eqs. (4.3), (4.4), (3.1) and (4.7), we have

$$\begin{aligned}
v_t(i) &= (1-s)\sum_{k=0}^{i}\sum_{\ell=0}^{i-k}\Big(k+(i-k-\ell)p+(i-k-\ell)\big((1-s)z-p\big)(1-\beta)^{-1}\big(1-\beta^{t-1}\big)\Big)f_{qr}(k,\ell|i) \\
&= (1-s)\Big(iq+ip(1-q-r)+i(1-q-r)\big((1-s)z-p\big)(1-\beta)^{-1}\big(1-\beta^{t-1}\big)\Big) \\
&= i(1-s)\big(q+(1-q-r)p\big)+i(1-s)(1-q-r)\big((1-s)z-p\big)(1-\beta)^{-1}\big(1-\beta^{t-1}\big) \\
&= i(1-s)z+i\beta\big((1-s)z-p\big)(1-\beta)^{-1}\big(1-\beta^{t-1}\big) \\
&= i\Big((1-s)z+\big((1-s)z-p\big)(1-\beta)^{-1}\big(\beta-\beta^t\big)\Big) \\
&= i\Big((1-s)z+\big((1-s)z-p\big)(1-\beta)^{-1}\Big(-(1-\beta)+\big(1-\beta^t\big)\Big)\Big) \\
&= i\Big((1-s)z-\big((1-s)z-p\big)+\big((1-s)z-p\big)(1-\beta)^{-1}(1-\beta^t)\Big) \\
&= i\big(p+\big((1-s)z-p\big)(1-\beta)^{-1}(1-\beta^t)\big). \quad ∎
\end{aligned}$$

F. Eq. (6.7)

Noting Eq. (3.1), we have

$$\begin{aligned}
r^i+\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)(1-p)^{i-\ell} &= r^i+\sum_{\ell=0}^{i-1}\binom{i}{\ell}r^\ell(1-q-r)^{i-\ell}(1-p)^{i-\ell} \\
&= \sum_{\ell=0}^{i}\binom{i}{\ell}r^\ell\Big((1-q-r)(1-p)\Big)^{i-\ell} = \Big(r+(1-q-r)(1-p)\Big)^i = (1-z)^i. \quad ∎
\end{aligned}$$

G. Eq. (6.32)

Noting $\lambda = 1-q-r$ and $z-p = q-p(q+r)$ due to Eqs. (4.7) and (3.1), we have

$$
\begin{aligned}
v_t(i) &= s + (1-s)\Big(r^i + \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(s\sum_{\tau=0}^{t-2}(1-s)^\tau\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^{i-\ell} + (1-s)^{t-1}\Big(\frac{r+(z-p)\lambda^{t-1}}{q+r}\Big)^{i-\ell}\Big)\Big) \\
&= s + (1-s)\Big(r^i + s\sum_{\tau=0}^{t-2}(1-s)^\tau \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^{i-\ell} \\
&\quad + (1-s)^{t-1}\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(\frac{r+(z-p)\lambda^{t-1}}{q+r}\Big)^{i-\ell}\Big) \\
&= s + (1-s)r^i + s(1-s)\sum_{\tau=0}^{t-2}(1-s)^\tau\Big(\sum_{\ell=0}^{i}\binom{i}{\ell}r^\ell\lambda^{i-\ell}\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^{i-\ell} - r^i\Big) \\
&\quad + (1-s)^t\Big(\sum_{\ell=0}^{i}\binom{i}{\ell}r^\ell\lambda^{i-\ell}\Big(\frac{r+(z-p)\lambda^{t-1}}{q+r}\Big)^{i-\ell} - r^i\Big) \\
&= s + (1-s)r^i + s\sum_{\tau=0}^{t-2}(1-s)^{\tau+1}\Big(r+\lambda\frac{r+q\lambda^\tau}{q+r}\Big)^i - s(1-s)r^i\sum_{\tau=0}^{t-2}(1-s)^\tau \\
&\quad + (1-s)^t\Big(r+\lambda\frac{r+(z-p)\lambda^{t-1}}{q+r}\Big)^i - (1-s)^t r^i \\
&= s\Big(1+\sum_{\tau=0}^{t-2}(1-s)^{\tau+1}\Big(\frac{r+q\lambda^{\tau+1}}{q+r}\Big)^i\Big) + (1-s)^t\Big(\frac{r+(z-p)\lambda^t}{q+r}\Big)^i \\
&\quad + (1-s)r^i - s(1-s)\frac{1-(1-s)^{t-1}}{s}r^i - (1-s)^t r^i \\
&= s\sum_{\tau=0}^{t-1}(1-s)^\tau\Big(\frac{r+q\lambda^\tau}{q+r}\Big)^i + (1-s)^t\Big(\frac{r+(z-p)\lambda^t}{q+r}\Big)^i. \quad \blacksquare
\end{aligned}
$$

H. Eq. (6.33)

From Eq. (6.30) we have

$$
\begin{aligned}
v(i) &= s - s + s(q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k q^{i-k}\Big(1-(1-s)\lambda^{i-k}\Big)^{-1} \\
&= s - s\Big(1-(q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k q^{i-k}\Big(1-(1-s)\lambda^{i-k}\Big)^{-1}\Big) \\
&= s - s\Big((q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k q^{i-k} - (q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k q^{i-k}\Big(1-(1-s)\lambda^{i-k}\Big)^{-1}\Big) \\
&= s - s(q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k q^{i-k}\Big(1-\Big(1-(1-s)\lambda^{i-k}\Big)^{-1}\Big) \\
&= s + s(1-s)(q+r)^{-i}\sum_{k=0}^{i}\binom{i}{k}r^k (q\lambda)^{i-k}\Big(1-(1-s)\lambda^{i-k}\Big)^{-1}. \quad \blacksquare
\end{aligned}
$$

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Reference

[1] F. Shi: Optimal hostage rescue problem where action can only be taken once – case where effect vanishes thereafter–. *Discussion Paper Series, No. 915. March, 2001 (Institute of Policy and Planning Sciences)*