No. 894

A New Method of Econometric Model Simulation: Resampling Approach

by

Tomonari Takeda and Tsunemasa Shiba

January 2001

A New Method of Econometric Model Simulation: Resampling Approach *

Tomonari Takeda[1]
Tsunemasa Shiba[2]
December 28, 2000

1 Introduction

Macroeconometric models have been used to make conditional predictions and policy simulations. Whatever the way they are used, we need to solve for endogenous variables. These solved values, or simulated values, are usually presented without any confidence interval, rather as point estimates or point predicted values. We call the simulated value obtained in this way "deterministically simulated " (abbreviated as ds[3]) value in this paper. Obviously ds endogenous variables are stochastic, and thus prediction intervals ought to be attached. But such is not the practice. If the econometric model consists of linear equations then their reduced forms would render prediction interval formula. Any macroeconometric model, currently in use, are nonlinear and dynamical, however. This makes it impossible to compute prediction confidence intervals to assess reliability of ds values.

Stochastic simulation (abbreviated as ss) is an obvious alternative to ds. The Cholesky decomposition of the variance-covariance matrix of the estimated residuals, was used to compute a set of ss values. Such ss method breaks down, when the sample size is less than the number of endogenous variables. McCarthy's (1972) method cleverly avoids the difficulty. His method seems to be the only practical ss method now in use.

In this paper, however, we point out that there are certain difficulties and problems associated with ss. We propose to use some resmpling techniques to alleviate the difficulties involved in ss. In doing so, we find that McCarthy's method lies somewhere in between our resampling method that uses the estimated residuals, and the usual ss that uses generated multivariate standard normal vector [4] . Using a macroeconometric model of the recent Japanese economy, we will demonstrate that the methods we propose are practical.

The outline of this paper is as follows. In the next section, we briefly review econometric model simulation methods including ds and ss. We

*We are grateful to Hiroki Tsurumi for penetrating comments. The second author's work has been supported in part by Zenkoku Ginkou Gakujyutsu Shinkou Zaidan Research Fund.

[1]Graduate Program in Policy and Planning Sciences, University of Tsukuba, Tsukuba-shi, Ibaraki-ken 305-8573 Japan. e-mail: ttakeda@sk.tsukuba.ac.jp

[2]Corresponding author: Graduate Group in Economics, Hitotsubashi University, 2-1 Naka, Kunitachi-shi, Tokyo 186–8601 Japan. e-mail: tshiba@stat.hit-u.ac.jp

[3]We use ds to imply a "determnistic simulation" or "deterministically simulated" values depending on the context. The same applies to our abbreviation ss that is introduced below.

[4]Using terminologies from bootstrap techniques, the usual stochastic simulation is a parametric bootstrapping, whereas our method can be viewed as a nonparametric bootstrapping. MacCarthy's method may be considered as a method somewhere in between the two.

propose various different bootstrap simulation methods in section 3, and compare performances of the proposed methods with the existing methods in section 4. In our concluding remarks given in section 5, we compare our proposed methods to some of the recent macroeconomic model evaluation methods such as DeJong *et al.* (2000).

2 Econometric Model Simulation Methods

Let us express a system of structural equations that is possibly nonlinear in variables as well as in parameters by

$$F(\mathbf{y}_t, \mathbf{z}_t; \mathbf{B}) = \mathbf{u}_t, \tag{1}$$

where $F(\cdot) = (f_1(\cdot), \ldots f_M(\cdot))' \sim M \times 1$ is a vector of stacked up structural equations,

M is the number of endogenous variables in the system,

$f_j(\mathbf{y}_t, \mathbf{z}_t; \beta_j) = u_{tj}$ for $j = 1, \ldots, M$;

$\mathbf{y}_t \sim 1 \times M$ lists all the endogenous variables at time t, thus $\mathbf{y}_t = (y_{t1}, \ldots, y_{tM})$;

$\mathbf{z}_t = (\mathbf{y}_{(t-)}\ \mathbf{x}_t) \sim 1 \times G$ is a vector of predetermined variables,

G is the number of predetermined variables in the system,

$\mathbf{y}_{(t-)} \sim 1 \times L$ lists all the lagged endogenous variables used in the system,

L is the number of lagged endogenous variables in the system,

$\mathbf{x}_t \sim 1 \times K$ lists all the exogenous variables used in the system,

K is the number of exogenous variables in the system, and $G = L+K$;

$\beta_j \sim 1 \times G$ is the vector of structural parameters in the jth equation and thus $\mathbf{B} = (\beta_1', \ldots, \beta_M')' \sim M \times G$;

u_{tj} is the jth scalar component of the disturbance vector, $\mathbf{u}_t = (u_{t1}, \ldots, u_{tM})' \sim M \times 1$ that is distributed independently over time, and has the first and second order moments $\mathbf{u}_t \sim (\mathbf{0}, \mathbf{\Sigma})$,

where $\mathbf{0} \sim M \times 1$ and $\mathbf{\Sigma} \sim M \times M$ positive definite [5] . It turns out that the notation $\mathbf{U} = (\mathbf{u}_1 \ldots \mathbf{u}_T) \sim M \times T$ is useful. When it is stacked up in the *vec* form, then $vec(\mathbf{U}) \sim (\mathbf{0}, \mathbf{I}_T \otimes \mathbf{\Sigma})$.

We now consider two econometric model simulation methods: the deterministic simulation, **ds**, and the stochastic simulation, **ss**. Let $\hat{\mathbf{B}}^*$ be a set of particular values of estimated $\mathbf{B}$, where $\hat{\mathbf{B}}^* = (\hat{\beta}_1^{*\prime}, \ldots, \hat{\beta}_G^{*\prime})' \sim M \times G$. Note that $\hat{\mathbf{B}}^*$ is a matrix of estimated value of $\mathbf{B}$. The left hand side of the system, then, may be written as

$$F(\mathbf{y}_t, \mathbf{z}_t; \hat{\mathbf{B}}^*). \tag{2}$$

[5] To avoid an unnecessary complication of notations, we use $\mathbf{y}_t$, $\mathbf{z}_t$ and $\mathbf{B}$ only, and do not introduce their subscripted version for single equation. That is, we assume that zero constraints are imposed on each of β_j's ($j = 1, \ldots, M$) so that each equation in the system is identified; economic theory imposes a set of zero restrictions on the elements of the $\mathbf{B}$ matrix. We also assume that each equation is normalized, *i.e.*, β_j assigns the value one to y_{tj} in $\mathbf{y}_t$.

Equation wise, this is equated to $\hat{\mathbf{u}}_t$, the estimated residual vector at time t. Ordinarily, single equation estimation methods are used to obtain

$$f_j(\mathbf{y}_t, \mathbf{z}_t; \hat{\beta}_j^*) = \hat{u}_{tj} \tag{3}$$

for $j = 1, \ldots, M$. When the elements of the $\mathbf{y}_t$ vector, are interpreted as the instruments in the second stage of the instrumental variables estimation, $\hat{y}_{ti}$ where $i \neq j$, and y_{tj} is unaltered, this identity holds for each equation. For the entire system, (2), the identity may not hold, however. This is why ds is needed. ds obtains a system solution conditioned on $\hat{\mathbf{B}}^*$. In the usual simulations, we set $\hat{\mathbf{u}}_t = 0$ to obtain a solution vector, $\mathbf{y}_t^d$, in the system

$$F(\mathbf{y}_t^d, \mathbf{z}_t; \hat{\mathbf{B}}^*) = 0. \tag{4}$$

This defines the ds, and it is the simulation method adopted in practice. $\hat{\mathbf{u}}_t$ is set to zero since taking expectation of both sides of "(2)= $\hat{\mathbf{u}}_t$" equation gives the right hand side, at least, equal to a zero vector[6]. In the left hand side, however, $E(F(y_t, z_t; \hat{\mathbf{B}}^*)) \neq F(E(y_t), E(z_t); E(\hat{\mathbf{B}}^*)))$[7], since the system could be nonlinear in variables and in parameters.

We have so far, argued that ds finds a solution vector $\mathbf{y}_t^d$ by arbitrarily setting $\hat{\mathbf{u}}_t$ to be zero in "(2)= $\hat{\mathbf{u}}_t$." This is permissible if our purpose is to investigate the performance of the system given by (3) for $j = 1, \ldots, M$. If we wish to take account of the stochastic nature of the true underlying system (1), however, we may replace $\hat{\mathbf{u}}_t$ in (2) by generating a vector, $\mathbf{u}_t^s$, as

$$\mathbf{u}_t^s \sim N_M(0, \hat{\mathbf{\Sigma}}^*), \tag{5}$$

where the superscript "s" indicates that the simulation method employed is a ss, and $\hat{\mathbf{\Sigma}}^*$ is a particular value, *i.e.,* an estimate, of $\hat{\mathbf{\Sigma}} = \hat{\mathbf{U}}\hat{\mathbf{U}}'/T$, which is an estimator of $Var(\mathbf{u}_t) = \mathbf{\Sigma}$. Our system, then, becomes

$$F(\mathbf{y}_t^s, \mathbf{z}_t; \hat{\mathbf{B}}^*) = \mathbf{u}_t^s, \tag{6}$$

where $\mathbf{y}_t^s$ now is a *stochastically* simulated solution to the system. The ss seems to be closer to the true underlying system, (1), compared to the ds value, $\mathbf{y}_t^d$, in that it takes care of the stochastic nature of $\mathbf{u}_t$.

We may qualify the ss on three points. One is: how to generate $\mathbf{u}_t^s$ in (6)? The second is: should the normality assumption in (5) be relaxed? The third is: what to do about the difference between $\mathbf{B}$ in (1) and $\hat{\mathbf{B}}^*$ in (6)?

To generate $\mathbf{u}_t^s \sim M \times 1$, suppose $T > M$. In this case $\hat{\mathbf{U}}$ has a full rank, *i.e.,* $\rho(\hat{\mathbf{U}}) = M$, hence $\hat{\mathbf{\Sigma}}^*$ is positive definite. Denoting the Choleski decomposition of $\hat{\mathbf{\Sigma}}^*$ by $\hat{\mathbf{\Sigma}}^{*\,1/2} \sim$ an $M \times M$ lower triangular matrix, then $\mathbf{u}_t^s$ may be generated as $\mathbf{u}_t^s = \hat{\mathbf{\Sigma}}^{*\,1/2} \epsilon_{(M)}$, where $\epsilon_{(d)} \sim N_d(0, \mathbf{I}_d) \sim d \times 1$, a multivariate standard normal vector with its dimension d. $\mathbf{u}_t^s$ generated in this way is distributed as $N_M(0, \hat{\mathbf{\Sigma}}^*)$ conditionally on $\hat{\mathbf{\Sigma}}^*$ and hence on $\hat{\mathbf{u}}_t$. Unfortunatly, this method breaks down if $T < M$, *i.e.,* when $\rho(\hat{\mathbf{U}}) < M$.

The use of the Choleski decomposition in the above, brings another problem to ss. This method of obtaining $\mathbf{u}_t^s$ suffers from the same type of

[6]Here, we temporarily regard $\hat{\mathbf{B}}^*$ and $\hat{\mathbf{u}}_t$ to be random variables.

[7]Jensen's inequality may be used to prove this inequality. Wallis (1995, p.312) discusses this point from a slightly different perspective.

ambiguity that we face, when analyzing the usual VAR system, too. That is, in the impulse response calculation, it is well known that the ordering of the endogenous variables in the system, matters, see *e.g.,* Hamilton (1994, p.322). When we use the Choleski decomposition to generate a $\mathbf{u}_t^s$ vector, we are introducing a certain causality in the random variables that we generate. This is a point neglected since the early days of ss. In the next section, we propose a resampling method to cope with this problem[8].

McCarthy (1972) generates $\mathbf{u}_t^s$ from $\mathbf{u}_t^s = \frac{1}{\sqrt{T}}\hat{\mathbf{U}}\epsilon_{(T)}$, where $\epsilon_{(T)} \sim N_T(\mathbf{0}, \mathbf{I}_T)$. Again, it is easy to see that such $\mathbf{u}_t^s$ is distributed as $N_M(\mathbf{0}, \hat{\mathbf{\Sigma}}^*)$ conditionally on $\hat{\mathbf{U}}$. When the system is as large as $T << M$, his method becomes the only practical way of generating pseudo random numbers for a large scale simultaneous equations system[9]. Letting $\mathbf{U}^s = (\mathbf{u}_1^s\ \mathbf{u}_2^s \ldots \mathbf{u}_T^s)$ be a $M \times T$ simulated disturbance term matrix, the plain ss obtains $\mathcal{U}^s = vec(\mathbf{U}^s)$ as

$$\mathcal{U}^s = (\mathbf{I}_T \otimes \hat{\mathbf{\Sigma}}^{1/2})\epsilon_{(MT)},$$

hence $\mathcal{U}^s|\hat{\mathbf{U}} \sim N_{MT}(0, \mathbf{I}_T \otimes \hat{\mathbf{\Sigma}})$, while McCarthy's method generates it as

$$\mathcal{U}^s = (\mathbf{I}_T \otimes \frac{1}{\sqrt{T}}\hat{\mathbf{U}})\epsilon_{(T^2)}.$$

McCarthy's $\mathcal{U}^s$ is distributed as the same $N_{MT}(0, \mathbf{I}_T \otimes \hat{\mathbf{\Sigma}})$ conditionally on $\hat{\mathbf{U}}$. By comparing these two equations, we may clarify how structural equation estimation results are used in the two different methods to generate $\mathbf{U}^s$. Since McCarthy's method uses $\hat{\mathbf{U}}$ directly to generate a $\mathbf{u}_t^s$ vector, and does not extract the second order moment information, it is a half way in between the method discussed in the previous paragraph and the nonparametric resampling method to be discussed in the next section that uses estimated $\hat{\mathbf{U}}$.

As to the normality assumption on $\mathbf{u}_t^s$ (the second problem), we are increasingly aware of a large number of literature that insist on the use of more fat-tailed distributions compared to the normal distribution. The Sterbenz and Calzolari (1990) paper is one step towards this direction in the ss literature. We may note here that we do not know whether all M elements of $\mathbf{u}_t$ should be generated as fat-tailed or some can be approximated by (multivariate) normal? Perhaps, we might be pushing things too further, if variables from a fat–tailed distribution are used to all endogenous variables. In the nonparametric bootstrap simulation method that we propose in the next section, we let the data decide the distribution [10] .

Now the third problem. If our purpose of an econometric model simulation is to investigate the performance of the estimated system then there is no problem *per se.* Ordinarily, the estimator that we employ is at least

[8]In any ss, we need to order equations to solve the system. This necessarily imposes an ordering of time wise independent multivariate pseudo random variables. Hence, we do not claim that in the resampling method proposed in the later sections, is totally free from arbitrary ordering of error terms. It only depends on the particular way of equations that we write down to solve the system; not on the way we generate pseudo random variables.

[9]Note that this somewhat resembles the situation Liu (1960) pointed out. At any rate, we shall denote the ss that uses McCarthy's method as *ss* hereafter.

[10]Freedman and Peters (1984b, p.150), in bootstrapping two stage least squares, acknowledge that "the bootstrap is distribution-free".

consistent, *i.e.,* plim $\hat{\mathbf{B}} = \mathbf{B}$, hence $\hat{\mathbf{B}}$ may be decomposed into $\mathbf{B}$ and the noise term asymptotically, $\hat{\mathbf{B}} = \mathbf{B} + \mathbf{V}$, where $\mathbf{V} = (v_1', \ldots, v_M')' \sim M \times G$ has a zero mean vector and a finite variance matrix. By fixing $\hat{\mathbf{B}}$ at a particular estimated value $\hat{\mathbf{B}}^*$, however, the plain ss fails to take account of the uncertainty involved in the parameter estimation (abbreviated as *pu* hereafter). To take account of *pu* properly, we may simulate $\mathbf{V}$ and denote it as $\mathbf{V}^s$. $\mathbf{V}^s$ has the same dimension as the $\mathbf{V}$ matrix defined above, and the distributional assumptions about v_i^s may be extracted from the estimated variance-covariance matrix of $\hat{\beta}_i^*$ assuming asymptotic normality. We add $\mathbf{V}^s$ to $\mathbf{B}$ to obtain $\hat{\mathbf{B}}^{*pu}$,

$$\hat{\mathbf{B}}^{*pu} = \hat{\mathbf{B}}^* + \mathbf{V}^s.$$

This is used in a ss:

$$F(\mathbf{y}_t^{sp}, \mathbf{z}_t; \hat{\mathbf{B}}^{*pu}) = \mathbf{u}_t^s, \tag{7}$$

where $\mathbf{y}_t^{sp}$ is meant to be a ss solution with the two sources of uncertainties: structural error term and *pu*. In equation (7), numerical values of $\mathbf{z}_t$, $\hat{\mathbf{B}}^{*pu}$ and $\mathbf{u}_t^s$ are available so that the system can be solved *e.g.,* using the Gauss-Seidel iteration. For each $\mathbf{u}_t^s$ replication we may generate one set of multivariate normal $\hat{\mathbf{B}}^{*s}$'s. This is essentially the way Fair (1994, p.186) among others, suggest to calibrate $\hat{\beta}_i^*$. We denote the simulation method of equation (6) as *spu* (a short hand notation for *ss* plus *pu*).

3 Bootstrapping Residuals for Simulation

To cope, at least partially, with the above listed problems of the usual stochastic simulation, we propose to resample error term vectors from the estimated error term matrix, $\hat{\mathbf{U}} \sim M \times T$. The jth component of $\hat{\mathbf{u}}_t$, one of the $M \times 1$ vectors in $\hat{\mathbf{U}}$, is the difference between the data on y_{tj} and its predicted value that uses $\hat{\mathbf{B}}^*$. We first sample a scalar number from a single $T \times 1$ vector $\tau = (1, .., r, .., T)'$ with replacement[11]. Suppose that element r was chosen. We then let the $\hat{\mathbf{u}}_r \sim M \times 1$ vector be $\mathbf{u}_1^s$, and so on. In other words, we resample a vector $\hat{\mathbf{u}}_r$ $(r = 1, \ldots, T)$ from a matrix $\hat{\mathbf{U}}$ with replacement. This method may be expressed as

$$F(\mathbf{y}_r^b, \mathbf{z}_r; \hat{\mathbf{B}}^*) = \hat{\mathbf{u}}_r, \tag{8}$$

where the superscript "b" in $\mathbf{y}_r^b$ indicates that it is a bootstrapped solution of the system. We abbreviate this method as *db* (a shorthand notation for direct boostrapping of estimated resuduals). By resampling a vector each time, we are able to retain the contemporaneous correlations that exist in $\hat{\mathbf{U}}$. Hence no need for the Choleski decomposition. Note also that the possibility of non-normal distribution has been appropriately dealt with. Therefore, we have taken care of the first two problems that we have cited in the previous section. Freedman (1984) and Freedman and Peters (1984a, 1984b) have outlined the resampling method that the *db* uses. But the *db* is a simulation

[11] Freedman (1984) implemented bootstrap methods to modify two stage least squares variance estimates. Freedman and Peters (1984a, 1984b) applied the method to actual econometric models. Their interest is not in investigating simulated properties of econometric models, however.

technique, and the main purpose of Freedman *et al.*'s papers is to obtain an estimate of coefficient estimators' variance.

Efron and Tibshirani (1993, pp.113-114) have pursuasively argued that in a single regression equation model, instead of resampling estimated residuals, resampling a vector of dependent varaible and explanatory variables at a time would be more robust to the specifications made for the regression model. In our case, however, the dependent variables have certain time series structures that we need to mimic in the resampling, whereas the error terms are assumed to be i.i.d. That is to say that the exchangeability assumption needed in bootstrap resampling seems to be satisfied only in the estimated residuals but not in the dependent variables and explanatory variables.

When bootstrapping is used to obtain an empirical distribution of estimators in a regression model, often estimated residuals are rescaled by a factor of $\sqrt{T/(T-K)}$ to obtain resampled residuals (Wu (1986)). In the context of econometric model simulation, we do not think that such rescaling is needed, because our interest is not in the variance or standard errors of estimators. For our purpose, we might want to be more concerned with possible heteroscedastic variance of estimated residuals. In the following we discuss a rescaling of estimated residuals so that the variance of to obtain a homoscedastic $\hat{\mathbf{u}}_r$. Consider a single regression equation, $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$, where $\mathbf{y}$ is a dependent variable vector, $\mathbf{X}$ is an explanatory variables matrix, β is a parameter vector, and $\mathbf{u}$ is an error term vector[12]. To sample from OLS estimated residuals, $\hat{\mathbf{u}} \sim T \times 1$, we should note that while $\mathbf{u}$ is assumed to possess $\mathrm{Var}(\mathbf{u}) = \sigma^2\mathbf{I}_T$, estimated residuals have $\mathrm{Var}(\hat{\mathbf{u}}) = \sigma^2\mathbf{M}_\mathbf{X}$, where $\mathbf{M}_\mathbf{X} = \mathbf{I}_T - \mathbf{P}_\mathbf{X}$, $\mathbf{P}_\mathbf{X} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, and $\rho(\mathbf{M}_\mathbf{X}) < T$, *i.e.*, less than the full rank. Hence, letting the tth diagonal element of $\mathbf{M}_\mathbf{X}$ to be $\mathbf{M}_{\mathbf{X}tt}$, what we should sample from is $\hat{u}_t/\sqrt{\mathbf{M}_{\mathbf{X}tt}}$ $(t = 1, \ldots, T)$, where $\hat{u}_t$ is the tth element of the OLS estimated residual vector $\hat{\mathbf{u}}$[13]. This provides an alternative way of bootstrapping solution and it may be expressed as

$$F(\mathbf{y}_r^{br}, \mathbf{z}_r; \hat{\mathbf{B}}^*) = \hat{\mathbf{u}}_r^{br}, \tag{9}$$

where the superscript, br, denotes rescaled bootstrapping. We use br to denote this method. In the following sections we use a medium sized econometric model to illustrate the usefulness of the bootstrapping method. The model is estimated by the two stage least squares (TSLS). The $\mathbf{X}$ matrix that constitutes the $\mathbf{M}_\mathbf{X}$ matrix for each equation in the above, should be replaced by the variables that include instruments.

In passing, we note that even in ss, where the second order moment information is extracted from estimated residuals, we may use the adjustment discussed in the previous paragraph, to alleviate the possible effects of heteroscedasticity. That is, we may rescale the residuals first to obtain the $\hat{\mathbf{U}}$ matrix that is used in McCarthy's method or in the traditional ss. Letting

[12]The notations used in the following several lines are local to this paragraph. These notations will not be carried over to other parts of this paper.

[13]See, *e.g.*, Hjorth (1994, p.186) for a description of this type of adjustment made to the regression residuals. Theil's BLUS (Theil (1971, pp.205– 206)) is an obvious alternative, but Freedman and Peters (1984b, p.151) do not think it worthwhile to implement these adjustments.

$\mathbf{u}_t^{sr}$ be the ss'ed rescaled residual vector, this method may be expressed as

$$F(\mathbf{y}_t^{sr}, \mathbf{z}_t; \hat{\mathbf{B}}^*) = \mathbf{u}_t^{sr}. \tag{10}$$

We call this method *sr* (a shorthand notation for ss based on rescaled residuals).

Instead of resampling from the estimated residuals of the equations, we could resample from the difference between the actual and the ds values, *i.e.*, $\tilde{\mathbf{u}}_t^d = \mathbf{y}_t - \mathbf{y}_t^d$. If our purpose is to examine the model's properties, then this method may be preferable. One thing to note is that the difference does not necessarily have the zero mean. We first describe a ss version of this method. Use $\tilde{\mathbf{u}}_t^d$ to compute $\boldsymbol{\Sigma}$. Based on this, generate $\mathbf{u}_t^{sd}$. Replace $\mathbf{u}_t^s$ in (6) by $\mathbf{u}_t^{sd}$. The simulation method becomes

$$F(\mathbf{y}_t^{sd}, \mathbf{z}_t; \hat{\mathbf{B}}^*) = \mathbf{u}_t^{sd}. \tag{11}$$

We denote this ss, *sd* (shorthand notation for stochastic simulation that uses $\hat{\boldsymbol{\Sigma}}$ calculated from ds residuals). A Bootstrap version of *sd* may be expressed as

$$F(\mathbf{y}_\tau^{bd}, \mathbf{z}_\tau; \hat{\mathbf{B}}^*) = \tilde{\mathbf{u}}_\tau^d. \tag{12}$$

We call this method *bd* (shorthand notation for bootstrap based on $\tilde{\mathbf{u}}_\tau^d$).

One quick way to deal with the third problem, estimated parameter uncertainty, is to partially use ss approach. We may generate stochastic $\hat{\mathbf{B}}^{*s}$'s for each of the resampled residual matrix, then our method becomes a hybrid of bootstrapping and the conventional ss. Let us use the notations that we used three paragraphs prior to this paragraph to illustarte this method. For each equation $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$, estimator of $\boldsymbol{\beta}$, $\hat{\boldsymbol{\beta}}$, would be presented with asymptotic variance-covariance matrix that is a function of $\mathbf{X}$, $\boldsymbol{\Omega}$. This $\boldsymbol{\Omega}$ matrix gives us the second order moment information to generate a multivariate normal vector. Each of the bootstrap method, when the *pu* is taken into account, becomes, *dbh*, *brh* and *bdh*, respectively for *db*, *br* and *bd*. In passing, we note that two of the newly introduced ss's could use this way of *pu* and each becomes *sdpu* and *srpu*, respectively for *sd* and *sr*.

Instead of the hybrid method, we may use the bootstrap approach all the way to deal with the estimated parameter uncertainty problem. Given the estimated original $\hat{\mathbf{B}}$ matrix, we first resample from the estimated residuals to obtain the pseudo y data. This pseudo data and the predetermined variables, z, are used to estimate a new set of $\hat{\mathbf{B}}$ matrix. Let us use $\hat{\mathbf{B}}^s$ to express this new estimates. Now use the resampled residuals that we used earlier and the $\hat{\mathbf{B}}^s$ matrix to solve or simulate the system. These steps are repeated to obtain a bootstrap empirical distribution of endogenous variables. We experiment with this method, only on *db*. This method is abbreviated as *dbbp* (shorthand notation for *db* with bootstrap reestimated parameters).

We suspect that this method requires a prohibitive computer time, when the system is large. But the main difficulty with this method is not in the computation time that it requires. It is in the fact that the method is very likely to produce a lot of meaningless coefficients, *i.e.*, that do not satisfy sign conditions and so on. If we are to employ the traditional "simple to

large" type econometrics methodology, then we do not allow such cases to happen. On the other hand, if our purpose is to simply investigate the properties of given system, then such cases may be permissible.

We have called our method a **bootstrap**, although we do not use empirical or simulated data to estimate something, *e.g.*, standard errors[14] . It might be that we should have called it a "resampling". Although, there are by now abundant applications of bootstrapping in econometrics[15], we find no application of bootstrap method to the stochastic simulation of econometric models, however.

It will be convenient to summarize the various simulation methods that have been discussed so far.

Table 1: Estimated Residuals Based Methods

	residuals adjusted	parameter uncertainty(pu)
ss	*sr*	*srpu*
		spu
db	*br*	*brh*
		dbh
		dbb

Note:"*srpu*" for example takes account of *pu* based on "*sr*".

Table 2: Deterministic Simulation Residuals Based Methods

	pu
sd	*sdpu*
bd	*bdh*

4 Bootstrap Simulation of an Econometric Model

In this section we shall apply our method to an econometric model to demonstrate that our methods are easy to use, and are also practical alternatives to ss. With a purely linear system, we suspect that the mean of stochastically simulated value should coincide the ds value. In other words, ss will not be very informative, when the system is purely linear. The econometric model that we deal with in this section, is fairly nonlinear and it has been shown

[14]A typical application of a bootstrap appears in Ohtani (2000). He obtains the distribution function of R^2, the coefficient of determination, but finds that the formula is plagued with unknown parameters. He then applies a bootstrap technique to compute confidence intervals of R^2.

[15]For an excellent survey on the theory and applications of resampling, see Davison and Hinkley (1997). Its asymptotic theory is conveniently surveyed in van der Vaart (1998) chapter 23. Bootstrap methods in econometrics are conveniently surveyed in Jeong and Maddala (1993) and Veall (1998).

that it could be used to make various policy simulations[16]. Klein's Model I (see, *e.g.*, Theil (1971, p.432 on)) has been econometricians' favorite tool kit to demonstrate usefulness of, *e.g.*, newly developed econometric methodologies. Unfortunately the model is purely linear in constant price variables. Our model has slightly less than 50 equations in total with about 30 structural equations. The system not only has log transformed variables but also interactions between the nominal price sector and the real price sector, hence the system as a whole is nonlinear. The sample period to estimate structural equations, is 1981 to 1993, annual.

We only present results of GDP rate of change, since the behavior of this variable should serve as a good summary of the whole system. ss results from 1982 to 1993 are presented in Figure 1[17] . Dynamics and nonlinearity

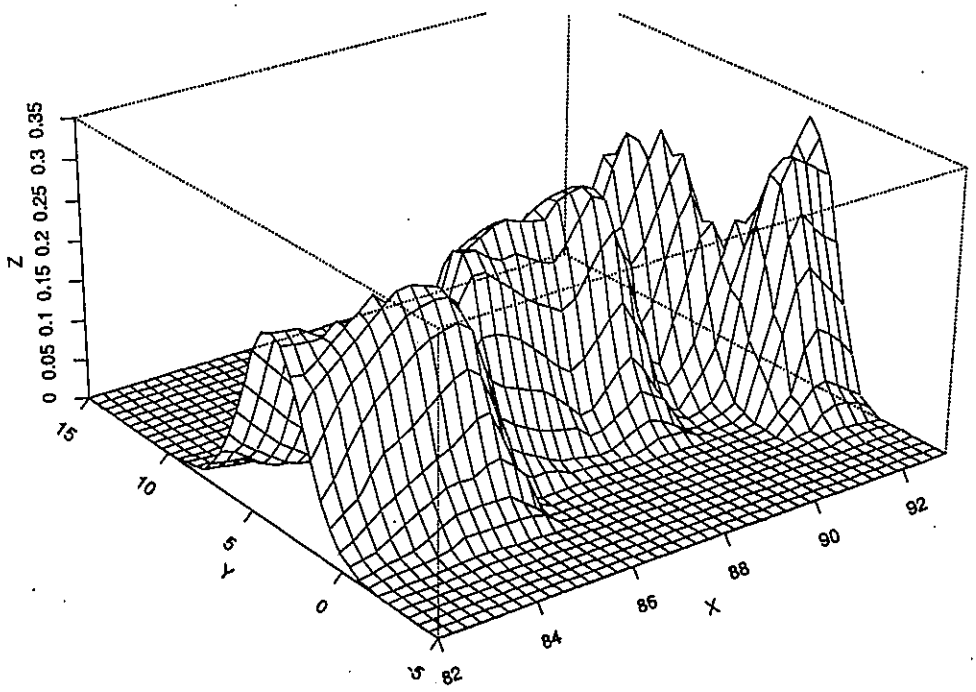

Figure 1: ss Results

of the system have produced multiple peaks and troughs in the GDP rate of change. Results of *db* are given in Figure 2[18] . The two charts seem to provide very similar distributions. It will be interesting to compare their

[16]An earlier version of this model is found in the first author's MA thesis (in Japanese) submitted to the University of Tsukuba in 1998. It was constructed in the Japan Center for Economic Research (JCER). We appreciate the many useful suggestions made by Hiromichi Muto of Gifu Shotoku Gakuen University and JCER, for the model construction. The current version of the model may be obtained by requesting to us.

[17]In all the figures below, the X axis denotes the simulation years, the Y axis denotes the GDP rate of change in percent, and Z axis denotes the frequency in the simulation.

[18]The number of bootstrap simulations is 1,000 in this paper. All simulations are path independent or static simulations. Initial values of the current endogenous variables and $y_{(t-)}$, lagged endogenous variables, come from ds calculated values. The use of ds values might seem odd. But if we carry out path dependent simulations, *i.e.*, dynamic simulations, then we end up creating prediction distribution densities that flattens (larger variance) as the forecasting horizon becomes longer.

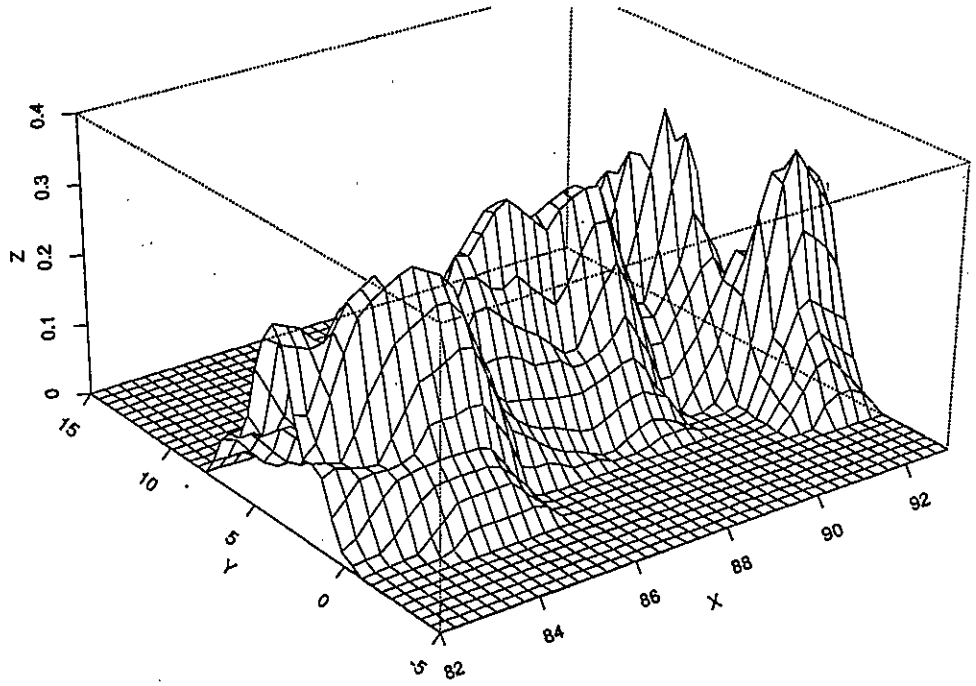

Figure 2: *db* Results

annual 95% confidence intervals. This is given in Table 3[19] .

The values in the table are fairly close. It is also interesting to observe that bootstrapping estimated residuals do not necessarily show departures from normality. We have not tested whether the simulation results can be viewed as normally distributed or not. Since ss is based on the normal distribution, we suspect that simulation results also be normal. On the other hand, *db* results might not be normal.

Evidently in figures 1 and 2, *e.g.,* annual variances of the simulation results seem to be changing. This is an obvious heteroscedasticity in the simulated values. Whether this is due to the fact that we have not employed the residual rescaling or not may be answered in Figure 3, case of *br*. This figure implements the rescaling in bootstrap, *br*. We see very little difference in figures 2 and 3, although the variances in figure 3 seem to be slightly larger than that of 2. Hence, at least for our model, residuals adjustment to homoscedasticity does not seem to be needed. The apparent heteroscedasticity of the simulation results are probably due to the system inherent nonlinearity and dynamics.

We next address the question: whether equation wise residuals should be used or the actuals less ds simulated values ought to be used? Figure 4 shows simulation results of *bd*, a simulation based on bootstrapping from $\tilde{u}_t^d = y_t - y_t^d$. We notice a great increase in variances; this is a remarkable change from the previous figures. Using the actuals less ds simulated values in the ss, *i.e.,* the case of *sd*, produced similar increase in the variances.

[19]As stated in the above, we have employed the ds values as our benchmark. Hence in the two cases, ds value ± 47.5% values provide the 95% interval. The median values of the two cases are very close to the ds values.

Table 3: 95% Confidence Intervals

lower limit	median	upper limit
0.496	3.754	6.927
−0.191	2.785	5.802
−0.930	2.177	5.050
−0.708	1.984	4.735
1.324	4.007	6.902
1.766	4.369	6.973
2.171	4.866	7.492
1.406	4.110	6.627
3.400	5.741	8.282
2.010	4.404	6.836
−0.835	1.457	3.665
−0.226	1.932	4.257

lower limit	median	upper limit
0.729	3.758	6.863
−0.157	2.811	5.794
−0.725	2.119	5.062
−0.802	1.976	4.834
1.227	4.016	6.929
1.682	4.389	7.135
2.163	4.801	7.477
1.663	4.136	6.678
3.275	5.860	8.404
2.070	4.476	6.868
−0.910	1.421	3.754
−0.401	1.933	4.287

Note: the table in the left is ss, while the one in the right is *db*.

We conclude that we may stick to the equation wise estimated residuals for econometric model simulation experiments.

The next issue is whether to take account of uncertainties involved in the estimation of parameters or not? In the ss this takes the form of generating multivariate normal random numbers to the coefficients, while in the bootstrapping, a hybrid of *db* and normally generated coefficients. In the previous section, we have proposed a bootstrap method of resampling from residuals plus reestimation of the parameters (in *dbbp* case) but we shall not experiment this method. Figure 5 presents the *spu* case, while figure 6 is the *dbh* case.

Although it might be important to take account of the uncertainty involved in coefficient estimation, simulated properties of a system may not require that the coefficients be generated.

5 Conclusions

In this paper, we proposed a new stochastic simulation method of econometric models by bootstrapping the model's estimated residual vector. We have explained that the method is easy to implement in that it does not require any matrix calculations such as the Choleski decomposition or McCarthy's (1972) generation of random numbers, and also the random number that we generate automatically possesses the contemporaneous correlation property that the estimated residual vector has. These properties of our method, should prove to be handy, when the scale of the econometric model becomes large.

As with other bootstrap applications, we do not know the appropriate number of bootstrap generations. We also need to investigate if the number changes as the degree of nonlinearity increases or as the scale of the system

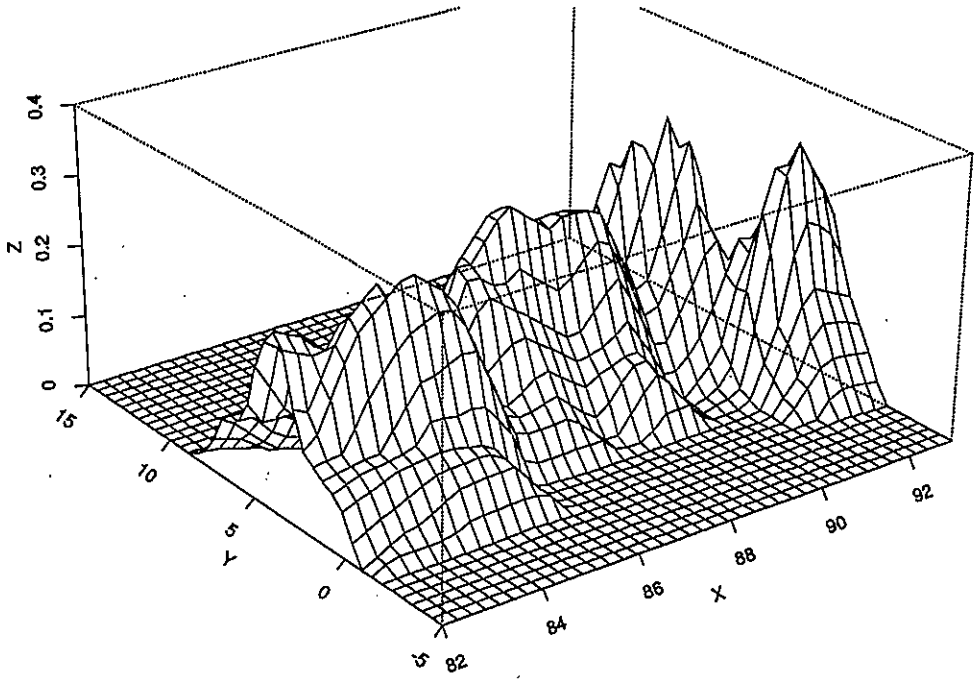

Figure 3: *br* Results

increases?

More importantly, we realize that we need to compare our methods with some of the recently proposed macroeconomic model evaluation methods such as DeJong *et al.* (2000) and Schorfheide (2000). In particular, importance sampling approach based DeJong *et al.*'s method that is essentially a ss, is worth looking into. They claim that their method is a Bayesian but it does not seem like they have followed an appropriate Bayesian predictive density approach. Our bootstrap results should be compared to those by Bayesain predictive density.

References

[1] Chernick, M.R., 1999, *Bootstrap methods,* John Wiley and Sons.

[2] Davison, A.C. and D.V. Hinkley, 1997, *Bootstrap methods and their application,* Cambridge University Press.

[3] DeJong, D.N., B.F. Ingram and C.H. Whiteman, 2000, "A Bayesian approach to dynamic macroeconomics," *Journal of Econometrics,* 98, 203–223.

[4] Fair, R.C., 1994, *Testing macroeconometric models,* Harvard University Press.

[5] Freedman, D, 1984"On bootstrapping two-stage least-squares estimates in stationary linear models," *Annals of Statistics,*, 827–842.

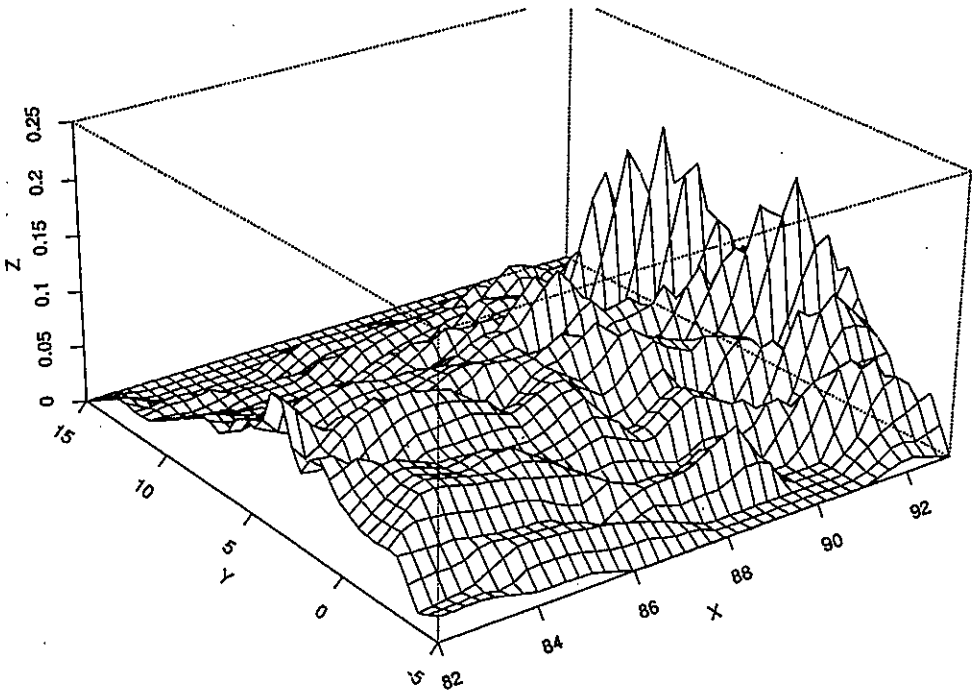

Figure 4: *bd* Results

[6] Freedman, D. and S.C. Peters, 1984a, "Bootstrapping a regression equation: some empirical results," *Journal of the American Statistical Association,* 97–106.

[7] Freedman, D. and S.C. Peters, 1984b, "Bootstrapping an econometric model: some empirical results," *Journal of Business and Economic Statistics,* 150–158.

[8] Hamilton, J.D., 1994, *Time series analysis,* Princeton University Press.

[9] Hjorth, J.S. Urban, 1994, *Computer intensive statistical methods,* Chapman and Hall.

[10] Jeong, J. and G.S. Maddala, 1993, "A perspective on application of bootstrap methods in econometrics," chapter 21 of *Handbook of Statistics* Vol.11, G.S. Maddala, C.R. Rao and H.D.Vinod eds., North-Holland.

[11] Liu, T.C., 1960, "Underidentification, structural estimation, and forecasting," *Econometrica,* 28, pp. 855–865.

[12] McCarthy, M.D., 1972, "Some notes on the generation of pseudo-structural errors for use in stochastic simulation studies," in pp. 185–191 of *Econometric Models of Cyclical Behavior,* B.G. Hickman ed., Columbia University Press.

[13] Ohtani, K., 2000, "Bootstrapping R^2 and adjusted R^2 in regression analysis," *Economic Modelling,* 17, 473–483.

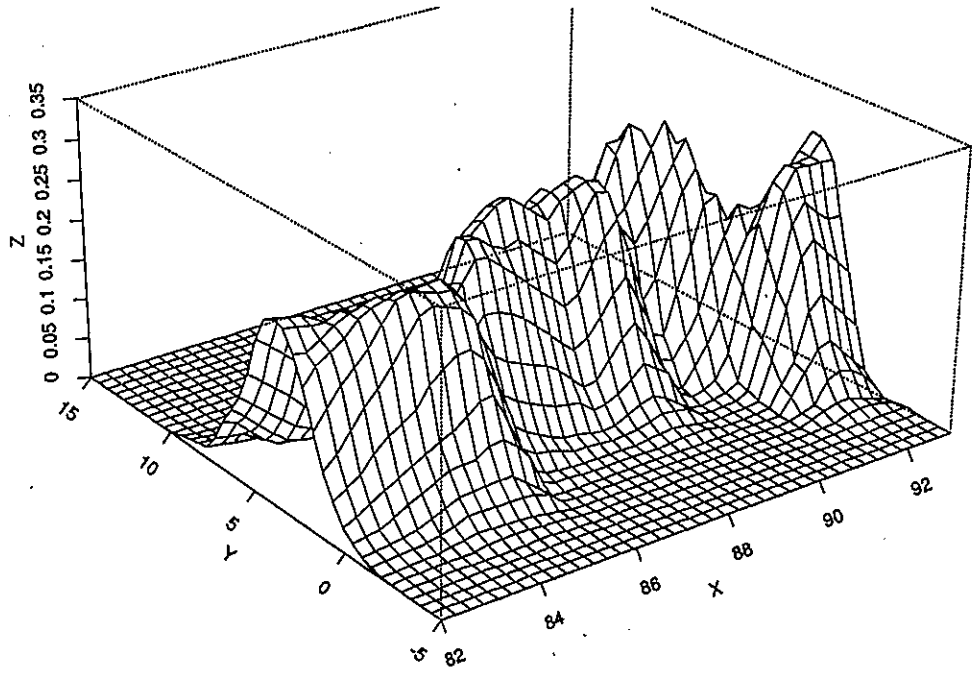

Figure 5: *spu* Results

[14] Schorfheide, F., 2000, "Loss function based evaluation of DSGE," Discussion Paper, Department of Economics, University of Pennsylvania.

[15] Sterbenz, F.P. and G. Calzolari, 1990, "Alternative specifications of the error process in the stochastic simulation of econometric models," *Journal of Applied Econometrics,* 5, pp.137–150.

[16] Theil, H., 1971, *Principles of econometrics,* North-Holland.

[17] van der Vaart, A.W., 1998, *Asymptotic statistics,* Cambridge University Press.

[18] Veall, M.R., 1998, "Applications of the bootstrap in econometrics and economic statistics," chapter 12 of *Handbook of Applied Economic Statistics,* A. Ullah and D.E.A. Giles, eds., Marcel Dekker.

[19] Wallis, K.F., 1995, "Large-Scale macroeconometric modeling," in pp. 312–355 of *Handbook of Applied Econometrics Macroeconomics,* M.H.Pesaran and M.R.Wickens, eds., Blackwell.

[20] Wu, C.F.J., 1986, "Jacknife, bootstrap and other resampling methods in regression analysis," *Annals of Statistics,* 14, 1261–1295.

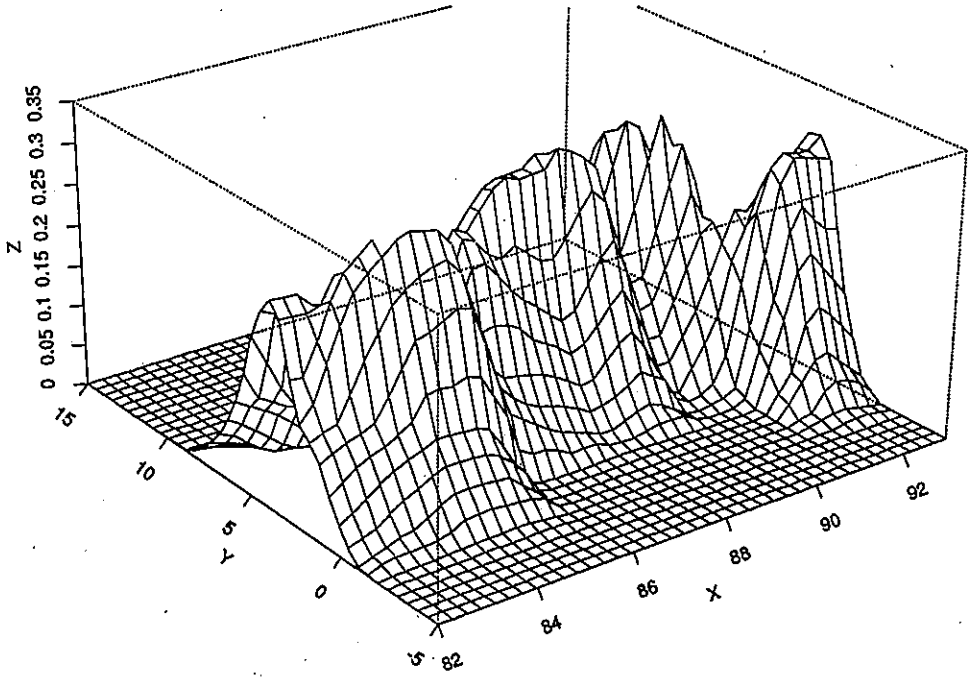

Figure 6: *dbh* Results