No. 932

Hynothesis Testing Based on Lagrange's Method

by

Yoshiko Nogami

June 2001

Hypothesis testing based on Lagrange's method

Yoshiko Nogami

Abstract.

The author has been considering two-sided tests based on Lagrange's method (see Y. Nogami(1992, 1995, 1997a, 1997b, 2000a, 2000b, 2000c, 2001a, 2001b)). This method can easily be extended to the one-sided tests. We also notice that the estimates used for the hypothesis testing there are all unbiased estimates.

§1. Introduction.

Based on the observations $X_1, \ldots, X_n$ from the density $f(x|\theta)$ the author so far introduced two-sided tests (see Y. Nogami(1992, 1995, 1997a, 1997b, 2000a 2000b, 2000c, 2001a, 2001b)) for testing the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ for a constant θ_0, inverting the shortest interval estimate for θ_0. To get the shortest interval estimate method of Lagrange's multiplier is used. One main point is to seek the estimate for deriving the shortest interval estimate. The estimates that the author has been used are all unbiased estimates (See Section 3). We may see a piece of coincidence of the idea of unbiased estimates in R. L. Anderson and T. A. Bancroft(1952; Section 10.3). However, there they emphasize goodness of the maximum likelihood estimates. In the papers by the author, merely unbiased estimates are important, and the estimates are not necessarily maximum likelihood estimates.

Besides, in the derivation of two-sided tests the generalized likelihood ratio (GLR) test has been used. However, as we can see in Y. Nogami(2000c; Sections 2 and 3) the GLR test does not lead to good property for testing the positional (or location) parameter of two-parameter exponential distribution, but the Lagrange's method does. We note that the two-sided test for the mean of normal distribution can be derived from both Lagrange's method and the GLR test.

This method can easily be extended to the one-sided tests. We explain this in Section 2 by using Y. Nogami(2001b) for example. We also show in Section 3 that the estimates used for testing hypotheses are all unbiased estimates.

§2. One-sided test based on Lagrange's method.

We consider the uniform distribution over the set $[\theta+\delta_1, \theta+\delta_2)$ with the density

$$(2.1)\qquad f(x|\theta)=\begin{cases} c^{-1}, & \text{for } \theta+\delta_1\leqq x<\theta+\delta_2, \\ 0, & \text{otherwise,} \end{cases}$$

where $-\infty<\theta<\infty$, δ_i $(i=1,2)$ are real numbers such that $\delta_1<\delta_2$ and $c=\delta_2-\delta_1(>0)$.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density (2.1). Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We consider the unbiased estimate $Y=(X_{(1)}+X_{(n)}-\delta_0)/2$ with $\delta_0=\delta_1+\delta_2$. We test the hypotheses $H_0:\theta\geqq\theta_0$ (or $\theta\leqq\theta_0$) versus $H_1:\theta<\theta_0$ (or $\theta>\theta_0$). In (2.2) of Y. Nogami(2001b) we replace α by

2α. Then, r in (2.4) in Y. Nogami(2001b) is replaced by

$$r^*=c(1-(2\alpha)^{1/n})/2.$$

Hence, our one-sided test becomes to reject $H_0:\theta\geqq\theta_0$ (or $\theta\leqq\theta_0$) if $Y\in(-\infty,\theta_0-r^*]$ (or if $Y\in[\theta_0+r^*,\infty)$) and accept $H_0:\theta\geqq\theta_0$ (or $\theta\leqq\theta_0$) if $Y\in(\theta_0-r^*,\infty)$ (or if $Y\in(-\infty,\theta_0+r^*)$).

In the next section we check unbiasedness of the estimates appeared in Y. Nogami(2000a, 2000b, 2000c, 2001b).

§3. Unbiased estimates.

In this section we consider the distributions with the densities:

(a) $f(x|\theta,\xi)=\xi\pi^{-1}\{\xi^2+(x-\theta)^2\}^{-1}$, for $-\infty<x<\infty$; $-\infty<\theta<\infty$, $\xi>0$,

(b) $f(x|\theta)=e^{-(x-\theta)}/\{1+e^{-(x-\theta)}\}^2$, for $-\infty<x<\infty$; $-\infty<\theta<\infty$,

(c) $f(x|\theta)=e^{-(x-\theta)}$, for $\theta<x<\infty$; $-\infty<\theta<\infty$,

(d) $f(x|\theta)=\begin{cases}(\delta_2-\delta_1)^{-1}, & \text{for } \delta_1\leqq x-\theta<\delta_2,\\ 0, & \text{otherwise.}\end{cases}$

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density $f(x|\theta)$ (or $f(x|\theta,\xi)$).

In (a) we consider n=2m+1 with m a nonnegetive integer and used the estimate $Y=X_{(m+1)}$ for θ when $\xi=1$ and the estimate $W=(\log_e|X|)_{(m+1)}$ for ξ when $\theta=0$. Then, from Y. Nogami(2000a)

$$E(Y)=k\int_{-\infty}^{\infty} y(F(y))^m(1-F(y))^m\, dF(y)$$

$$=k\int_0^1 F^{-1}(u)u^m(1-u)^m\, du$$

where the second equality follows from a variable transformation u=F(y).

Since $F^{-1}(u)=\tan[(u-2^{-1})\pi]+\theta$ for $0\leq u\leq 1$, applying a variable transformation $v=1-u$ to the second term of the right hand side of the first equality below leads to

$$(3.1)\quad k^{-1}(E(Y)-\theta)=\int_0^{1/2}\tan[(u-2^{-1})\pi]u^m(1-u)^m\,du+\int_{1/2}^{1}\tan[(u-2^{-1})\pi]u^m(1-u)^m\,du$$

$$=\int_0^{1/2}\tan[(u-2^{-1})\pi]u^m(1-u)^m\,du+\int_0^{1/2}\tan[(2^{-1}-v)\pi](1-v)^m v^m\,dv.$$

Noticing that $\tan(-a)=-\tan a$ we obtain that the extreme right hand side of (3.1) $=0$. Hence, $E(Y)=\theta$.

Secondly, we consider $W=(\log_e|X|)_{(m+1)}$. Then, from Y. Nogami(2000a)

$$E(W)=k\int_{-\infty}^{\infty}w(Q_Z(w))^m(1-Q_Z(w))^m\,dQ_Z(w)$$

$$=k\int_0^1 Q_Z^{-1}(u)u^m(1-u)^m\,du,$$

where the second equality follows by taking the variable transformation $u=Q_Z(w)$. Since $Q_Z^{-1}(u)=\log_e\{\tan((\pi u)/2)\}+\log_e\xi$, we have that

$$(3.2)\quad k^{-1}(E(W)-\log_e\xi)=\int_0^1\log_e\{\sin((\pi u)/2)\}u^m(1-u)^m\,du$$

$$-\int_0^1\log_e\{\cos((\pi u)/2)\}u^m(1-u)^m\,du.$$

Applying a variable transformation $v=1-u$ to the second term of the right hand side of (3.2) leads to

$$(3.2)=\int_0^1\log_e\{\sin((\pi u)/2)\}u^m(1-u)^m\,du$$

$$-\int_0^1\log_e\{\cos(\pi(1-v)/2)\}(1-v)^m v^m\,dv.$$

Since $\cos((\pi/2)-b)=\sin b$, $(3.2)=0$ and hence $E(W)=\log_e\xi$.

In (b) we consider n=2m+1 with m a nonnegative integer and used the estimate $Y=X_{(m+1)}$ for θ. Then, from Y. Nogami(2000b)

$$E(Y)= k\int_{-\infty}^{\infty} y(F(y))^m(1-F(y))^m\, dF(y)$$

$$= k\int_0^1 F^{-1}(w)w^m(1-w)^m\, dw$$

where the second equality follows by taking a variable transformation w=F(y).

Since $F^{-1}(w)=\theta-\log_e[(1-w)/w]$ and $\int_0^1 \log_e[(1-w)/w]\, w^m(1-w)^m\, dw=0$, we can easily see that $E(Y)=\theta$.

In (c) we used the estimate $Y=(\Sigma_{i=1}X_i/n)-1=\bar{X}-1$ for θ. From (3) in Y. Nogami (2000c)

$$E(Y+1-\theta)=(\Gamma(n))^{-1}n^n\int_0^{\infty} t^n e^{-nt}\, dt=1.$$

Hence, $E(Y)=\theta$.

In (d) we used the estimate $Y=(X_{(1)}+X_{(n)}-\delta_0)/2$. From (2.1) of Y. Nogami (2001b) we get

$$E(Y)=\int_{\theta-c/2}^{\theta+c/2} nc^{-n}y(c-2|y-\theta|)^{n-1}\, dy$$

$$=\int_{-c/2}^{c/2} nc^{-n}v(c-2|v|)^{n-1}\, dv +\theta$$

where the second equality follows by a variable transformation $v=y-\theta$. Since v is an odd function of v, the first term of the extreme right hand side is zero. Thus, $E(Y)=\theta$.

REFERENCE.

R. L. Anderson and T. A. Bancroft(1952). Statistical Theory in Research. Mcgraw-Hill Book Company, Inc.

Y. Nogami(1992). A statistical inference under the uniform distribution U[θ, θ+1). Discussion Paper Series No. 507, Institute of Socio-Economic Planning, University of Tsukuba, December.

-------(1995). 一様分布 U[θ, θ+1)の統計的推測II (D.P. No.507の改定及び追加補足版) Discussion Paper Series No. 627, Institute of Socio-Economic Planning, University of Tsukuba, May.

------(1997a). A new view on statistical inference (Part I)--case of the uniform distribution U[θ, θ+1)--. Discussion Paper Series No. 713, Institute of Policy and Planning Sciences, University of Tsukuba, February.

------(1997b). A supplement for D. P. No. 713., Discussion Paper Series No. 718 Institute of Policy and Planning Sciences, University of Tsukuba, March.

------(2000a). Unbiased tests for location and scale parameters--case of Cauchy distribution--. Discussion Paper Series No. 856, Institute of Policy and Planning Sciences, University of Tsukuba, April

-------(2000b). Unbiased test for a location parameter--case of Logistic distribution--., Discussion Paper Series No. 857, Institute of Policy and Planning Sciences, University of Tsukuba, April

-------(2000c). Optimal two-sided tests for the positional and proportional parameters of the exponential distribution--comparison with the generalized likelihood-ratio tests--. Discussion Paper Series No. 893, Institute of Policy and Planning Sciences, University of Tsukuba, December.

-------(2001a). Supplement for Discussion Paper Series No.'s 856, 857 and 893., Discussion Paper Series No. 913, Institute of Policy and Planning Sciences, University of Tsukuba, March

------(2001b). Optimal two-sided test for the location parameter of the uniform distribution based on Lagrange's method., Discussion Paper Series No. 931, Institute of Policy and Planning Sciences, University of Tsukuba, June