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Optimal Two-Sided Test for the Location Parameter of the Uniform Distribution Based on Lagrange's Method

by

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Abstract.

In this paper we deal with the uniform distribution with the density

$$f(x|\theta)=\begin{cases}(\delta_2-\delta_1)^{-1}, & \text{for } \delta_1\leq x-\theta<\delta_2\\ 0, & \text{otherwise}\end{cases}$$

where $-\infty<\theta<\infty$ and δ_i (i=1, 2) are real values such that $\delta_1<\delta_2$. Based on a random sample X_1, ... ,X_n from $f(x|\theta)$ we construct the two-sided test for testing the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ for a constant θ_0, inverting the shortest interval estimate for θ_0. We show that the power function of this test is minimized at θ_0 and exhibit its exact form. We also prove that this test has the greatest power among the tests symmetric about θ_0.

§1. Introduction.

In this paper we deal with the uniform distirbution over the interval $[\theta+\delta_1, \theta+\delta_2)$ with the density

$$(1.1)\qquad f(x|\theta)=\begin{cases}(\delta_2-\delta_1)^{-1}, & \text{for } \delta_1 \leqq x-\theta<\delta_2 \\ 0, & \text{otherwise}\end{cases}$$

where $-\infty<\theta<\infty$ and $\delta_i(i=1,2)$ are real values such that $\delta_1<\delta_2$. Based on a random sample $X_1,\ldots,X_n$ from $f(x|\theta)$ we test the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for a constant θ_0. Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We first estimate θ by an unbiased estimate

$$(1.2)\qquad Y=(X_{(1)}+X_{(n)}-\delta_0)/2$$

with $\delta_0=\delta_1+\delta_2$.

The author needs to remark on the estimator Y. Although the author had been noticed in Y. Nogami(1992) (see p.1) that the unbiased estimator Y could be a good estimator, miscaluculation in deriving the probability density function (p.d.f.) of Y had prevented her from reaching Y. Later, in Y. Nogami(1995), the author corrected this and used Y to obtain the two-sided test based on the shortest interval estimate. This paper is the generalization of Sections 1 and 2 of Y. Nogami(1995) to the density (1.1) and introduce in Section 4 extra property which is the generalization of Y. Nogami(1997).

Let $0<\alpha<1$. We call (U_1,U_2) a $(1-\alpha)$ interval estimate for the parameter ν if $P_\nu[U_1<\nu<U_2]=1-\alpha$. We use Lagrange's method to get the shortest $(1-\alpha)$ interval estimate for the parameter θ based on Y. Inverting this interval for θ_0 we obtain the acceptance region of the two-sided test in Section 2.

As the bibliography the problem of testing hypotheses $H_0:\theta=0$ versus $H_1:\theta\neq0$ is treated by A. Birnbaum(1954). He treats the case of $\delta_2=-\delta_1=2^{-1}$ and states in his paper that there exists no uniformly most powerful unbiased test of $H_0: \theta=0$. However, he had not mentioned the estimator Y. Also, the alternative hypohtesis $H_1:0<|\theta|<2^{-1}(1-\alpha^{1/n})$ in his Example 2 is too close to the hypothesis $H_0:\theta=0$ as $n\to\infty$ because $\alpha^{1/n}$ becomes close to 1 when n is large. So, the author still believes goodness of our test introduced in Section 2. In fact, in Section 3 we see that the power function of this test is minimized at θ_0 with the minimum value α and exhibit its exact form. (see (3.1).) Furthermore, in Section 4 we state the theorem that this test has the greatest power among size-α tests symmetric about θ_0 and prove it by usage of generalized Neyman-Pearson Lemma.

Hereafter, we let $c=\delta_2-\delta_1(>0)$. Let $\doteq$ be the defining property.

§2. The two-sided test for θ.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density (1.1). We find the shortest $(1-\alpha)$ interval estimate for θ based on Y defined by (1.2) using Lagrange's method and invert this interval estimate for θ_0 to get the two-sided test for testing hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$.

We first find the probability density function(p.d.f.) of Y. Applying the variable transformation $Y=(X_{(1)}+X_{(n)}-\delta_0)/2$ and $Z=X_{(1)}$ to the joint p.d.f. of $(X_{(1)}, X_{(n)})$ and taking the marginal p.d.f. we obtain the p.d.f. of Y as follows:

$$(2.1) \qquad g_Y(y|\theta)=\begin{cases} nc^{-n}(c-2|y-\theta|)^{n-1}, & \text{for } -c/2<y-\theta<c/2 \\ 0, & \text{otherwise.} \end{cases}$$

To get the shortest $(1-\alpha)$ interval estimate for θ we shall find real numbers r_1 and r_2 $(r_1<r_2)$ which minimize r_2-r_1 subject to

$$(2.2) \qquad P_\theta[r_1<Y-\theta<r_2]=\int_{\theta+r_1}^{\theta+r_2} g_Y(y|\theta)\,dy=1-\alpha.$$

Letting λ be a real number we define

$$L\triangleq L(r_1, r_2;\lambda)\triangleq r_2-r_1-\lambda\{\int_{\theta+r_1}^{\theta+r_2} g_Y(y|\theta)\,dy-1+\alpha\}.$$

By Lagrange's method $\partial L/\partial r_1=0=\partial L/\partial r_2$, which leads to

$$(2.3) \qquad g_Y(\theta+r_1|\theta)=g_Y(\theta+r_2|\theta)(=\lambda^{-1}), \qquad \forall\theta.$$

Since $\partial L/\partial\lambda=0$ is equivalent to (2.2), (2.2) and (2.3) lead to $r_2=-r_1(\triangleq r)$. Substituting these into (2.2), making a variable change $u=y-\theta$ and performing further calculations leads to

$$\text{the extreme left hand side of (2.2)}=2\int_0^r nc^{-n}(c-2u)^{n-1}\,du=1-(1-(2r/c))^n.$$

Solving $\alpha=(1-(2r/c))^n$ we get

$$(2.4) \qquad r=c(1-\alpha^{1/n})/2.$$

Hence, the shortest $(1-\alpha)$ interval estimate for θ is

(2.5) $(Y-r, Y+r)$

where r is given by (2.4).

Therefore, by inverting the shortest $(1-\alpha)$ interval estimate (2.5) for θ_0 we get the acceptance region (θ_0-r, θ_0+r). Namely, our test is to reject H_0 if $Y\in(-\infty, \theta_0-r]\cup[\theta_0+r, \infty)$ and accept H_0 if $Y\in(\theta_0-r, \theta_0+r)$. Let $\phi(y)$ be a randomized test which chooses between two decisions, rejection or acception of H_0 with probabilities $\phi(y)$ and $1-\phi(y)$, respectively. Let $\phi^*(y)$ be a randomized test defined by

$$(2.6)\qquad \phi^*(y)=\begin{cases}1, & \text{if } y\leqq\theta_0-r \text{ or } \theta_0+r\leqq y\\ 0, & \text{if } \theta_0-r<y<\theta_0+r.\end{cases}$$

In the next section we show that the power function of our test is minimized at θ_0 and exhibit its exact form.

§3. The power function.

Let $y_1=\theta_0-r$ and $y_2=\theta_0+r$ where r is given by (2.4). We define the power function $\pi(\theta)$ of our test as follows:

$$\pi(\theta)\doteq E_\theta(\phi^*(Y))=1-\int_{y_1}^{y_2} g_Y(y|\theta)\,dy$$

where $g_Y(y|\theta)$ is defined by (2.1). From (2.2) with $r_2=-r_1(\doteq r)$ we have that $\pi(\theta_0)=\alpha$.

From (2.3) and $d\pi(\theta)/d\theta=g_Y(y_2|\theta)-g_Y(y_1|\theta)$ we obtain that $[d\pi(\theta)/d\theta]_{\theta=\theta_0}=0$. Furthermore, from $d\pi(\theta)/d\theta=g_Y(y_2|\theta)-g_Y(y_1|\theta)$ and the definitions of $g_Y(y_i|\theta)$ $(i=1,2)$ we can easily check that $d\pi(\theta)/d\theta<0$ for $\theta<\theta_0$ and $d\pi(\theta)/d\theta>0$ for $\theta_0<\theta$. Since $\pi(+\infty)=\pi(-\infty)=1$ and $\pi(\theta_0)=\alpha$, $\pi(\theta)$ is minimized at $\theta=\theta_0$ with the minimum value $\pi(\theta_0)=\alpha$.

We can find the power function directly as follows:

$$(3.1)\qquad \pi(\theta)=\begin{cases}1, & \text{for } \theta\leqq\theta_0-r-c/2\\ 1-\{1-2c^{-1}(\theta_0-r-\theta)\}^n/2, & \text{for } \theta_0-r-c/2\leqq\theta<\theta_0+r-c/2\\ 1-\{1-2c^{-1}(\theta_0-r-\theta)\}^n/2+\{1-2c^{-1}(\theta_0+r-\theta)\}^n/2, & \text{for } \theta_0+r-c/2\leqq\theta<\theta_0-r\\ \{1-2c^{-1}(\theta-\theta_0+r)\}^n/2+\{1-2c^{-1}(\theta_0+r-\theta)\}^n/2, & \text{for } \theta_0-r\leqq\theta<\theta_0+r\\ 1-\{1-2c^{-1}(\theta-\theta_0-r)\}^n/2+\{1-2c^{-1}(\theta-\theta_0+r)\}^n/2, & \text{for } \theta_0+r\leqq\theta<\theta_0-r+c/2\\ 1-\{1-2c^{-1}(\theta-\theta_0-r)\}^n/2, & \text{for } \theta_0-r+c/2\leqq\theta<\theta_0+r+c/2\\ 1, & \text{for } \theta_0+r+c/2\leqq\theta.\end{cases}$$

Table. The Values of $\alpha^{1/n}$.

α	.10	.05	.01	.001
n=4	.56	.47	.32	.18
5	.63	.55	.40	.25
6	.69	.61	.46	.32
7	.72	.65	.52	.37
10	.79	.74	.63	.50
50	.95	.94	.91	.87
100	.98	.97	.95	.93

Here, we remark that for ($\alpha \geqq 0.001; n \geqq 10$), ($\alpha \geqq 0.01; n \geqq 7$), ($\alpha \geqq 0.05; n \geqq 5$) and ($\alpha \geqq 0.10; n \geqq 4$), we have that $\alpha^{1/n} \geqq 1/2$. (See Table.) So, calculations led to (3.1) depend on α and n such that $\alpha^{1/n} \geqq 1/2$.

In the last section we show that our test has the greatest power among size-α tests symmetric about θ_0. To do so we use the generalized Neyman-Pearson lemma.

§4. Best symmetric two-sided test.

In this section we shall prove the following theorem:

Theorem. Let $X_1, \ldots, X_n$ be a random sample from the p.d.f. $f(x|\theta)$ given by (1.1). The test ϕ^* given by (2.6) for testing the hypothesis $H_0: \theta=\theta_0$ versus the alternative hypothesis $H_1: \theta \neq \theta_0$ has the greatest power among size-α tests symmetric about θ_0.

Proof.) We first introduce a size-α test ϕ symmetric about θ_0. Using the generalized Neyman-Pearson Lemma we obtain the test ϕ given by (4.8) which has the greatest power. Hence, the test ϕ^* given by (2.6) will be the best symmetric two-sided test of size-α.

Let $X_{(i)}$ be the i-th smallest observation such that $X_{(1)} \leqq X_{(2)} \leqq \ldots \leqq X_{(n)}$. Let Y be as defined by (1.2). The p.d.f. $g_Y(y|\theta)$ of Y is given by (2.1). Let ϕ be a size-α test symmetric about θ_0 (namely, $\phi(y)=\phi(2\theta_0-y)$, $\forall y$). Then, it follows that

(4.1) $\quad E_{\theta_0}(\phi(Y))=\alpha$

and

(4.2) $\quad E_{\theta_0}(Y\phi(Y))=\theta_0 E_{\theta_0}(\phi(Y))=\theta_0 \alpha.$

Above (4.2) holds because $E_{\theta_0}((Y-\theta_0)\phi(Y))=0$ and (4.1) holds.

Hence, by generalized Neyman-Pearson Lemma ϕ defined by (4.8) maximizes the integral

$\int \phi(y)\, g_Y(y|\theta')\, dy$ for $\theta' \neq \theta_0$

out of all functions ϕ, $0 \leqq \phi \leqq 1$, satisfying (4.1) and (4.2), when there exist real constants k_1 and k_2 such that in almost everywhere sense of either of two probability measures

$$\phi(y)=\begin{cases} 1, & \text{if } (c-2|y-\theta'|)^{n-1} I_{(-c/2,\,c/2)}(y-\theta') \\ & \qquad \geqq (k_2+k_1 y)(c-2|y-\theta_0|)^{n-1} I_{(-c/2,\,c/2)}(y-\theta_0) \quad (4.3) \\ 0, & \text{if } (c-2|y-\theta'|)^{n-1} I_{(-c/2,\,c/2)}(y-\theta') \\ & \qquad < (k_2+k_1 y)(c-2|y-\theta_0|)^{n-1} I_{(-c/2,\,c/2)}(y-\theta_0). \quad (4.4) \end{cases}$$

We check existence of such k_1 and k_2 and show that the test ϕ is of form (4.8). We first check existence of such k_1 and k_2 until the fifth line below (4.7).

When $\theta' \leq \theta_0 - c/2$ or $\theta_0 + c/2 \leq \theta'$, we take $k_1=0$ and $k_2=1$. When $\theta' \leq \theta_0 - c$ or $\theta_0 + c \leqq \theta'$, the ienquality (4.3) or (4.4) trivially holds. When $\theta_0 - c < \theta' \leq \theta_0 - c/2$, the inequality (4.4) holds for $(\theta_0+\theta')/2 < y < \theta_0 + c/2$ and the inequality (4.3) holds for $\theta' - c/2 < y \leqq (\theta_0+\theta')/2$. When $\theta_0 + c/2 \leq \theta' < \theta_0 + c$, the inequality (4.4) holds for $\theta_0 - c/2 < y \leq (\theta_0+\theta')/2$ and the inequality (4.3) holds for $(\theta_0+\theta')/2 < y < \theta' + c/2$.

We now consider the case of $\theta_0 - c/2 < \theta' < \theta_0 + c/2$. Let $\theta_0 < \theta' < \theta_0 + c/2$. Then, for $\theta_0 - c/2 < y < \theta' - c/2$, the inequality (4.4) holds when $k_1=0$ and $k_2=1$. On the other hand, for $\theta_0 + c/2 < y < \theta' + c/2$, the inequality (4.3) always holds for any k_1 and k_2. Let $\theta_0 - c/2 < \theta' < \theta_0$. Then, for $\theta' - c/2 < y < \theta_0 - c/2$, the inequality (4.3) holds for any real k_1 and k_2. On the other hand, for $\theta' + c/2 < y < \theta_0 + c/2$, the inequality (4.4) holds when $k_1=0$ and $k_2=1$. Henceforth, it is enough to consider the y's in $(\theta' - c/2, \theta_0 + c/2)$ for $\theta_0 < \theta' < \theta_0 + c/2$ or in $(\theta_0 - c/2, \theta' + c/2)$ for $\theta_0 - c/2 < \theta' < \theta_0$. We let

$$h(y)=(c-2|y-\theta'|)/(c-2|Y-\theta_0|) \quad (\geqq 0)$$

and

$$z(y)=(k_2+k_1 y)^{1/(n-1)} \quad (\geqq 0).$$

We also let

(4.5) $\quad y_0=-k_2/k_1.$

For $\theta_0<\theta'<\theta_0+c/2$, take y_0 such that $\theta'-c/2<y_0<\theta_0$. For $\theta_0-c/2<\theta'<\theta_0$, take y_0 such that $\theta_0<y_0<\theta'+c/2$. Let p be a given number such that $0<p<1$. Take $y_0=(1-p)\theta_0+p(\theta'-c/2)$ for $\theta_0<\theta'<\theta_0+c/2$ and take $y_0=(1-p)\theta_0+p(\theta'+c/2)$ for $\theta_0-c/2<\theta'<\theta_0$. Then, from (4.5), k_2 is taken as follows:

$$(4.6)\qquad k_2=\begin{cases}-k_1\{(1-p)\theta_0+p(\theta'-c/2)\}, & \text{for } \theta_0<\theta'<\theta_0+c/2\\ -k_1\{(1-p)\theta_0+p(\theta'+c/2)\}, & \text{for } \theta_0-c/2<\theta'<\theta_0.\end{cases}$$

Substituting these values into z(y) we have that

$$(4.7)\qquad z(y)=\begin{cases}[k_1\{y-((1-p)\theta_0+p(\theta'-2^{-1}c))\}]^{1/(n-1)}, & \text{for } \theta_0<\theta'<\theta_0+2^{-1}c,\\ [k_1\{y-((1-p)\theta_0+p(\theta'+2^{-1}c))\}]^{1/(n-1)}, & \text{for } \theta_0-2^{-1}c<\theta'<\theta_0\end{cases}$$

which are drawn by stripe lines in the figure below. Since we must accept H_0 for $y=\theta_0$, we must have $h(\theta_0)<z(\theta_0)$. We take $k_1=2(pc)^{-1}$ for $\theta_0<\theta'<\theta_0+2^{-1}c$ and $k_1=-2(pc)^{-1}$ for $\theta_0-2^{-1}c<\theta'<\theta_0$. Then, $(z(\theta_0))^{n-1}=h(\theta_0)$. Hence, we have $h(\theta_0)<z(\theta_0)$ because $0<1-2|\theta_0-\theta'|/c<1$. k_2 is obtained by substituting these values of k_1 into (4.6).

To show that the test ϕ is of form (4.8) we check the existence of two intersection points of h(y) and z(y). Since for $\theta_0<\theta'<\theta_0+c/2$

$$h(y)=\begin{cases}1-\{2(\theta'-\theta_0)/(2y+c-2\theta_0)\}, & \text{for } \theta'-c/2<y\leq\theta_0,\\ -1-\{2(c-(\theta'-\theta_0))/(2y-c-2\theta_0)\}, & \text{for } \theta_0<y\leq\theta',\\ 1-\{2(\theta'-\theta_0)/(2y-c-2\theta_0)\}, & \text{for } \theta'<y<\theta_0+c/2,\end{cases}$$

h(y) is an increasing function for $\theta'-c/2<y<\theta_0+c/2$. Since for $\theta_0-c/2<\theta'<\theta_0$

$$h(y)=\begin{cases}1+\{2(\theta_0-\theta')/(2y+c-2\theta_0)\}, & \text{for } \theta_0-c/2<y\leqq\theta',\\ -1+\{2c-2(\theta_0-\theta')\}/(2y+c-2\theta_0), & \text{for } \theta'<y\leqq\theta_0,\\ 1+\{2(\theta_0-\theta')/(2y-c-2\theta_0)\}, & \text{for } \theta_0<y<\theta'+c/2,\end{cases}$$

h(y) is a decreasing function for $\theta_0-c/2<y<\theta'+c/2$. On the other hand, when $\theta_0<\theta'<\theta_0+c/2$ $dz(y)/dy>0$ for all $y>y_0$ and when $\theta_0-c/2<\theta'<\theta_0$ $dz(y)/dy<0$ for all $y<y_0$. Since $0=z(y_0)<h(y_0)(<1)$ and since for $\theta_0<\theta'<\theta_0+c/2$ $z((\theta_0+c/2)-)<\lim_{y\to(\theta_0+c/2)-}h(y)=+\infty$ and for $\theta_0-c/2<\theta'<\theta_0$ $z((\theta_0-c/2)+)<\lim_{y\to(\theta_0-c/2)+}h(y)=+\infty$, in view of the fact that $h(\theta_0)<z(\theta_0)$ there must exist two intersection points of h(y) and z(y) for $\theta_0<\theta'<\theta_0+c/2$ and for $\theta_0-c/2<\theta'<\theta_0$, respectively. (See Figure.)

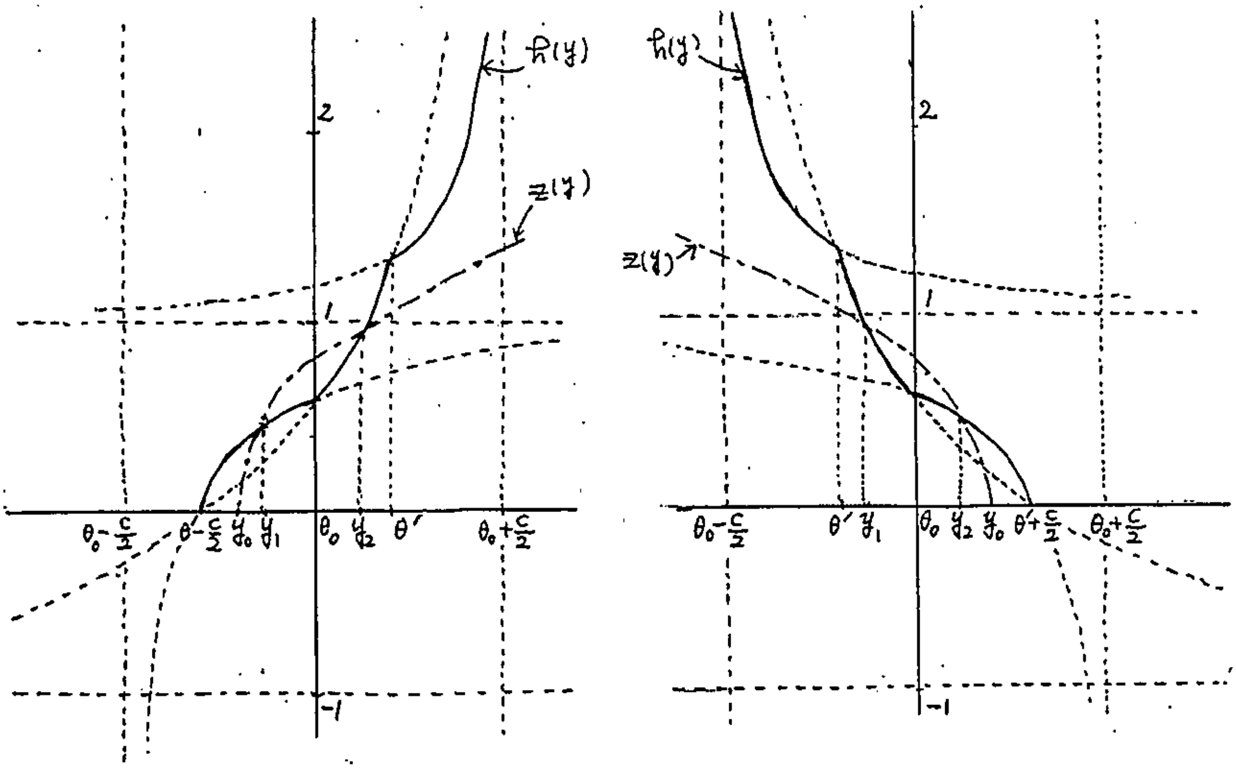
Let y_1 and y_2 be such y-coordinates of these intersection points with $y_1<y_2$. Then, we finally have the optimal test of form

$$(4.8)\qquad \phi(y)=\begin{cases}1, & \text{if } y\leqq y_1 \text{ or } y\geqq y_2\\ 0, & \text{if } y_1<y<y_2.\end{cases}$$

Hence, the test provided by (2.6) is size-α test with the greatest power among size-α tests symmetric about θ_0.

Figure. The graphs of h(y) and z(y)

$(\theta_0<\theta'<\theta_0+2^{-1}c)$ $\qquad$ $(\theta_0-2^{-1}c<\theta'<\theta_0)$



(q.e.d.)

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