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by

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Abstract

Let $\phi(\theta)$ be the power function of the two-sided test for testing hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ for some constant θ_0. Let α be a real number such that $0<\alpha<1$. The two-sided test of size α is unbiased if $[d\phi(\theta)/d\theta]_{\theta=\theta_0}=0$, $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0$ and $\phi(\theta_0)=\alpha$. In this paper the author shows $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0$ in the problems of Discussion Paper Series No.'s 856, 857 and 893.

§1. Introduction.

In Discussion Paper Series(D. P. S.) No.'s 856, 857 and 893, we considered the two-sided test for testing the hypotheses $H_0:\theta=\theta_0$ versus $H_1: \theta\neq\theta_0$ for some constant θ_0 with independent observations $X_1, \ldots, X_n$ distributed according to the density $f(x|\theta)$.

Let A be the acceptance region for the test. Then, the power function of the test is given by $\phi(\theta)=1-P_\theta(A)$. Let α be a real number such that $0<\alpha<1$. The two-sided test of size α is unbiased when $\phi(\theta)$ takes the minimum value at $\theta=\theta_0$ with $\phi(\theta_0)=\alpha$. Hence, to show that the two-sided test of size α is unbiased it is sufficient to show that $[d\phi(\theta)/d\theta]_{\theta=\theta_0}=0$, $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0$ and $\phi(\theta_0)=\alpha$. Unfortunately, the author forgot to show that $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0$ in D. P. S. No.'s 856, 857 and 893. So, in this paper the author proves this fact for the problems in D. P. S. No.'s 856, 857 and 893.

D. P. S. No. 856 treats Cauchy distribution $C(\theta,\xi)$ with location parameter θ and scale parameter ξ. D. P. S. No. 857 treats Logistic distribution with location parameter θ, and D. P. S. No. 893 treats Exponential distribution with positional parameter θ.

Let m be some nonnegative integer. In Sections 2, 3 and 4 we denote by $\beta(\alpha/2)$ the lower $100(\alpha/2)\%$ point of the Beta distribution with $(m+1, m+1)$ degrees of freedom. Without loss of generality we assume that $0<\beta(\alpha/2)<2^{-1}$.

§2. Unbiased test for θ in D. P. S. No. 856.

In Section 3 in D. P. S. No. 856 the author introduced the two-sided tests for θ.

Let $n=2m+1$. In this case the author showed that the two-sided test with the acceptance region (θ_0-r, θ_0+r) where $r=\tan[(2^{-1}-\beta(\alpha/2))\pi]$ had the property that $[d\phi(\theta)/d\theta]_{\theta=\theta_0}=0$ and $\phi(\theta_0)=\alpha$. To show the unbiasedness of this test we need to show the remaining condition $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0$.

Theorem 1. When $n=2m+1$,

$$[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0.$$

Proof.) Let $y_1=\theta_0-r$ and $y_2=\theta_0+r$. Since $d\phi(\theta)/d\theta=g_Y(y_2|\theta)-g_Y(y_1|\theta)$, we have that

(1) $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0} = [dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0} - [dg_Y(y_1|\theta)/d\theta]_{\theta=\theta_0}$.

On the other hand, by (4) in D.P.S. No. 856 we have that

(2) $$dg_Y(y|\theta)/d\theta = kmf(y|\theta)(dF(y)/d\theta)(F(y))^{m-1}(1-F(y))^{m-1}(1-2F(y))$$

$$+ k(F(y))^m(1-F(y))^m(df(y|\theta)/d\theta).$$

Since $[F(y_1)]_{\theta=\theta_0} = [1-F(y_2)]_{\theta=\theta_0} = \beta(\alpha/2)$ and $dF(y)/d\theta = -f(y|\theta)$ and since $[df(y_2|\theta)/d\theta]_{\theta=\theta_0} = -[df(y_1|\theta)/d\theta]_{\theta=\theta_0} = 2r\pi(f(y_2|\theta_0))^2$, and $f(y_1|\theta_0) = f(y_2|\theta_0)$, it follows by (1) that

$$[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0} = k(f(y_2|\theta_0))^2(1-\beta(\alpha/2))^{m-1}(\beta(\alpha/2))^{m-1}\{m(1-2\beta(\alpha/2)) +$$

$$2r\pi\beta(\alpha/2)(1-\beta(\alpha/2))\}$$

which is positive for $0<\beta(\alpha/2)<2^{-1}$.

(q.e.d.)

Thus, unbiasedness of our test is proved. In the next section we consider the two-sided test for the scale parameter ξ.

§3. Unbiased test for ξ in D.P.S. No. 856.

In Section 5 in D.P.S. No. 856 the author introduced the two-sided tests for the problem of testing the hypotheses $H_0: \xi=\xi_0$ versus $H_1: \xi\neq\xi_0$ with some constant ξ_0. Let $n=2m+1$. In this case the author showed that the two-sided test with the acceptance region $(\xi_0^*-\ln r_2, \xi_0^*-\ln r_1)$ where $\xi_0^*=\ln \xi_0$, $r_1=[\tan\{2^{-1}\pi(1-\beta(\alpha/2))\}]^{-1}$ and $r_2=[\tan\{2^{-1}\pi\beta(\alpha/2)\}]^{-1}$ had the property that $[d\phi(\xi)/d\xi]_{\xi=\xi_0}=0$ and $\phi(\xi_0)=\alpha$. To show the unbiasedness of this test we need to show the remaining condition $[d^2\phi(\xi)/d\xi^2]_{\xi=\xi_0}>0$.

Theorem 2. When n=2m+1,

$$[d^2\phi(\xi)/d\xi^2]_{\xi=\xi_0} > 0.$$

Proof.) Let $y_1=\xi_0^*-\ln r_2$ and $y_2=\xi_0^*-\ln r_1$. Since $d\phi(\xi)/d\xi=\xi^{-1}\{g_Y(y_2|\xi) - g_Y(y_1|\xi)\}$ and $[d\phi(\xi)/d\xi]_{\xi=\xi_0}=0$, we have that

(3) $$[d^2\phi(\xi)/d\xi^2]_{\xi=\xi_0}=\xi_0^{-1}\{[dg_Y(y_2|\xi)/d\xi]_{\xi=\xi_0}-[dg_Y(y_1|\xi)/d\xi]_{\xi=\xi_0}\}.$$

But, by (30) in D.P.S. No. 856 and in view of (2) and $dQ_z(y)/d\xi=-\xi^{-1}q_z(y)$ we have that

$$dg_Y(y|\xi)/d\xi=-km\xi^{-1}(q_z(y))^2(Q_z(y))^{m-1}(1-Q_z(y))^{m-1}(1-2Q_z(y))$$

$$+k(Q_z(y))^m(1-Q_z(y))^m\ (dq_z(y)/d\xi).$$

Since $dq_z(y)/d\xi=2(\pi\xi)^{-1}e^{y-\xi^*}(e^{2(y-\xi^*)}-1)(1+e^{2(y-\xi^*)})^{-2}$, we have that $[dq_z(y_2)/d\xi]_{\xi=\xi_0}=(2\pi\xi_0)^{-1}\sin(2\pi\beta(\alpha/2))=-[dq_z(y_1)/d\xi]_{\xi=\xi_0}$. We also have that $[Q_z(y_1)]_{\xi=\xi_0}=1-[Q_z(y_2)]_{\xi=\xi_0}=\beta(\alpha/2)$ and $[q_z(y_1)]_{\xi=\xi_0}=\pi^{-1}\sin(\pi\beta(\alpha/2))=[q_z(y_2)]_{\xi=\xi_0}$. Putting these together leads to

$$[dg_Y(y_2|\xi)/d\xi]_{\xi=\xi_0}=km(\xi_0\pi^2)^{-1}\sin^2(\pi\beta(\alpha/2))(1-\beta(\alpha/2))^{m-1}(\beta(\alpha/2))^{m-1}(1-2\beta(\alpha/2))$$

$$+k(\beta(\alpha|2))^m(1-\beta(\alpha/2))^m(2\pi\xi_0)^{-1}\sin(2\pi\beta(\alpha/2))$$

and $[dg_Y(y_1|\xi)/d\xi]_{\xi=\xi_0}=-[dg_Y(y_2|\xi)/d\xi]_{\xi=\xi_0}$. Therefore, noticing that $\sin(2\pi\beta(\alpha/2))>0$ for $0<\beta(\alpha/2)<2^{-1}$, we have in view of (3) that $[d^2\phi(\xi)/d\xi^2]_{\xi=\xi_0}>0$.

(q.e.d.)

Thus, unbiasedness of our test is proved.

In the next section we consider the two-sided test for location parameter θ in the Logistic distribution demonstrated in D.P.S. No. 857.

§4. Unbiased test for θ in D.P.S. No. 857.

In Section 3 of D.P.S. No. 857 the author introduced the two-sided tests for the problem of testing hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ with some constant θ_0. Let $n=2m+1$. For this case the author showed that the two-sided test with the acceptance region (θ_0-r, θ_0+r) where $r=\ln\{(1-\beta(\alpha/2))/\beta(\alpha/2)\}$ had the property $[d\phi(\theta)/d\theta]_{\theta=\theta_0}=0$ and $\phi(\theta_0)=\alpha$. To show the unbiasedness of this test we need to show the remaining condition $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0$.

<u>Theorem 3.</u> When $n=2m+1$,

$$[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0.$$

<u>Proof.)</u> Let $y_1=\theta_0-r$ and $y_2=\theta_0+r$. Since $d\phi(\theta)/d\theta=g_Y(y_2|\theta)-g_Y(y_1|\theta)$, we have that

$$(4)\qquad [d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}=[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}-[dg_Y(y_1|\theta)/d\theta]_{\theta=\theta_0}.$$

But, by (3) in D. P. S. No. 857 and in view of (2) and $dF(y)/d\theta=-f(y|\theta)$ we have that

$$dg_Y(y|\theta)/d\theta = -km(f(y|\theta))^2(F(y))^{m-1}(1-F(y))^{m-1}(1-2F(y)) \\ +k(F(y))^m(1-F(y))^m(df(y|\theta)/d\theta).$$

Since $df(y|\theta)/d\theta=e^{-(y-\theta)}(1-e^{-(y-\theta)})(1+e^{-(y-\theta)})^{-3}$, we have that $[df(y_2|\theta)/d\theta]_{\theta=\theta_0}=(1-2\beta(\alpha/2))f(y_2|\theta_0)=-[df(y_1|\theta)/d\theta]_{\theta=\theta_0}$. We also have that $[F(y_1)]_{\theta=\theta_0}=\beta(\alpha/2)$ $1-[F(y_2)]_{\theta=\theta_0}$ and $f(y_1|\theta_0)=f(y_2|\theta_0)=\beta(\alpha/2)(1-\beta(\alpha/2))$. Putting these together leads to

$$[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}=k(1-\beta(\alpha/2))^{m+1}(\beta(\alpha/2))^{m+1}(1-2\beta(\alpha/2))(m+1)$$

and $[dg_Y(y_1|\theta)/d\theta]_{\theta=\theta_0}=-[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}$. Therefore, in view of (4) we have that $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}>0$ for $0<\beta(\alpha/2)<2^{-1}$. (q. e. d.)

Thus, the proof of unbiasedness of our test is completed.

In the next section we consider the two-sided test for the positional parameter θ in the Exponential distribution demonstrated in D. P. S. No. 893.

§5. Unbiased test for θ in D.P.S. No. 893.

In Section 2 of D.P.S. No. 893 the author introduced the two-sided test for the problem of testing hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ with some constant θ_0.

Let $h_T(t)$ be the function of t defined by (3) in D.P.S. No. 893. Let t_1 and t_2 be positive numbers satisfying $h_T(t_1)=h_T(t_2)$ and $\int_{t_1}^{t_2} h_T(t)\,dt=1-\alpha$. The author showed that the two-sided test with the acceptance region $(\theta_0+t_1-1, \theta_0+t_2-1)$ had the property that $[d\pi(\theta)/d\theta]_{\theta=\theta_0}=0$ and $\pi(\theta_0)=\alpha$. (In this section we use $\pi(\theta)$ in stead of $\phi(\theta)$.) To complete the proof of unbiasedness of this test we need to show the remaining condition $[d^2\pi(\theta)/d\theta^2]_{\theta=\theta_0}>0$.

Theorem 4.

$$[d^2\pi(\theta)/d\theta^2]_{\theta=\theta_0}>0.$$

Proof.) Let $y_1=\theta_0+t_1$ and $y_2=\theta_0+t_2$. By the sixth line from the bottom in p.4 of D.P.S. No.893, we have that

$$\pi(\theta)=1-\int_{y_1-\theta}^{y_2-\theta} h_T(t)\,dt.$$

Since $d\pi(\theta)/d\theta=h_T(y_2-\theta)-h_T(y_1-\theta)$, we have that

$$(5)\qquad [d^2\pi(\theta)/d\theta^2]_{\theta=\theta_0}=[dh_T(y_2-\theta)/d\theta]_{\theta=\theta_0}-[dh_T(y_1-\theta)/d\theta]_{\theta=\theta_0}.$$

Since $dh_T(y_i-\theta)/d\theta=(\Gamma(n))^{-1}n^n\{-(n-1)(y_i-\theta)^{n-2}e^{-n(y_i-\theta)}+n(y_i-\theta)^{n-1}e^{-n(y_i-\theta)}\}\cdot I_{[0,\infty)}(y_i-\theta)$ $(i=1,2)$, we have that for $i=1,2$

$$[dh_T(y_i-\theta)/d\theta]_{\theta=\theta_0}=(\Gamma(n))^{-1}n^n\{-(n-1)t_i^{n-2}e^{-nt_i}+nt_i^{n-1}e^{-nt_i}\}I_{[0,\infty)}(t_i).$$

Hence, by (5) and $h_T(t_1)=h_T(t_2)$ we have that

$$(6)\qquad [d^2\pi(\theta)/d\theta^2]_{\theta=\theta_0}=(\Gamma(n))^{-1}n^n(n-1)\{t_1^{n-2}e^{-nt_1}I_{[0,\infty)}(t_1)-t_2^{n-2}e^{-nt_2}I_{[0,\infty)}(t_2)\}.$$

On the other hand, since $[dh_T(t)/dt]_{t=(n-1)/n}=0$ and since we must have that $t_1<(n-1)/n<t_2$, it follows that $[dh_T(t)/dt]_{t=t_1}>0$ and $[dh_T(t)/dt]_{t=t_2}<0$. These together with $h_T(t_1)=h_T(t_2)$ lead to

$$(n-1)t_1^{n-2}e^{-nt_1}I_{[0,\infty)}(t_1)> nt_1^{n-1}e^{-nt_1}I_{[0,\infty)}(t_1)=nt_2^{n-1}e^{-nt_2}I_{[0,\infty)}(t_2)$$

$$>(n-1)t_2^{n-2}e^{-nt_2}I_{[0,\infty)}(t_2).$$

Therefore, we obtain that (6)>0, which completes the proof. (q. e. d.)

Thus, unbiasedness of our test is proved.

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