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Optimal two-sided tests for the Cauchy distribution in two-sample problem based on Lagrange's method.

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Yoshiko Nogami

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Abstract.

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two independent samples randomly taken from the Cauchy distributions $C(\mu_1, \xi_1)$ and $C(\mu_2, \xi_2)$, respectively. Let α be a real number such that $0<\alpha<1$. We obtain the unbiased test of size α for testing the hypothesis $H_0: \mu_1=\mu_2$ versus the alternative hypothesis $H_1: \mu_1 \neq \mu_2$ using Lagrange's method.

In the same way, we also obtain the unbiased test of size α for testing hypotheses $H_0: \xi_1=\xi_2$ versus $H_1: \xi_1 \neq \xi_2$.

§1. Introduction.

In this paper we deal with the Cauchy distribution $C(\mu,\xi)$ with the density

$$f(x|\mu,\xi)=\xi\pi^{-1}\{\xi^2 + (x-\mu)^2\}^{-1}, \qquad \text{for } -\infty<x<\infty$$

where $-\infty<\mu<\infty$ and $\xi>0$.

Let $X_1, \ldots ,X_n$ be a random sample taken from the Cauchy distribution $C(\mu_1, \xi_1)$. Let $Y_1, \ldots ,Y_n$ be another independent sample randomly taken from the Cauchy distribution $C(\mu_2,\xi_2)$. We first consider the problem to test the hypothesis $H_0: \mu_1=\mu_2$ versus the alternative hypothesis $H_1:\mu_1\neq\mu_2$ when ξ_1 and ξ_2 are known. We secondly consider the problem to test the hypotheses $H_0:\xi_1=\xi_2$ versus $H_1:\xi_1\neq\xi_2$ when either μ_1 and μ_2 are known or $\mu_1=\mu_2$.

Let A be the acceptance region of the hypothesis $H_0:\nu=\nu_0$. Let $\zeta(\nu)\doteq P_\nu(A)$. We call $\zeta(\nu)$ the operating (characteristic) function. Let α be a real number such that $0<\alpha<1$. The two-sided test of size α is unbiased if $\zeta(\nu)$ is maximized at $\nu=\nu_0$ and $\zeta(\nu_0)=1-\alpha$. In both problems we show that our two-sided tests of size α are unbiased.

We assume that mn is odd. If mn is not odd, then we discard extra observations. We form mn differences X_i-Y_j for i=1, ... ,m and j=1, ... ,n. Let $W_1, \ldots ,W_{mn}$ be such differences. Since the characteristic function of W is of form

$$E(e^{itW})=\exp\{i(\mu_1-\mu_2)t-(\xi_1+\xi_2)|t|\}, \quad \forall \text{ real } t,$$

where $i=\sqrt{-1}$, W has the Cauchy distribution $C(\mu_1-\mu_2,\xi_1+\xi_2)$. We use this fact for our analyses.

We call (U_1,U_2) a $(1-\alpha)$ random interval for a parameter ν if $P_\nu[U_1<\nu<U_2]= 1-\alpha$.

Let $\doteq$ be the defining property. Hereafter, we let $\theta\doteq\mu_1-\mu_2$ and $\delta\doteq\xi_1+\xi_2$.

In Section 2 we find the test for testing the hypotheses $H_0:\theta=0$ versus $H_1: \theta\neq0$. In Section 3 we show that the test obtained in Section 2 is unbiased. In Section 4, letting ξ be a known number we find the test for testing the hypotheses $H_0:\xi_1=\xi_2(=\xi)$ versus $H_1:\xi_1\neq\xi_2$.

§2. The two-sided test for θ.

In this section we assume that ξ_1 and ξ_2 are known. To test the hypothesis $H_0:\theta=0$ versus the alternative hypothesis $H_1:\theta\neq 0$ we first construct the shortest $(1-\alpha)$ random interval using Lagrange's method which is similar method to obtaining the two-sided tests for θ in Nogami(2000).

Let $W_{(1)}\leq \cdots \leq W_{(mn)}$ denote the ordered values of $W_1, \ldots, W_{mn}$. Let p be a nonnegative integer. If $mn=2p+1$, then we estimate θ by $W_{(p+1)}$. Let $U\doteq W_{(p+1)}$. Then, by letting $f_W(u)\doteq f(u|\theta,\delta)$ the density of U is given by

(1) $$g_U(u|\theta)\doteq k(F_W(u))^p(1-F_W(u))^p f_W(u), \qquad -\infty<u<\infty,$$

where

(2) $$k\doteq\Gamma(2p+2)/\{\Gamma(p+1)\}^2$$

and

(3) $$F_W(u)\doteq\pi^{-1}\tan^{-1}\{\delta^{-1}(u-\theta)\} + 2^{-1}, \qquad -\infty<u<\infty.$$

Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the shortest $(1-\alpha)$ random interval for θ we want to minimize r_2-r_1 subject to

(4) $$P_\theta[r_1<U-\theta<r_2]=1-\alpha.$$

By a variable transformation $V\doteq F_W(U)$ we have that

(5) $$\text{the left hand side of (4)}=P_\theta[F_W(r_1+\theta)<V<F_W(r_2+\theta)]=1-\alpha.$$

Hence, we want to minimize r_2-r_1 subject to (5). To do so we use Lagrange's multiplier. Let λ be a real number and define

$$L\doteq L(r_1,r_2;\lambda)\doteq r_2-r_1-\lambda\{\int_{F_W(r_1+\theta)}^{F_W(r_2+\theta)} h_V(v)\,dv -1+\alpha\}$$

where with k given by (2)

$$h_V(v) \doteq kv^p(1-v)^p, \quad \text{for } 0<v<1.$$

Since by Lagrange's method we have that $\partial L/\partial r_1 = 0 = \partial L/\partial r_2$, we get that

(6) $$h_V(F_W(r_1+\theta))f_W(r_1+\theta) = h_V(F_W(r_2+\theta))f_W(r_2+\theta)(=\lambda^{-1}), \quad \forall\theta.$$

Let $\beta(\alpha/2)$ be a positive number such that

$$\int_0^{\beta(\alpha/2)} h_V(v)\,dv = \alpha/2 .$$

Without loss of generality we assume that $0<\beta(\alpha/2)<2^{-1}$. When we take that

(7) $$F_W(r_1+\theta) = \beta(\alpha/2) \quad \text{and} \quad F_W(r_2+\theta) = 1-\beta(\alpha/2),$$

$\partial L/\partial\lambda = 0$ or equivalently (5) is satisfied and furthermore we obtain by (3) that $r_1 = -r_2 \doteq -r$ where

(8) $$r = F_W^{-1}(1-\beta(\alpha/2)) - \theta = \delta\tan\{(2^{-1}-\beta(\alpha/2))\pi\}.$$

From (7) and the fact that $r_1 = -r_2 \doteq -r$ we have that $h_V(F_W(-r+\theta)) = h_V(F_W(r+\theta))$. We also have that $f_W(-r+\theta) = f_W(r+\theta)$ by the definition. Hence, (6) with $r_1 = -r_2 \doteq -r$ is satisfied. Therefore, from (5), (6) and the fact that $r_1 = -r_2 \doteq -r$ the shortest $(1-\alpha)$ random interval for θ is given by $(U-r, U+r)$ with r given by (8).

Hence, by inverting this interval for $\theta=0$ our two-sided test of size α is to reject H_0 if $U\in(-\infty,-r]\cup[r,+\infty)$ and to accept H_0 if $U\in(-r,r)$.

In the next section we prove unbiasedness of this test.

§3. Unbiasedness of the test in §2.

To see the unbiasedness of the two-sided test of size α obtained in Section 2 we define the operating (characteristic) function $\zeta(\theta)$ associated with the acceptance region $(-r,r)$ as follows:

$$\zeta(\theta) = \int_{-r}^{r} g_U(u|\theta)\,du$$

where $g_U(u|\theta)$ is given by (1). Since from (4) and the fact that $r_1=-r_2\doteq-r$ $\zeta(0)=1-\alpha$, we show that $[d\zeta(\theta)/d\theta]_{\theta=0}=0$ and $[d^2\zeta(\theta)/d\theta^2]_{\theta=0}<0$.

Because $g_U(u|\theta)=h_V(F_W(u))f_W(u), \forall u$ and (6) holds for $\theta=0$ and $r_1=-r_2\doteq-r$, we have that

(9) $$[d\zeta(\theta)/d\theta]_{\theta=0}=[g_U(-r|\theta)-g_U(r|\theta)]_{\theta=0}=0.$$

We now show that $[d^2\zeta(\theta)/d\theta^2]_{\theta=0}<0$.

Theorem.

$$[d^2\zeta(\theta)/d\theta^2]_{\theta=0} < 0.$$

Proof.) Since $d\zeta(\theta)/d\theta=g_U(-r|\theta)-g_U(r|\theta)$, we have that

(10) $$[d^2\zeta(\theta)/d\theta^2]_{\theta=0}=[dg_U(-r|\theta)/d\theta]_{\theta=0} - [dg_U(r|\theta)/d\theta]_{\theta=0}.$$

By (1) and the fact that $dF_W(u)/d\theta=-f_W(u)$ we have that

$$dg_U(u|\theta)/d\theta=-kp(f_W(u))^2(F_W(u))^{p-1}(1-F_W(u))^{p-1}(1-2F_W(u))$$

$$+k(F_W(u))^p(1-F_W(u))^p(df_W(u)/d\theta).$$

Since $[F_W(-r)]_{\theta=0}=1-[F_W(r)]_{\theta=0}=\beta(\alpha/2)$ and since $[df_W(r)/d\theta]_{\theta=0}=-[df_W(-r)/d\theta]_{\theta=0}$ $=2\delta^{-1}\chi r[(f_W(r))^2]_{\theta=0}$ and $[f_W(-r)]_{\theta=0}=[f_W(r)]_{\theta=0}$, putting these together leads to

$$[dg_U(r|\theta)/d\theta]_{\theta=0}=k[(f_W(r))^2]_{\theta=0}(1-\beta(\alpha/2))^{p-1}(\beta(\alpha/2))^{p-1}$$

$$\{p(1-2\beta(\alpha/2))+2\delta^{-1}\chi r(1-\beta(\alpha/2))\beta(\alpha/2)\}$$

and $[dg_U(-r|\theta)/d\theta]_{\theta=0}=-[dg_U(r|\theta)/d\theta]_{\theta=0}$. Thus, in view of (10) we obtain that $[d^2\zeta(\theta)/d\theta^2]_{\theta=0}<0$ for $0<\beta(\alpha/2)<2^{-1}$. (q.e.d.)

Therefore, from (9), Theorem and the fact that $\zeta(0)=1-\alpha$ our test of size α is unbiased.

In the next section we deal with the problem to test the hypotheses $H_0:\xi_1=\xi_2$ versus $H_1:\xi_1\neq\xi_2$ when either μ_1 and μ_2 are known or $\mu_1=\mu_2$.

§4. Optimal two-sided test for $H_0:\xi_1=\xi_2$.

Let ξ be a known number. To test the hypotheses $H_0:\xi_1=\xi_2(=\xi)$ versus $H_1:\xi_1\neq\xi_2$ we first construct the shortest $(1-\alpha)$ random interval using Lagrange's multiplier which is similar method to obtaining the two-sided tests for the scale parameter in Nogami(2000).

Let $W_{(1)}\leq\cdots\leq W_{(mn)}$ denote the ordered values of $W_1, \ldots, W_{mn}$ in Section 1. Let p be a nonnegative integer. Assume that mn=2p+1. Let $Z\doteq\ln|W-\theta|$.

We beforehand derive the distribution of Z. Let $\delta^*\doteq\ln\delta$. Since $w=e^z+\theta$ for $w>\theta$; $w=\theta-e^z$ for $w<\theta$; $z=-\infty$ for $w=\theta$, and since W is distributed according to the Cauchy distribution $C(\theta,\delta)$, a variable transformation $z=\ln|w-\theta|$ leads to the density of Z as follows:

$$q_Z(z)\doteq q_Z(z|\delta)=f_W(e^z+\theta)|d(e^z+\theta)/dz|+f_W(\theta-e^z)|d(\theta-e^z)/dz|$$

$$=2\pi^{-1}\exp\{z-\delta^*\}[1+\exp\{2(z-\delta^*)\}]^{-1}, \qquad -\infty<z<\infty$$

which is the same form as (28) in Nogami(2000) with ξ there replaced by δ.

We now estimate δ^* by $U\doteq Z_{(p+1)}$. Going through the same process as those until (37) in Nogami(2000), we also obtain optimal $(1-\alpha)$ random interval for δ as follows:

(11) $\quad (r_1e^U, r_2e^U)$

where

(12) $\quad r_1=[\tan\{2^{-1}\pi(1-\beta(\alpha/2))\}]^{-1}$ and $r_2=[\tan\{2^{-1}\pi\beta(\alpha/2)\}]^{-1}$.

Hence, by inverting the above $(1-\alpha)$ random interval (11) for $\delta_0=2\xi$ our two-sided test is to reject H_0 if $U\in(-\infty,\delta_0^*-\ln r_2]\cup[\delta_0^*-\ln r_1,\infty)$ and to accept H_0 if $U\in(\delta_0^*-\ln r_2,\delta_0^*-\ln r_1)$ where r_1 and r_2 are given by (12).

Unbiasedness of this test of size α is proved in the same way as those in

Section 5 of Nogami(2000) and Section 3 of Nogami(2001), so the author omits the proof of it.

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