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Best symmetric two-sided test for the positional parameter of the uniform distribution

by

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Abstract.

Let α be a real number such that $0<\alpha<1$. In this paper we show that the two-sided test obtained in Nogami(2000, §3) has the greatest power among the size-α tests symmetric about θ.

Let α be a real number such that $0<\alpha<1$. In this paper we show that the two-sided test obtained in Nogami(2000, §3) has the greatest power among size-α tests symmetric about θ.

We shall prove the following theorem:

Theorem. Let $X_1, \ldots, X_n$ be a random sample from the p.d.f. $f(x|\theta)=c^{-1}$, for $\theta+\delta_1<x<\theta+\delta_2$; $=0$ otherwise where $\delta_1<\delta_2$ and $c=\delta_2-\delta_1$. The test ϕ^* given by (9) with θ_0 some constant for testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ has the greateat power among size-α tests symmetric about θ.

Proof.) Let $X_{(i)}$ be the i-th smallest observation such that $X_{(1)}\leq X_{(2)}\leq \cdots \leq X_{(n)}$. Let $Y=(X_{(1)}+X_{(n)}-\delta_0)/2$ with $\delta_0=\delta_1+\delta_2$. As we have seen in Nogami(2000) the p.d.f. of Y is given by

$$g(y|\theta)=nc^{-n}(c-2|y-\theta|)^{n-1}I_{(-c/2,\, c/2)}(y-\theta)$$

where for a set A $I_A(x)=1$ for $x\in A$; $=0$ for $x\notin A$.

Let ϕ be a size-α test symmetric about θ (namely, $\phi(y)=\phi(2\theta-y), \forall y$). Then, it follows that

$$(1) \qquad E_\theta(\phi(Y))=\alpha$$

and

$$(2) \qquad E_\theta(Y\phi(Y))=\theta E_\theta(\phi(Y))=\theta\alpha.$$

Above (2) holds because $E_\theta((Y-\theta)\phi(Y))=0$ and (1) holds.

Hence, by generalized Neyman-Pearson Lemma ϕ maximizes the integral

$$\int \phi(y)\, g(y|\theta')\, dy \qquad \text{for } \theta'\neq\theta_0$$

out of all functions ϕ, $0\leq\phi\leq 1$, satisfying (1) and (2), when there exist real constants k_1 and k_2 such that

$$(3)\qquad \phi(y)=\begin{cases}1, & \text{if } (c-2|y-\theta'|)^{n-1}I_{(-c/2,\,c/2)}(y-\theta') \\ & \geq (k_2+k_1y)(c-2|y-\theta_0|)^{n-1}I_{(-c/2,\,c/2)}(y-\theta_0) \\ 0, & \text{if } (c-2|y-\theta'|)^{n-1}I_{(-c/2,\,c/2)}(y-\theta')\end{cases}$$

$$(4)\qquad < (k_2+k_1y)(c-2|y-\theta_0|)^{n-1}I_{(-c/2,\,c/2)}(y-\theta_0).$$

We check the existence of such k_1 and k_2 and show that the test ϕ is of form (8). We first check the existence of such k_1 and k_2 until the fifth line below (7).

When $\theta'<\theta_0-2^{-1}c$ or $\theta_0+2^{-1}c<\theta'$, we take $k_1=0$ and $k_2=1$. Let $\theta'<\theta_0-2^{-1}c$. Let $a\vee b$ be the maximum of a and b. Then, the inequality (4) holds for $(\theta_0-2^{-1}c)\vee((\theta_0+\theta')/2)<y<\theta_0+2^{-1}c$ and the inequality (3) holds otherwise. Let $\theta_0+2^{-1}c<\theta'$. Let $a\wedge b$ be the minimum of a and b. Then, (4) holds for $\theta_0-2^{-1}c<y<((\theta_0+\theta')/2)\wedge(\theta_0+2^{-1}c)$ and (3) holds otherwise.

Let $\theta_0<\theta'<\theta_0+2^{-1}c$. Then, for $\theta_0-2^{-1}c<y<\theta'-2^{-1}c$, the inequality (4) holds when $k_1=0$ and $k_2=1$. On the other hand, for $\theta_0+2^{-1}c<y<\theta'+2^{-1}c$, the inequality (3) always holds for any real k_1 and k_2.

Let $\theta_0-2^{-1}c<\theta'<\theta_0$. Then, for $\theta'-2^{-1}c<y<\theta_0-2^{-1}c$, the inequality (3) always holds for any real k_1 and k_2. On the other hand, for $\theta'+2^{-1}c<y<\theta_0+2^{-1}c$, the inequality (4) holds when $k_1=0$ and $k_2=1$.

Henceforth, it is enough to consider the y's in $(\theta'-2^{-1}c,\theta_0+2^{-1}c)$ for $\theta_0<$ $<\theta'<\theta_0+2^{-1}c$ or in $(\theta_0-2^{-1}c,\theta'+2^{-1}c)$ for $\theta_0-2^{-1}c<\theta'<\theta_0$. We let

$$h(y)=(c-2|y-\theta'|)/(c-2|y-\theta_0|)\quad(\geq 0)$$

and

$$z(y)=(k_2+k_1y)^{1/(n-1)}\qquad(\geq 0).$$

We also let

(5) $y_0=-k_2/k_1$.

For $\theta_0<\theta'<\theta_0+2^{-1}c$, take y_0 such that $\theta'-2^{-1}c<y_0<\theta_0$. For $\theta_0-2^{-1}c<\theta'<\theta_0$, take y_0 such that $\theta_0<y_0<\theta'+2^{-1}c$. Let p be a given number such that $0<p<1$. Take $y_0=(1-p)\theta_0+p(\theta'-2^{-1}c)$ for $\theta_0<\theta'<\theta_0+2^{-1}c$ and take $y_0=(1-p)\theta_0+p(\theta'+2^{-1}c)$ for $\theta_0-2^{-1}c<\theta'<\theta_0$. Then, from (5), k_2 is taken as follows:

$$(6)\qquad k_2=\begin{cases}-k_1\{(1-p)\theta_0+p(\theta'-2^{-1}c)\}, & \text{for } \theta_0<\theta'<\theta_0+2^{-1}c\\ -k_1\{(1-p)\theta_0+p(\theta'+2^{-1}c)\}, & \text{for } \theta_0-2^{-1}c<\theta'<\theta_0.\end{cases}$$

Substituting these values into z(y) we have

$$(7)\qquad z(y)=\begin{cases}[k_1\{y-((1-p)\theta_0+p(\theta'-2^{-1}c))\}]^{1/(n-1)}, & \text{for } \theta_0<\theta'<\theta_0+2^{-1}c,\\ [k_1\{y-((1-p)\theta_0+p(\theta'+2^{-1}c))\}]^{1/(n-1)}, & \text{for } \theta_0-2^{-1}c<\theta'<\theta_0\end{cases}$$

which are drawn by stripe lines in the figure below. Since we must accept H_0 for $y=\theta_0$, we must have $h(\theta_0)<z(\theta_0)$. We take $k_1=2(pc)^{-1}$ for $\theta_0<\theta'<\theta_0+2^{-1}c$ and $k_1=-2(pc)^{-1}$ for $\theta_0-2^{-1}c<\theta'<\theta_0$. Then, $(z(\theta_0))^{n-1}=h(\theta_0)$. Hence, we have $h(\theta_0)<z(\theta_0)$ because $|\theta_0-\theta'|<2^{-1}c$. k_2 is obtained by substituting these values of k_1 into (6).

To show that the test ϕ is of form (8) we check the existence of two intersection points of h(y) and z(y). Since for $\theta_0<\theta'<\theta_0+2^{-1}c$

$$h(y)=\begin{cases}1-\{2(\theta'-\theta_0)/(2y+c-2\theta_0)\}, & (\theta'-2^{-1}c<y\leq\theta_0)\\ -1-\{2(c-(\theta'-\theta_0))/(2y-c-2\theta_0)\}, & (\theta_0<y\leq\theta')\\ 1-\{2(\theta'-\theta_0)/(2y-c-2\theta_0)\}, & (\theta'<y<\theta_0+2^{-1}c),\end{cases}$$

h(y) is an increasing function for $\theta'-2^{-1}c<y<\theta_0+2^{-1}c$.
Since for $\theta_0+2^{-1}c<\theta'<\theta_0$

$$h(y)=\begin{cases} 1+\{2(\theta_0-\theta')/(2y+c-2\theta_0)\}, & (\theta_0-2^{-1}c<y\leq\theta') \\ -1+\{2c-2(\theta_0-\theta')\}/(2y+c-2\theta_0), & (\theta'<y\leq\theta_0) \\ 1+\{2(\theta_0-\theta')/(2y-c-2\theta_0)\}, & (\theta_0<y<\theta'+2^{-1}c), \end{cases}$$

$h(y)$ is a decreasing function for $\theta_0-2^{-1}c<y<\theta'+2^{-1}c$. On the other hand, when $\theta_0<\theta'<\theta_0+2^{-1}c$ $dz(y)/dy>0$ for all $y>y_0$ and when $\theta_0-2^{-1}c<\theta'<\theta_0$ $dz(y)/dy<0$ for all $y<y_0$. Since $0=z(y_0)<h(y_0)(<1)$ and since for $\theta_0<\theta'<\theta_0+2^{-1}c$ $z((\theta_0+2^{-1}c)-)<\lim_{y\to(\theta_0+2^{-1}c)-}h(y)=+\infty$ and for $\theta_0-2^{-1}c<\theta'<\theta_0$ $z((\theta_0-2^{-1}c)+)<\lim_{y\to(\theta_0-2^{-1}c)+}h(y)=+\infty$, in view of the fact that $h(\theta_0)<z(\theta_0)$ there must exist two intersection points of $h(y)$ and $z(y)$ for $\theta_0<\theta'<\theta_0+2^{-1}c$ and for $\theta_0-2^{-1}c<\theta'<\theta_0$, respectively. (See Figure.)

Let y_1 and y_2 be such y-coordinates of these intersection points with $y_1<y_2$. Then, we finally have the optimal test of form

$$(8)\qquad \phi(y)=\begin{cases} 1, & \text{if } y\leq y_1 \text{ or } y\geq y_2 \\ 0, & \text{if } y_1<y<y_2. \end{cases}$$

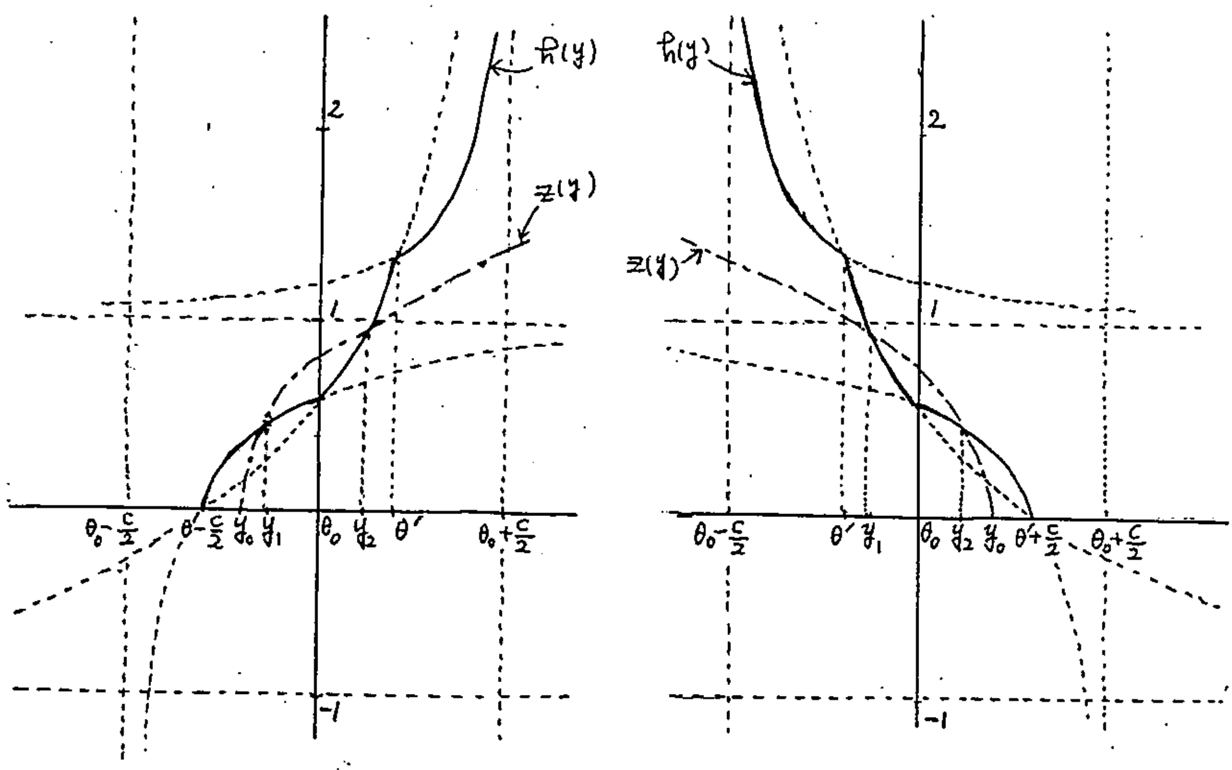
Hence, the test in Nogami(2000, §3) provided by

$$(9)\qquad \phi^*(y)=\begin{cases} 1, & \text{if } y\leq\theta_0-r \text{ or } y\geq\theta_0+r \\ 0, & \text{if } \theta_0-r<y<\theta_0+r, \end{cases}$$

where $r=c(1-\alpha^{1/n})/2$, is the size-α test with the greatest power among size-α tests symmetric about θ.

Figure. The graphs of h(y) and z(y)

($\theta_0 < \theta' < \theta_0 + 2^{-1}c$) ($\theta_0 - 2^{-1}c < \theta' < \theta_0$)



(q. e. d.)

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