No. 893

Optimal two-sides tests for the positional and proportional parameters of the exponential distribution.
--comparison with the generalized likelihood-ratio tests--

by

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December 2000

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Abstract.

The author has been considering goodness of the two-sided tests derived from the Lagrange's methods (cf. [1], [2], [3]). In this paper we consider the exponential distribution with density

$$f(x|\theta,\xi)=\begin{cases}\xi e^{-\xi(x-\theta)}, & \text{for } \theta<x<\infty\\ 0, & \text{otherwise.}\end{cases}$$

Based on i.i.d. observations $X_1, \ldots, X_n$ we construct two two-sided tests for testing constantness of θ when $\xi=1$ and for testing constantness of ξ when $\theta=0$. We compare these with the generalised likelihood-ratio tests.

§1. Introduction.

The author has been considering goodness of the two-sided tests with the acceptance region derived from inverting the shortest random interval (R. I.) for the parameter of the underlined distribution. (See e.g. [1], [2], [3].) This paper is on the same lines of such research. The author would like to call these tests as the tests by the Lagrange's method.

Let $I_A(x)$ be an indicator function so that for a set A $I_A(x)=1$ if $x\in A$; $=0$ if $x\notin A$. In this paper we consider as the underlined distribution the exponential distribution with the density

$$(1) \qquad f(x|\theta,\xi)=\xi e^{-\xi(x-\theta)} I_{[\theta,\infty)}(x)$$

where $-\infty<\theta<\infty$ and $\xi>0$.

We first let $\xi=1$ and introduce the two-sided test for the problem of testing the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0. To compare with the generalised likelihood-ratio(GLR) test we constructed it and see that this test is not unbiased.

We also let $\theta=0$ and consider the problem of testing $H_0:\xi=\xi_0$ versus $H_1:\xi\neq\xi_0$ for some constant ξ_0. However, we unfortunately know that our test is not unbiased, but the GLR test is unbiased. We already know that the same thing happens for the problem of testing constantness of the variance of a normal distribution (See [4].).

Let $\doteq$ be the defining property.

§2. Optimal two-sided test for θ.

In this section we deal with the density

$$(2) \qquad f(x|\theta,1) = e^{-(x-\theta)} I_{[\theta,\infty)}(x)$$

where $f(x|\theta,\xi)$ is defined by (1). Let $X_1, \ldots, X_n$ be a random sample of size n taken from $f(x|\theta,1)$. We consider the problem of testing the hypothesis H_0: $\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$. To construct the test we first find the shortest R. I. for θ using the Lagrange's method and then construct the acceptance region derived from inverting this R. I. for θ_0.

To get an estimate for θ we take $Y \doteq \bar{X}-1$ where $\bar{X} \doteq \Sigma_{i=1}^{n} X_i/n$. We can easily check $E(Y)=\theta$. Finding the joint density of variables $W=X_1+\ldots+X_n$, $Z_1=X_1$, ..., $Z_{n-1}=X_{n-1}$ and taking the marginal density $g_W(w|\theta)$ of W we obtain

$$g_W(w|\theta)=(\Gamma(n))^{-1}(w-n\theta)^{n-1}e^{-(w-n\theta)}I_{[n\theta,\infty)}(w).$$

Noticing $Y=n^{-1}W-1$ we get the density of Y as follows:

$$h_Y(y|\theta)=g_W(n(y+1)|\theta)n$$

$$=(\Gamma(n))^{-1}n^n(y+1-\theta)^{n-1}e^{-n(y+1-\theta)}I_{[\theta-1,\infty)}(y).$$

Furthermore, letting $t=y+1-\theta$ we have the density of T so that

$$(3)\qquad h_T(t)=(\Gamma(n))^{-1}n^n t^{n-1}e^{-nt}I_{[0,\infty)}(t)$$

which is the gamma distribution with parameters n and n.

Let α be a real number so that $0<\alpha<1$. Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the shortest R. I. for θ at precision coefficient $1-\alpha$ we want to minimize r_2-r_1 subject to

$$(4)\qquad P_\theta[r_1<Y-\theta<r_2] = 1-\alpha.$$

But, it follows by a variable transformation $t=y+1-\theta$ that

$$(5)\qquad \text{the left hand side of (4)}=P_\theta[r_1+1<T<r_2+1]=1-\alpha.$$

Hence, we want to minimize t_2-t_1 with $t_i=r_i+1$ $(i=1,2)$ subject to the condition (5). To do so we use the Lagrange's method. Let γ be a real number and define

$$L \doteq L(t_1,t_2;\gamma) \doteq t_2-t_1-\gamma\{\int_{t_1}^{t_2} h_T(t)\,dt-1+\alpha\}$$

where $h_T(t)$ is defined by (3). Then, we have that

(6) $$\begin{cases} \partial L/\partial t_1 = -1+\gamma h_T(t_1)=0 \\ \partial L/\partial t_2 = 1-\gamma h_T(t_2)=0 \end{cases}$$

By (6) we get

(7) $$h_T(t_1)=h_T(t_2)(=\gamma^{-1}).$$

Taking t_1 and t_2 which satisfy (7) and noticing that $t_1<T=Y+1-\theta<t_2$ we obtain the shortest R. I. $(Y+1-t_2, Y+1-t_1)$ for θ.

Therefore, our test is to reject H_0 if $Y\leq\theta_0+t_1-1$ or $Y\geq\theta_0+t_2-1$ and to accept H_0 if $\theta_0+t_1-1<Y<\theta_0+t_2-1$. By using a test function we can write this test by

$$\phi(y)=\begin{cases} 1, & \text{if } y\leq\theta_0+t_1-1 \text{ or } \theta_0+t_2-1\leq y \\ 0, & \text{if } \theta_0+t_1-1<y<\theta_0+t_2-1. \end{cases}$$

To check unbiasedness of this test we obtain the power function as follows:

$$\pi(\theta)\doteq E_\theta(\phi(Y))=1-\int_{\theta_0+t_1-1}^{\theta_0+t_2-1} h_Y(y|\theta)\,dy = 1-\int_{\theta_0-\theta+t_1}^{\theta_0-\theta+t_2} h_T(t)\,dt$$

where the last equality follows by a variable transformation $t=y+1-\theta$. Then, we have

$$\begin{aligned} d\pi(\theta)/d\theta\big|_{\theta=\theta_0} &= h_T(\theta_0-\theta+t_2)-h_T(\theta_0-\theta+t_1)\big|_{\theta=\theta_0} \\ &= h_T(t_2)-h_T(t_1)=0 \end{aligned}$$

where the last equality holds because of (7). Therefore, our test ϕ is unbiased.

In the next section we construct the GLR test and see that this test is not unbiased.

§3. The GLR test for θ.

In this section we again consider the problem of testing $H_0:\theta=\theta_0$ versus $H_1: \theta\neq\theta_0$ and $\xi=1$. We take a random sample $X_1, \dots, X_n$ from $f(x|\theta,1)$ given by (2). Let $X_{(1)}$ be the smallest observation among $X_1, \dots, X_n$. We can easily check that $Y\doteq X_{(1)}$ is the maximum likelihood estimator for θ. Hence, by letting $L(\theta)=\prod_{i=1}^{n} f(x_i|\theta,1)$, the GLR test is to reject H_0 if

$$L(\theta_0)/L(y)= e^{-n(y-\theta_0)} I_{[\theta_0,\infty)}(y) \leq k$$

where $0<k<1$. Since the p.d.f. of Y is given by

$$g_Y(y|\theta)=ne^{-n(y-\theta)} I_{[\theta,\infty)}(y),$$

the GLR test is defined by using the test function as follows:

$$\phi(y) = \begin{cases} 1, & \text{if } y\leq\theta_0 \text{ or } \lambda_k\leq y \\ 0, & \text{if } \theta_0<y<\lambda_k \end{cases}$$

where λ_k is determined by the equality

$$\int_{\theta_0}^{\lambda_k} g_Y(y|\theta_0)\, dy = 1-\alpha.$$

To check the unbiasedness of this test we define the power function as follows:

$$\pi(\theta)\doteq E_\theta(\phi(Y))=1-\int_{\theta_0}^{\lambda_k} g_Y(y|\theta)\, dy$$

$$=1-\int_{n(\theta_0-\theta)}^{n(\lambda_k-\theta)} e^{-t} I_{[0,\infty)}(t)\, dt$$

where the last equality follows by a variable transformation $t=n(y-\theta)$. Hence, $d\gamma(\theta)/d\theta|_{\theta=\theta_0}\neq 0$ because $0<\lambda_k-\theta_0<+\infty$. Therefore, the GLR test is not unbiased.

The results in Sections 2 and 3 are not true for the problem of testing the proportional parameter ξ of the exponential distribution. We shall see this in next two sections.

§4. Two-sided test for ξ.

In this section we deal with the density

(8) $$f(x|0,\xi)=\xi e^{-\xi x}I_{[0,\infty)}(x)$$

where $\xi>0$. Let $X_1, \ldots ,X_n$ be a random sample of size n taken from $f(x|0,\xi)$. We consider the problem of testing the hypothesis $H_0:\xi=\xi_0$ versus the alternative hypothesis $H_1:\xi\neq\xi_0$ for some constant ξ_0. To find our test we use the maximum likelihood estimator $Y\doteq n/\Sigma_{i=1}^n X_i$ for ξ.

To derive the density of Y we use the density of $Z\doteq\Sigma_{i=1}^n X_i$ defined by

(9) $$g_Z(z|\xi)\doteq(\Gamma(n))^{-1}\xi^n z^{n-1}e^{-\xi z}I_{[0,\infty)}(z).$$

By a variable transformation $Y\doteq n/Z$ the density of Y is derived as follows:

$$h_Y(y|\xi)=g_Z(ny^{-1}|\xi)|ny^{-2}|$$

$$=(\Gamma(n))^{-1}n^n\xi^n y^{-(n+1)}e^{-n\xi/y}I_{[0,\infty)}(y).$$

Again by a variable transformation $V=\xi Y^{-1}$ it follows that

(10) $$h_V(v)=h_Y(\xi v^{-1}|\xi)|\xi v^{-2}|$$

$$=(\Gamma(n))^{-1}n^n v^{n-1}e^{-nv}I_{[0,\infty)}(v)$$

which is the gamma distribution with parameters n and n.

Let α be a real number such that $0<\alpha<1$. Let v_1 and v_2 be real numbers so that $v_1<v_2$. To find the shortest R. I. for θ at precision coefficient $1-\alpha$ we want to minimize v_2-v_1 subject to

$$P_\theta[v_1Y<\xi<v_2Y]=P[v_1<V<v_2]=1-\alpha.$$

To do so we use the Lagrange's method. Let γ be a real number and define

$$L\doteq L(v_1,v_2;\gamma)\doteq v_2-v_1-\gamma\{\int_{v_1}^{v_2} h_V(v)\,dv-1+\alpha\},$$

where $h_V(v)$ is defined by (10). The Lagrange's method leads to

$$(11)\qquad \begin{cases} \partial L/\partial v_1 = -1+\gamma h_V(v_1)=0 \\ \partial L/\partial v_2 = 1-\gamma h_V(v_2)=0 \end{cases}$$

By (11) we get

$$(12)\qquad h_V(v_1)=h_V(v_2)\,(=\gamma^{-1}).$$

Taking v_1 and v_2 which satisfy (12) we obtain the shortest R. I. (v_1Y, v_2Y) for ξ.

Therefore, our test is to reject H_0 if $Y\leq\xi_0v_2^{-1}$ or $\xi_0v_1^{-1}\leq Y$ and to accept H_0 if $\xi_0v_2^{-1}<Y<\xi_0v_1^{-1}$. By using a test function we can write

$$\phi(y)=\begin{cases} 1, & \text{if } y\leq\xi_0v_2^{-1} \text{ or } \xi_0v_1^{-1}\leq y \\ 0, & \text{if } \xi_0v_2^{-1}<y<\xi_0v_1^{-1}. \end{cases}$$

To check the unbiasedness of this test we obtain the power function as follows:

$$\pi(\xi)\doteq E_\xi(\phi(Y))=1-\int_{\xi_0v_2^{-1}}^{\xi_0v_1^{-1}} h_Y(y|\xi)\,dy=1-\int_{\xi v_1/\xi_0}^{\xi v_2/\xi_0} h_V(v)\,dv$$

where the last equality holds by a variable transformation $v=\xi y^{-1}$. But, since we have

$$d\pi(\xi)/d\xi\big|_{\xi=\xi_0} = -v_2\xi_0^{-1}h_V(\xi v_2/\xi_0)+v_1\xi_0^{-1}h_V(\xi v_1/\xi_0)\big|_{\xi=\xi_0}$$

$$= -v_2\xi_0^{-1}h_V(v_2)+v_1\xi_0^{-1}h_V(v_1) \neq 0,$$

unbiasedness of $\pi(\xi)$ does not hold.

In the next section we construct the GLR test.

§5. The GLR test for ξ.

In this section we consider the same problem as that in Section 4. Let $X_1, \ldots, X_n$ be a random sample of size n taken from $f(x|0,\xi)$ given by (8). The maximum likelihood estimator for ξ is $\hat{\xi}=n/\Sigma_{i=1}^{n}X_i$. Hence, by letting $L(\xi)=\prod_{i=1}^{n}f(x_i|0,\xi)$ and $Y=(\xi_0\Sigma_{i=1}^{n}X_i)/n$ the GLR test is to reject H_0 if

$$L(\xi_0)/L(\hat{\xi})=y^n e^{-n(y-1)} \le k$$

for some k with $0<k<1$. Letting $\zeta(y)=y^n e^{-n(y-1)}$ we see that there exist y_1 and y_2 such that $\zeta(y_1)=\zeta(y_2)=k$. Thus, the GLR test is to reject H_0 if $Y\le y_1$ or $y_2\le Y$ and to accept H_0 if $y_1<Y<y_2$.

We check unbiasedness of this test. Since the distribution of $Z=\Sigma_{i=1}^{n}X_i$ is given by (9), by a variable transformation $Z=nY\xi_0^{-1}$ applied to the second equality below the power function of this test is given by

$$\pi(\xi)=1-P_\xi[y_1<Y<y_2]=1-\int_{ny_1/\xi_0}^{ny_2/\xi_0} g_Z(z|\xi)\,dz=1-\int_{ny_1\xi/\xi_0}^{ny_2\xi/\xi_0} h_U(u)\,du \tag{13}$$

where the last equality of (13) follows by a variable transformation $U=\xi Z$ and

$$h_U(u)=(\Gamma(n))^{-1}u^{n-1}e^{-u}I_{[0,\infty)}(u).$$

Hence, it follows that

$$d\pi(\xi)/d\xi\big|_{\xi=\xi_0} = -ny_2\xi_0^{-1}h_U(ny_2\xi/\xi_0)+ny_1\xi_0^{-1}h_U(ny_1\xi/\xi_0)\big|_{\xi=\xi_0}$$

$$= -ny_2\xi_0^{-1}h_U(ny_2)+ny_1\xi_0^{-1}h_U(ny_1)$$

$$= -n^n\{\Gamma(n)\xi_0 e^n\}^{-1}(\zeta(y_2)-\zeta(y_1))=0$$

where the last equality follows because $\zeta(y_1)=\zeta(y_2)$. Therefore, the GLR test is unbiased.

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