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by

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Abstract

We deal with interior point methods (IPMs) for solving a class of so-called $P_*(\kappa)$ complementarity problems (CPs). First of all, several elementary results about $P_*(\kappa)$ mappings and $P_*(\kappa)$ CPs are presented. Then we extend some notions introduced recently by Peng, Roos and Terlaky [22] for linear optimization problems to the case of CPs. New large-update IPMs for solving CPs are introduced based on the so-called *self-regular* proximities. To build up the complexity of these new algorithms, we impose a new smoothness condition on the underlying mapping and this condition can be viewed as a natural generalization of the *relative Lipschitz* condition for convex programs introduced by Jarre [6]. By utilizing various appealing properties of *self-regular* proximities, we will show that if the undertaken problem satisfies certain conditions, then these new large-update IPMs for solving CPs have polynomial $O\left(n^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right)$ iteration bounds where q is a constant, the so-called barrier degree of the corresponding proximity.

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1 Introduction

We consider the standard nonlinear complementarity problem(CP):

$$\text{(CP) Find } (x,s) \quad \text{such that } \; s = f(x),\; (x,s) \geq 0,\; xs = 0.$$

Here f is a continuous mapping from $\Re^n$ to $\Re^n$ and xs denotes the componentwise product of the vectors x and s. To be more specific, we also call it a linear complementarity problem (LCP) if the involved mapping $f(x)$ is affine, i.e., $f(x) = Mx + c$ for some $M \in \Re^{n\times n}, c \in \Re^n$; otherwise we call it a nonlinear complementarity problem (NCP) when $f(x)$ is nonlinear.

CPs have a broad range of associations with different areas. First, at its infancy in 1960s, the LCP was closely associated with linear and quadratic optimization (LO and QO) problems. Now it is known that CPs cover fairly general classes of mathematical programming problems with versatile applications in engineering, economics, and science. For instance, by exploiting the first-order optimality conditions of the underlying optimization problem, any general convex optimization problem satisfying certain constraint qualifications (e.g., Slater constraint qualification [14]) can be modeled as a monotone CP. Closely related to CPs is a large class of problems: Variational Inequality Problems (VIPs) that are widely used in the study of equilibrium in, e.g., economics, transportation planning and game theory. As a result of its wide association with optimization and equilibrium problems, the study on CPs has attracted much attention from many researchers in different fields such as operations research, mathematics, computer science, economics and engineering for long since its introduction. Several monographs [2, 5] and surveys [3, 4, 19] have documented the basic theory, algorithms and applications of NCPs and their role in optimization theory. It is worthwhile to mention that many classical numerical algorithms for solving CPs are based on approaches for optimization problems or equation systems.

Besides its many meaningful applications, CPs contain one common feature that is crucial to the study of general mathematical and equilibrium programming problems. This is the concept of complementarity. Actually, the concept of complementarity plays an important role in the design and analysis of numerical algorithms, particularly IPMs for solving large classes of problems.

Since Karmarkar's epoch-making paper [10], the study of IPMs has flourished and thousands of papers have been published about IPMs. At the early stage of research on IPMs, people focused mainly on algorithms for LO and QO. Due to the close connection between LCPs and LO and QO, IPMs for LCPs were soon suggested as a direct extension of primal-dual IPMs for LO. According to records, the first IPM for LCPs was proposed by Kojima, Mizuno and Yoshise [13] and their algorithm was originated in the primal-dual IPMs for LO. Later Kojima, Megiddo, Noma and Yoshise [11] set up a framework of IPMs for tracing the central path of a class of LCPs. Independent of the works by this Japanese group, Monteiro and Alder [16] also proposed an IPM for convex quadratic optimization problems which could indeed be applied to monotone LCPs. Since then, the study of IPMs for CPs has paralleled to that for LO. A general and unified analysis about path-following methods for VIPs and CPs was given by Nesterov and Nemirovskii in [18]. The survey by Yoshise [24] gave a comprehensive review about the major developments in IPMs for CPs and listed lots of available references up to that time.

Let us briefly describe how an IPM works for CPs. First let us consider the following relaxed system of CP

$$s = f(x), s > 0, x > 0, xs = \mu e,$$

where μ is a positive constant and e denotes the all-one vector. It has been shown [11, 15] that the above system has a unique solution $(x(\mu), s(\mu))$ if the considered CP satisfies certain conditions. This solution set forms a path as μ goes to zero which is called the central path. Most of IPMs for CPs follow this path appropriately and approximate the solution set of the problem as μ reduces to zero.

To trace the central path approximately, various strategies have been introduced to keep the iterative sequence staying in a certain neighborhood of the central path as well as reducing the parameter μ. These strategies have played an important role both in the analysis and practice of IPMs. It is worth to point out that there are two general strategies used in IPMs with respect to the update of the parameter μ. These are the so-called small-update and large-update IPMs. It has been proven and generally accepted that the worst-case iteration bound of small-update IPMs is better than that for large-update IPMs while the later ones perform much more efficiently in practice (see the discussion in the introduction of [22]). This is a big gap between the theory and practice of IPMs.

Recently, the first three authors of this paper introduced the concept of *self-regular* functions in the positive orthant and the cone of positive definite matrices [22] as well as *self-regular* proximities which are used in IPMs to keep control on the distance of an iterative sequence to the central path and define the corresponding search directions. By using some new analysis tools developed in [21, 22] and employing new search directions, we were able to show that new large-update IPMs for LO have polynomial $\mathcal{O}\left(n^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right)$ iteration bounds where q is a constant, the so-called barrier degree of the proximity. This is a significant improvement over the known $\mathcal{O}\left(n \log \frac{n}{\epsilon}\right)$ iteration bound of large-update IPMs before.

The present work aims at extending the results of [22] to large classes of CPs. As we will see in our later analysis, this is far from a trivial task. The reason for this is that, the convergence rate of IPMs has been established only for classes of problems which satisfy certain Lipschitz conditions such as the self-concordant condition posed by Nesterov and Nemirovskii [18], the relative Lipschitz condition introduced by Jarre [6, 7] and the scaled Lipschitz condition by Zhu [27]. For CPs, Jansen [8], Jansen et'al [9] introduced a smoothness condition which can be viewed as a straightforward extension of the scaled Lipschitz condition. In this paper, to establish the complexity of our algorithm, we will introduce a new smoothness condition for the considered problem. This new condition can be regarded as a generalization of Jarre's condition. Via using extensively the properties of *self-regular* proximities, we will prove that if the considered CP satisfies several assumptions, then our new large-update IPMs have polynomial $\mathcal{O}\left(n^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right)$ iteration bound.

The class of CPs we will discuss in this paper is the class of so-called $P_*(\kappa)$ CPs. It is worth to mention that there exists some inconsistency about the definition of a nonlinear $P_*(\kappa)$ mapping in the CP literature. For instance, in some references (see [25, 26]) a $P_*(\kappa)$ mapping is defined according to certain specific properties of the mapping itself while in some other references such as [9], it is required that the Jacobian of the considered mapping to be a $P_*(\kappa)$ matrix. The reason for these different definitions is that in the study of some properties such as the feasibility of the problem, the definition based on the mapping itself is more direct and more applicable, while in the estimation of the complexity of IPMs for CPs, the Jacobian matrix plays a much more important role. In this paper, we will consider this issue first and show that these different definitions are equivalent if the undertaken mapping $f(x)$ is continuously differentiable.

An important ingredient in IPMs for CPs is the existence of the central path, since otherwise we can't apply IPMs to the problem. For $P_*(\kappa)$ LCPs, Kojima et'al [12] had already proven that if a $P_*(\kappa)$ LCP is strictly feasible, then the central path is uniquely defined and converges to the solution set of the problem. We also mention that in [11], the authors considered the homotopy path for several classes of CPs under certain assumptions. However they did not specify their results to nonlinear $P_*(\kappa)$ CPs. Slightly to our surprise, during the preparation of this paper, the authors also noted that albeit there have already quite a number of papers [9] dealing with IPMs for $P_*(\kappa)$ CPs, none of them discussed explicitly the existence of the central path for nonlinear $P_*(\kappa)$ CPs. In the present paper, we will discuss this question for general nonlinear $P_*(\kappa)$ CPs under certain assumption. Our results is a direct extension of those in [12] for $P_*(\kappa)$ LCPs.

The paper is organized as follows. In Section 2, we will first state some assumptions about the problem and then give several fundamental results about $P_*(\kappa)$ mappings and CPs. In Section 3 we describe the new algorithm based on a *self-regular* proximity and introduce a new smoothness condition for the underlying mapping $f(x)$. Section 4 is devoted to study the complexity of the algorithm. Finally we close this paper by some concluding remarks in Section 5.

2 Preliminary Results on $P_*(\kappa)$ Mappings and $P_*(\kappa)$ CPs

In the present section, we will first state some basic assumptions about the considered class of CPs and give several elementary results about CPs under those assumptions. We start with some basic definitions of classes of matrices[12].

Definition 2.1 *Let κ be a nonnegative constant. A matrix $M \in \Re^{n\times n}$ is said to be a $P_*(\kappa)$ matrix if and only if there holds*

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+(x)} x_i[Mx]_i + \sum_{i\in\mathcal{I}_-(x)} x_i[Mx]_i \geq 0, \quad \forall x \in \Re^n,$$

where

$$\mathcal{I}_+(x) = \{i \in \mathcal{I} : x_i[Mx]_i \geq 0\}, \quad \mathcal{I}_-(x) = \{i \in \mathcal{I} : x_i[Mx]_i < 0\};$$

and

$$\mathcal{I} = \{1, 2, \cdots, n\}.$$

We remark that the index sets $\mathcal{I}_+(x)$ and $\mathcal{I}_-(x)$ depend not only on $x \in \Re^n$ but also the matrix M. The class of $P_*(\kappa)$ matrices includes as specific case the class of positive semidefinite matrices where the constant $\kappa = 0$.

We denote by P_* the union of all $P_*(\kappa)$ matrices with $\kappa \geq 0$. We next introduce the definitions of P and P_0 matrices [2].

Definition 2.2 *A matrix $M \in \Re^{n\times n}$ is said to be a P (or P_0) matrix if and only if for any $x \neq 0 \in \Re^n$, there exists at least one index $i \in \mathcal{I}$ such that $x_i(Mx)_i > 0$ (or $x_i(Mx)_i \geq 0$).*

From the above definition, one can easily see that $P \subset P_*(\kappa) \subset P_* \subset P_0$. For more discussion about the relations among the class of $P_*(\kappa)$ matrices and other classes of matrices we refer to [11].

The following technical result about P and P_0 matrices will be used in our later discussion. We have

Lemma 2.3 *If M is a P matrix, then there exists a vector x such that*

$$Mx > 0, \quad x > 0;$$

If M is a P_0 matrix, then there exists a nonzero vector x such that

$$Mx \geq 0, \quad x \geq 0.$$

Proof: The first statement of the lemma is precisely the same as Corollary 3.3.5 in [2], thus its proof is omitted here. To prove the second statement of the lemma, we observe that if M is a P_0 matrix, then the matrix $M + \epsilon E$ is a P matrix, here we denote by E the identity matrix in $\Re^{n\times n}$. Thus, from the first statement of the lemma, we know that for any $\epsilon > 0$, there exists a vector $x_\epsilon > 0$ with $\|x_\epsilon\| = 1$ such that

$$Mx_\epsilon > 0, \quad x_\epsilon > 0.$$

Therefore, there must exist an accumulation point x^* of the sequence x_{ϵ_k} as ϵ_k reduces to to zero. By taking limits if necessary, one can see that

$$Mx^* \geq 0, \quad x^* \geq 0,$$

which completes the proof of the second result of the lemma. □

We next progress to define the notion of a $P_*(\kappa)$ mapping.

Definition 2.4 *Let κ be a nonnegative constant. A mapping $f(x) : \Re^n \to \Re^n$ is said to be a $P_*(\kappa)$-mapping if for any $x \neq y \in \Re^n$, the relation*

$$(1+4\kappa) \sum_{i\in\mathcal{I}_+^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i\in\mathcal{I}_-^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \geq 0, \qquad (1)$$

holds, where

$$\begin{aligned} \mathcal{I}_+^f(x,y) &= \{i \in \mathcal{I} : (x_i - y_i)(f_i(x) - f_i(y)) \geq 0\}; \\ \mathcal{I}_-^f(x,y) &= \{i \in \mathcal{I} : (x_i - y_i)(f_i(x) - f_i(y)) < 0\}. \end{aligned}$$

The mapping $f(x)$ is said to be a strict $P_*(\kappa)$ mapping if inequality (1) holds strictly for any $x \neq y \in \Re^n$.

It follows directly from the above two definitions that if $f(x) = Mx + q$, then $f(x)$ is a $P_*(\kappa)$ mapping if and only if its Jacobian matrix M is a $P_*(\kappa)$ matrix. In the sequel we consider an extension of this observation in case that $f(x)$ is nonlinear and continuously differentiable. We proceed by introducing a specific subclass of $P_*(\kappa)$ mappings.

Definition 2.5 *Let κ be a nonnegative constant. A mapping $f(x) : \Re^n \to \Re^n$ is said to be a $P_*(\kappa, \beta)$ mapping if for any $x \neq y \in \Re^n$, the relation*

$$(1+4\kappa) \sum_{i\in\mathcal{I}_+^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i\in\mathcal{I}_-^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \geq \beta \|x - y\|^2$$

holds, where

$$\begin{aligned} \mathcal{I}_+^f(x,y) &= \{i \in \mathcal{I} : (x_i - y_i)(f_i(x) - f_i(y)) \geq 0\}; \\ \mathcal{I}_-^f(x,y) &= \{i \in \mathcal{I} : (x_i - y_i)(f_i(x) - f_i(y)) < 0\}. \end{aligned}$$

From this definition it follows immediately that a $P_*(\kappa, \beta)$ mapping with $\beta > 0$ is a strict $P_*(\kappa)$ mapping. Similarly, we also define

Definition 2.6 *Suppose that κ is a nonnegative constant. A matrix M is said to be a $P_*(\kappa, \beta)$ matrix if*

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+(x)} x_i (Mx)_i + \sum_{i \in \mathcal{I}_-(x)} x_i (Mx)_i \geq \beta \|x\|^2, \quad \forall x \neq 0 \in \Re^n$$

holds, where

$$\begin{aligned} \mathcal{I}_+(x) &= \{i \in \mathcal{I} : x_i (Mx)_i \geq 0\}; \\ \mathcal{I}_-(x) &= \{i \in \mathcal{I} : x_i (Mx)_i < 0\}. \end{aligned}$$

Our next result characterizes the interrelation between $P_*(\kappa)$ and $P_*(\kappa, \beta)$ mappings. We have

Lemma 2.7 *Let κ be a nonnegative constant. Then a mapping $f(x) : \Re^n \to \Re^n$ is a $P_*(\kappa)$ mapping if and only if for any positive $\beta > 0$, the mapping $f_\beta(x) = f(x) + \beta x$ is a $P_*(\kappa, \beta)$ mapping.*

Proof: The necessary part of the lemma is trivial. Since if $f(x)$ is a $P_*(\kappa)$ mapping with $\kappa \geq 0$, then for any $x, y \in \Re^n$ we know that the set $\mathcal{I}_+^f(x,y)$ is nonempty. Further, it is easy to see that, for any $\beta > 0$, the inclusions $\mathcal{I}_+^f(x,y) \subseteq \mathcal{I}_+^{f_\beta}(x,y)$ and $\mathcal{I}_-^{f_\beta}(x,y) \subseteq \mathcal{I}_-^f(x,y)$ hold. Therefore it follows directly

$$\begin{aligned} &(1+4\kappa) \sum_{i \in \mathcal{I}_+^{f_\beta}(x,y)} (x_i - y_i)\left((f_\beta)_i(x) - (f_\beta)_i(y)\right) + \sum_{i \in \mathcal{I}_-^{f_\beta}(x,y)} (x_i - y_i)\left((f_\beta)_i(x) - (f_\beta)_i(y)\right) \\ &\geq (1+4\kappa) \sum_{i \in \mathcal{I}_+^{f}(x,y)} (x_i - y_i)\left((f_\beta)_i(x) - (f_\beta)_i(y)\right) + \sum_{i \in \mathcal{I}_-^{f}(x,y)} (x_i - y_i)\left((f_\beta)_i(x) - (f_\beta)_i(y)\right) \\ &\geq \beta \|x-y\|^2 + (1+4\kappa) \sum_{i \in \mathcal{I}_+^{f}(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right) + \sum_{i \in \mathcal{I}_-^{f}(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right) \\ &\geq \beta \|x-y\|^2, \end{aligned}$$

where the first two inequalities follow from the assumption that $\kappa \geq 0$ and the fact that $\mathcal{I}_+^{f_\beta}(x,y)$ is nonempty, and the last inequality is given by the definition of $P_*(\kappa)$ mapping.

To prove the sufficient part of the lemma, let us assume that $f_\beta(x)$ is a $P_*(\kappa, \beta)$ mapping for any sufficiently small $\beta > 0$. Suppose that the statement of the lemma is false, i.e., $f(x)$ is not a $P_*(\kappa)$ mapping. Then, from Definition 2.4, we deduce that there exist $x, y \in \Re^n$ such that

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+^{f}(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right) + \sum_{i \in \mathcal{I}_-^{f}(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right) < 0.$$

Let us denote

$$\begin{aligned} \beta_0 &= \frac{(1+4\kappa)\sum_{i \in \mathcal{I}_+^f(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right) + \sum_{i \in \mathcal{I}_-^f(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right)}{\|x-y\|^2}; \\ \beta_1 &= \max_{i \in \mathcal{I}_-^f(x,y)} \frac{(x_i - y_i)(f_i(x) - f_i(y))}{\|x-y\|^2}. \end{aligned}$$

From their choices, one can easily verify that both β_0 and β_1 are negative. Let

$$\beta_2 = \frac{1}{2}\min\left(\frac{-\beta_0}{1+4\kappa}, -\beta_1\right).$$

Obviously $\beta_2 > 0$ holds. Let us define $f_{\beta_2}(x) = f(x) + \beta_2 x$ for any $x \in \Re^n$. For this specific mapping f_{β_2}, it is straightforward to check that $\mathcal{I}_-^{f_{\beta_2}}(x,y) = \mathcal{I}_-^f(x,y)$ and hence $\mathcal{I}_+^{f_{\beta_2}}(x,y) = \mathcal{I}_+^f(x,y)$. Therefore, one has

$$\begin{aligned}
&(1+4\kappa)\sum_{i\in\mathcal{I}_+^{f_{\beta_2}}(x,y)}(x_i-y_i)\left((f_{\beta_2})_i(x)-(f_{\beta_2})_i(y)\right)+\sum_{i\in\mathcal{I}_-^{f_{\beta_2}}(x,y)}(x_i-y_i)\left((f_{\beta_2})_i(x)-(f_{\beta_2})_i(y)\right)\\
&=(1+4\kappa)\sum_{i\in\mathcal{I}_+^{f}(x,y)}(x_i-y_i)\left((f_{\beta_2})_i(x)-(f_{\beta_2})_i(y)\right)+\sum_{i\in\mathcal{I}_-^{f}(x,y)}(x_i-y_i)\left((f_{\beta_2})_i(x)-(f_{\beta_2})_i(y)\right)\\
&=(1+4\kappa)\sum_{i\in\mathcal{I}_+^{f}(x,y)}(x_i-y_i)\left(f_i(x)-f_i(y)\right)+\sum_{i\in\mathcal{I}_-^{f}(x,y)}(x_i-y_i)\left(f_i(x)-f_i(y)\right)\\
&\quad+(1+4\kappa)\beta_2\sum_{i\in\mathcal{I}_+^{f}(x,y)}(x_i-y_i)^2+\beta_2\sum_{i\in\mathcal{I}_-^{f}(x,y)}(x_i-y_i)^2\\
&\le \beta_0\|x-y\|^2+(1+4\kappa)\beta_2\|x-y\|^2\le\frac{\beta_0}{2}\|x-y\|^2<0,
\end{aligned}$$

where the first inequality is true since $\kappa \geq 0$, and the last inequality follows from the choice of β_2. The above discussion means that the mapping $f_{\beta_2}(x)$ is not a $P_*(\kappa, \beta)$ mapping. This contradicts to our assumption that $f_\beta(x)$ is a $P_*(\kappa,\beta)$ mapping for any positive $\beta > 0$. Thus $f(x)$ must be a $P_*(\kappa)$ mapping which completes the proof of the lemma. □

Note that in the above proof, we indeed show that if $f(x)$ is not a $P_*(\kappa)$ mapping, then there is a sufficiently small $\beta > 0$ such that $f_\beta(x)$ is not a $P_*(\kappa)$ mapping. Since a $P_*(\kappa,\beta)$ mapping is obviously a $P_*(\kappa)$ mapping, thus we obtain readily the following corollary.

Corollary 2.8 *Let κ be a nonnegative constant. Then a mapping $f(x) : \Re^n \to \Re^n$ is a $P_*(\kappa)$ mapping if and only if for any positive $\beta > 0$, the mapping $f_\beta(x) = f(x) + \beta x$ is a $P_*(\kappa)$ mapping.*

One can prove the following results for $P_*(\kappa)$ and $P_*(\kappa,\beta)$ matrices similarly, by specifying the mapping $f(x)$ to $f(x) = Mx + q$.

Corollary 2.9 *Let κ be a nonnegative constant. Then a matrix $M \in \Re^{n\times n}$ is a $P_*(\kappa)$ matrix if and only if for any positive $\beta > 0$, the matrix $M + \beta E$ is a $P_*(\kappa)$ (or $P_*(\kappa,\beta)$) matrix.*

We progress to present some relations between a differentiable $P_*(\kappa)$ mapping and its Jacobian matrix $\nabla f(x)$. One has

Lemma 2.10 *Suppose that $f(x) : \Re^n \to \Re^n$ is continuously differentiable and $\kappa \geq 0, \beta > 0$. If $f(x)$ is a $P_*(\kappa)$ (or $P_*(\kappa,\beta)$) mapping, then for any $x \in \Re^n$, $\nabla f(x)$ is a $P_*(\kappa)$ (or $P_*(\kappa,\beta)$) matrix.*

Proof: We consider first the case for $P_*(\kappa,\beta)$ mappings. To prove the statement of the lemma, for any $x, u \in \Re^n$, let us consider a sequence $\left\{x + \frac{1}{j}u : j = 1, 2, \dots\right\}$. Since $f(x)$ is a $P_*(\kappa,\beta)$ mapping, there exist two sequences of index sets $\mathcal{I}_+^j := \mathcal{I}_+^f(x + \frac{1}{j}u, x)$ and $\mathcal{I}_-^j := \mathcal{I}_-^f(x + \frac{1}{j}u, x)$ such that

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+^j} \frac{1}{j} u_i \left(f_i(x + \frac{1}{j}u) - f_i(x) \right) + \sum_{i \in \mathcal{I}_-^j} \frac{1}{j} u_i \left(f_i(x + \frac{1}{j}u) - f_i(x) \right) \geq \frac{\beta}{j^2} \|u\|^2 .$$

By the finiteness of $\mathcal{I}$, there exist two index sets $\mathcal{I}'_+(x,u)$ and $\mathcal{I}'_-(x,u)$ and a subsequence J such that for all $j \in J$, $\mathcal{I}_+^j = \mathcal{I}'_+(x,u)$ and $\mathcal{I}_-^j = \mathcal{I}'_-(x,u)$ hold. Therefore, for any $j \in J$, we have

$$\frac{1}{j} u_i \left(f_i(x + \frac{1}{j}u) - f_i(x) \right) \geq 0, \quad \forall i \in \mathcal{I}'_+(x,u),$$
$$\frac{1}{j} u_i \left(f_i(x + \frac{1}{j}u) - f_i(x) \right) < 0, \quad \forall i \in \mathcal{I}'_-(x,u),$$

and

$$(1+4\kappa) \sum_{i \in \mathcal{I}'_+(x,u)} \frac{1}{j} u_i \left(f_i(x + \frac{1}{j}u) - f_i(x) \right) + \sum_{i \in \mathcal{I}'_-(x,u)} \frac{1}{j} u_i \left(f_i(x + \frac{1}{j}u) - f_i(x) \right) \geq \frac{\beta}{j^2} \|u\|^2 .$$

Taking the limits $j \to \infty$ for $j \in J$, we obtain

$$u_i[\nabla f(x)u]_i \geq 0 \ (i \in \mathcal{I}'_+(x,u)), \quad u_i[\nabla f(x)u]_i \leq 0 \ (i \in \mathcal{I}'_-(x,u))$$

and

$$(1+4\kappa) \sum_{i \in \mathcal{I}'_+(x,u)} u_i[\nabla f(x)u]_i + \sum_{i \in \mathcal{I}'_-(x,u)} u_i[\nabla f(x)u]_i \geq \beta \|u\|^2$$

which implies that $\nabla f(x)$ is a $P_*(\kappa,\beta)$ matrix. The proof for $P_*(\kappa)$ mappings follows similarly. □

In what follows we consider a converse case of the above lemma, namely discuss the properties of a continuously differentiable mapping $f(x)$ under the condition that $\nabla f(x)$ is a $P_*(\kappa,\beta)$ matrix for any $x \in \Re^n$. One has

Lemma 2.11 *Suppose that $f(x) : \Re^n \to \Re^n$ is continuously differentiable and $\kappa \geq 0$. If the Jacobian matrix $\nabla f(x)$ is a $P_*(\kappa,\beta)$ matrix with $\beta > 0$ for any $x \in \Re^n$, then $f(x)$ is a strict $P_*(\kappa)$ mapping.*

Proof: The proof takes a similar recipe as that in [17] for P mapping. For self-completeness, we give a detailed proof here. The proof is inductive. We first observe that the result is trivial if $n = 1$. Hence we can assume hereafter that the statement holds for some $n - 1 \geq 1$.

Suppose that $\nabla f(x)$ is a $P_*(\kappa,\beta)$ matrix for any $x \in \Re^n$. Let us suppose that the statement of the lemma is not true, i.e., there exist two points in $x \neq y \in \Re^n$ such that

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i \in \mathcal{I}_-^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \leq 0. \qquad (2)$$

We first consider the case that there exists some $i \in \mathcal{I}$ such that $x_i = y_i$. For simplicity we can assume that $i = n$ and consider the subfunction

$$h_i(\chi_1, \cdots, \chi_{n-1}) = f_i(\chi_1, \cdots, \chi_{n-1}, y_n), \quad i = 1, \cdots, n-1.$$

Since $\nabla h(\chi_1, \cdots, \chi_{n-1})$ is again a $P_*(\kappa, \beta)$ matrix for any $(\chi_1, \cdots, \chi_{n-1}) \in \Re^{n-1}$, the induction hypothesis implies that h is a strict $P_*(\kappa)$ mapping in $\Re^{n-1}$ and therefore for any $x \neq y \in \Re^n$ with some $x_i = y_i$ for $i \in \mathcal{I}$, there holds

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i \in \mathcal{I}_-^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) > 0. \tag{3}$$

This relation contradicts to (2).

Thus it remains to consider the case that $x_i \neq y_i$ for all $i \in \mathcal{I}$. For any fixed $y \in \Re^n$, let us denote by Ω_y the set given by

$$\{x \in \Re^n : (1+4\kappa) \sum_{i \in \mathcal{I}_+^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i \in \mathcal{I}_-^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \leq 0, x > y\}.$$

We proceed to show that Ω_y is empty. Suppose to the contrary that Ω_y is nonempty. Let us consider any convergent sequence $x^k \in \Omega_y$ with $x^k \to x$. It follows readily $x \geq y$ and

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i \in \mathcal{I}_-^f(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \leq 0.$$

Now we have three cases, namely (i): $x \neq y$ but $x_i = y_i$ for some $i \in \mathcal{I}$; (ii): $x = y$; (iii): $x > y$. The first case (i) is impossible since otherwise from the first part of our proof we already know that inequality (3) holds if there are some $x_i = y_i$ and $x \neq y$. If case (ii) holds, then one has

$$\lim_{k \to \infty} \frac{1}{\|x^k - y\|} \Big(f(x^k) - f(y) - \nabla f(y)(x^k - y) \Big) = 0.$$

Let us denote $\Lambda^k = \text{diag}\,\Big(x_1^k - y_1, x_2^k - y_2, \cdots, x_n^k - y_n \Big)$, it follows

$$\lim_{k \to \infty} \frac{1}{\|x^k - y\|^2} \Lambda^k \Big(f(x^k) - f(y) - \nabla f(y)(x^k - y) \Big) = 0.$$

Observe that the sequence $\{\frac{x^k - y}{\|x^k - y\|}\}$ is bounded and thus has at least an accumulation point. Without loss of generality, we can further assume that

$$\lim_{k \to \infty} \frac{x^k - y}{\|x^k - y\|} = u, \quad \|u\| = 1.$$

Denote

$$\mathcal{I}_-^f(x,u) = \{i \in \mathcal{I} : u_i(\nabla f(y)u)_i < 0\}; \quad \mathcal{I}_+^f(x,u) = \{i \in \mathcal{I} : u_i(\nabla f(y)u)_i \geq 0\}.$$

Then one can easily see that there exists a sufficiently large integer $\tilde{k}$ such that for any $k \geq \tilde{k}$, there hold

$$\mathcal{I}_-^f(x^k, y) \subseteq \mathcal{I}_-^f(x,u); \quad \mathcal{I}_+^f(x^k, y) \supseteq \mathcal{I}_+^f(x,u).$$

Since $\kappa \geq 0$, we have

$$\lim_{k\to\infty} \frac{(1+4\kappa)\sum_{i\in\mathcal{I}_+^f(x^k,y)}(x_i^k - y_i)(f_i(x^k) - f_i(y)) + \sum_{i\in\mathcal{I}_-^f(x^k,y)}(x_i^k - y_i)(f_i(x^k) - f_i(y))}{\|x^k - y\|^2}$$
$$\geq \lim_{k\to\infty} \frac{(1+4\kappa)\sum_{i\in\mathcal{I}_+^f(u,y)}(x_i^k - y_i)(f_i(x^k) - f_i(y)) + \sum_{i\in\mathcal{I}_-^f(u,y)}(x_i^k - y_i)(f_i(x^k) - f_i(y))}{\|x^k - y\|^2}$$
$$= (1+4\kappa) \sum_{i\in\mathcal{I}_+^f(u,y)} u_i(\nabla f(y)u)_i + \sum_{i\in\mathcal{I}_-^f(u,y)} u_i(\nabla f(y)u)_i \geq \beta\|u\|^2 = \beta,$$

where the last inequality is implied by the assumption in the lemma that ∇f is a $P_*(\kappa,\beta)$ matrix with $\beta > 0$. The above relation implies that for sufficiently large k, the inequality

$$(1+4\kappa) \sum_{i\in\mathcal{I}_+^f(x^k,y)} (x_i^k - y_i)(f_i(x^k) - f_i(y)) + \sum_{i\in\mathcal{I}_-^f(x^k,y)} (x_i^k - y_i)(f_i(x^k) - f_i(y)) > 0$$

holds, which contradicts to the assumption $x^k \in \Omega_y$. This implies that y doesn't belong to the boundary of Ω_y. Our above discussion shows the cases (i) and (ii) are impossible. Hence only case (iii) remains to deal with. In this situation, we have $x \in \Omega_y$ which further implies Ω_y is closed. Let us define

$$u = \text{argmin}_{x\in\Omega_y} \|x - y\|. \tag{4}$$

If Ω_y is nonempty, then we know that u is (might not uniquely) well-defined. Moreover, for any u satisfying relation (4), one can easily prove the following conclusion

$$\text{if } x \in \Omega_y \text{ and } x \leq u \quad \Longrightarrow \quad x = u. \tag{5}$$

Since $\nabla f(u)$ is a $P_*(\kappa,\beta)$ matrix and thus a P matrix, by Lemma 2.3 there is a vector $h < 0$ such that $\nabla f(u)h < 0$. It follows immediately

$$\lim_{t\to 0} \frac{1}{t}\left(f(u+th) - f(u)\right) = \nabla f(u)h < 0.$$

Since $y < u \in \Omega_y$, one can choose sufficiently small $t > 0$ such that the relations $u > u + th > y$ and $f(u+th) - f(u) < 0$ hold. From the continuity of $f(x)$ it follows

$$(1+4\kappa) \sum_{i\in\mathcal{I}_+^f(u+th,y)} (u_i + th_i - y_i)\left(f_i(u+th) - f_i(y)\right)$$
$$+ \sum_{i\in\mathcal{I}_-^f(u+th,y)} (u_i + th_i - y_i)\left(f_i(u+th) - f_i(y)\right) < 0.$$

The above discussion means that $y < u + th < u$ and $u + th \in \Omega_y$ for sufficiently small $t > 0$ which contradicts the statement (5). Hence case (iii) can't be true and this further implies that Ω_y is empty.

Now suppose that $x \neq y$ satisfies (2). Then $x_i \neq y_i$ for any $i \in \mathcal{I}$; otherwise it will contradict to the first part of the proof. Denote $\Lambda = \text{diag}\left(\text{sign}\left(x_1 - y_1\right), \cdots, \text{sign}\left(x_n - y_n\right)\right)$, and let $\tilde{f}(x) = \Lambda f(\Lambda x)$. Then for any $x \in \Re^n$, $\nabla \tilde{f}(x)$ is a $P_*(\kappa,\beta)$ matrix since the diagonal matrix Λ is nonsingular. Moreover, by the construction of $\tilde{f}(x)$, the relations $\tilde{x} = \Lambda x > \Lambda y = \tilde{y}$ hold. It follows directly

$$(1+4\kappa) \sum_{i\in\mathcal{I}_+^{\tilde{f}}(\tilde{x},\tilde{y})} (\tilde{x}_i - \tilde{y}_i)\left(\tilde{f}_i(\tilde{x}) - \tilde{f}_i(\tilde{y})\right) + \sum_{i\in\mathcal{I}_-^{\tilde{f}}(\tilde{x},\tilde{y})} (\tilde{x}_i - \tilde{y}_i)\left(\tilde{f}_i(\tilde{x}) - \tilde{f}_i(\tilde{y})\right)$$
$$= (1+4\kappa) \sum_{i\in\mathcal{I}_+^f(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right) + \sum_{i\in\mathcal{I}_-^f(x,y)} (x_i - y_i)\left(f_i(x) - f_i(y)\right) \leq 0,$$

which is a contradiction to the second part of our proof. From our above discussions we have seen that for any $x \neq y \in \Re^n$, the inequality (2) does not hold. Therefore $f(x)$ is a strict $P_*(\kappa)$ mapping. This completes the proof of the lemma. □

Now we are ready to state one of the main results in this section which is a combination of Lemma 2.10 and Lemma 2.11.

Lemma 2.12 *Suppose that $f(x) : \Re^n \to \Re^n$ is continuously differentiable. Then $f(x)$ is a $P_*(\kappa)$ mapping if and only if $\nabla f(x)$ is a $P_*(\kappa)$ matrix for any $x \in \Re^n$.*

Proof: The necessary part of the lemma follows from Lemma 2.10. Hence it remains to prove the sufficient part. Since $\nabla f(x)$ is a $P_*(\kappa)$ matrix for any $x \in \Re^n$, it is trivial to see that for any $\beta > 0$ and $x \in \Re^n$, the matrix $\nabla f(x) + \beta E$ is a $P_*(\kappa, \beta)$ matrix. Therefore, by Lemma 2.11, we deduce that the mapping $f(x) + \beta x$ is a $P_*(\kappa)$ mapping for any $\beta > 0$. Now let us recall Corollary 2.8, one can conclude that $f(x)$ is a $P_*(\kappa)$ mapping. This completes the proof of the lemma. □

Our above lemma clarify some unclear arguments in the definition of $P_*(\kappa)$ mappings in the literature.

In the rest of this section we discuss the existence of the central path for nonlinear $P_*(\kappa)$ CPs. For this we first impose two assumptions on the considered CP which will be used throughout this paper.

A.1 *Interior Point Condition: there exists a known point (x^0, s^0) which satisfies*

$$s^0 = f(x^0), \quad (x^0, s^0) > (0, 0).$$

A.2 *f is a continuously differentiable $P_*(\kappa)$ mapping with $\kappa \geq 0$.*

We remark that these assumptions are quite general and mild assumptions in the IPM literature for CPs. The class of $P_*(\kappa)$ CPs is a rather general class of CPs which covers CPs with P and monotone mappings. In the case of LCPs, it reduces to class of LCPs introduced by [12] which is to date the largest set of LCPs that could be solved by IPMs in polynomial time. Assumption A.1 is generally required in the study of feasible IPMs for CPs. It is worthwhile to point out that for monotone CPs, by using an augmented homogeneous model described by Andersen and Ye in [1], we can always get a strictly feasible point for the reconstructed CP. For $P_*(\kappa)$ LCPs, one can apply the big-$\mathcal{M}$ method introduced in the monograph [12] to get a strictly feasible initial point. However, as observed by Peng, Roos and Terlaky [20], Andersen and Ye's homogeneous model can not be applied to a $P_*(\kappa)$ CP since there is no guarantee that the new formulated CP is still in the class of $P_*(\kappa)$ CPs. We also mention that as shown by Zhao and Li (Theorem 4.2 in [26]), a $P_*(\kappa)$ complementarity problem is strictly feasible if and only if its solution set is nonempty and bounded.

We now progress to show that the central path exists uniquely if the considered CP satisfies Assumptions A.1 and A.2. For this let us first introduce some definitions and notations.

Definition 2.13 *A mapping $f(x) : \Re^n \to \Re^n$ is said to be a P_0 mapping if for every $x, y \in \Re^n$ with $x \neq y$, there exists an index $i \in \mathcal{I}$ such that*

$$x_i - y_i \neq 0 \text{ and } (x_i - y_i)(f_i(x) - f_i(y)) \geq 0.$$

Let $\mathcal{D}$ be a subset of $\Re^{2n}$ and define

$$r(x,s) := s - f(x), \ \ r(\Re^{2n}_{++}) := \{u \in \Re^n : \ u = s - f(x), \ (x,s) \in \Re^{2n}_{++}\},$$
$$F(x,s) := (xs, r(x,s)), \ \ F^{-1}(\mathcal{D}) := \{(x,s) \in \Re^{2n}_{+} : F(x,s) \in \mathcal{D}\}.$$

In the paper [11], the authors showed that the central path exists if the CP satisfies A.1 and the conditions below:

#A.1 *The set $F^{-1}(\mathcal{D})$ is bounded for every compact subset $\mathcal{D}$ of $\Re^n_+ \times r(\Re^{2n}_{++})$.*

#A.2 *f is a P_0 mapping.*

Since a $P_*(\kappa)$ mapping is obviously a P_0 mapping, the condition #A.2 is implied by A.2. In what follows we will show that the condition #A.1 holds under the assumptions A.1 and A.2.

Lemma 2.14 *If a CP satisfies the conditions A.1 and A.2 then the condition #A.1 holds as well.*

Proof: The proof is very similar to Section 3 of [11], for self-completeness, we write it out here. Suppose that the set $F^{-1}(\mathcal{D})$ is unbounded for a compact subset $\mathcal{D}$ of $\Re^n_+ \times r(\Re^{2n}_{++})$. Then, we can take a sequence $\{(x^k, s^k) : \ k = 1,2,\ldots\} \subset \Re^{2n}_+$ such that $\lim_{k\to\infty} \|(x^k, s^k)\| = \infty$ and $\lim_{k\to\infty}(s^k - f(x^k)) = \bar{u} \in \mathcal{D}$. Since $r(\Re^{2n}_{++})$ is an open subset of $\Re^n$, we can find a vector $\tilde{u} \in r(\Re^{2n}_{++})$ such that $s^k - f(x^k) \geq \tilde{u}$ for every sufficiently large k. In addition, the definition of $r(\Re^{2n}_{++})$ ensures the existence of an $(\tilde{x}, \tilde{s}) \in \Re^{2n}_{++}$ satisfying $\tilde{s} - f(\tilde{x}) = \tilde{u}$. Because the set $\{(x^k s^k, s^k - f(x^k)) : \ k = 1,2,\ldots\} \subset \mathcal{D}$ is bounded, we can find positive numbers η and ζ such that the following inequalities

$$(1+4\kappa) \sum_{i\in\mathcal{I}^f_+(x^k,\tilde{x})} x_i^k s_i^k + \sum_{i\in\mathcal{I}^f_-(x^k,\tilde{x})} x_i^k s_i^k \leq (1+4\kappa)n \max_{i\in\mathcal{I}}\{x_i^k s_i^k\} \leq \eta$$

and

$$(1+4\kappa) \sum_{i\in\mathcal{I}^f_+(x^k,\tilde{x})} \tilde{x}_i \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right) + \sum_{i\in\mathcal{I}^f_-(x^k,\tilde{x})} \tilde{x}_i \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right)$$
$$\leq (1+4\kappa)n \max_{i\in\mathcal{I}} \left\{\tilde{x}_i \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right)\right\} \leq \zeta$$

hold. Since $\tilde{s} - \tilde{u} = f(\tilde{x})$, by a simple calculation, we have

$$\begin{aligned}
&(x_i^k - \tilde{x}_i)\left(f_i(x^k) - f_i(\tilde{x})\right) \\
&= (x_i^k - \tilde{x}_i)\left(s_i^k - (s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i)\right) \\
&= x_i^k s_i^k - \tilde{x}_i s_i^k - x_i^k \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right) + \tilde{x}_i \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right)
\end{aligned}$$

for each $i \in \mathcal{I}$. Using the facts $x^k \geq 0$ and $s^k - f(x^k) - \tilde{u} \geq 0$, for each $i \in \mathcal{I}$, we deduce

$$\begin{aligned}
&x_i^k s_i^k - \tilde{x}_i s_i^k - x_i^k \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right) + \tilde{x}_i \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right) \\
&\leq x_i^k s_i^k - \tilde{x}_i s_i^k - x_i^k \tilde{s}_i + \tilde{x}_i \left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right).
\end{aligned}$$

From the above observations it follows directly that

$$
\begin{aligned}
0 &\le (1+4\kappa) \sum_{i\in\mathcal{I}^f_+(x^k,\tilde{x})} (x_i^k - \tilde{x}_i)\left(f_i(x^k) - f_i(\tilde{x})\right) + \sum_{i\in\mathcal{I}^f_-(x^k,\tilde{x})} (x_i^k - \tilde{x}_i)\left(f_i(x^k) - f_i(\tilde{x})\right) \\
&\le (1+4\kappa) \sum_{i\in\mathcal{I}^f_+(x^k,\tilde{x})} \left\{x_i^k s_i^k - \tilde{x}_i s_i^k - x_i^k \tilde{s}_i + \tilde{x}_i\left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right)\right\} \\
&\quad + \sum_{i\in\mathcal{I}^f_-(x^k,\tilde{x})} \left\{x_i^k s_i^k - \tilde{x}_i s_i^k - x_i^k \tilde{s}_i + \tilde{x}_i\left(s_i^k - f_i(x^k) - \tilde{u}_i + \tilde{s}_i\right)\right\} \\
&\le \eta + \zeta \\
&\quad - \left((1+4\kappa) \sum_{i\in\mathcal{I}^f_+(x^k,\tilde{x})} \tilde{x}_i s_i^k + \sum_{i\in\mathcal{I}^f_-(x^k,\tilde{x})} \tilde{x}_i s_i^k\right) \\
&\quad - \left((1+4\kappa) \sum_{i\in\mathcal{I}^f_+(x^k,\tilde{x})} x_i^k \tilde{s}_i + \sum_{i\in\mathcal{I}^f_-(x^k,\tilde{x})} x_i^k \tilde{s}_i\right) \\
&\le \eta + \zeta - \left(\tilde{x}^T s^k + (x^k)^T \tilde{s}\right),
\end{aligned}
$$

and hence

$$\tilde{x}^T s^k + (x^k)^T \tilde{s} \le \eta + \zeta$$

for every sufficiently large k. Since $(x^k, s^k) \in \Re^{2n}_+ (k = 1, 2, \ldots)$, the above inequality implies that the sequence $\{(x^k, s^k)\}$ lies in the bounded set $\{(x, s) \in \Re^{2n}_+ :\ \tilde{x}^T s + x^T \tilde{s} \le \eta + \zeta\}$. This contradicts the assumption $\lim_{k\to\infty} \|(x^k, s^k)\| = \infty$ and the proof is completed. □

The following result is a direct consequence of Lemma 2.14 and Lemma 4.2 in [11].

Proposition 2.15 *Suppose that a CP satisfies assumptions A.1 and A.2. Then the central path of the underlying CP exists.*

3 New Interior-Point Methods for $P_*(\kappa)$ CPs

In the present section we introduce some new IPMs for solving $P_*(\kappa)$ CPs. These new IPMs are based on the so-called *self-regular* functions and *self-regular* proximities introduced in [22]. We start with the basic definition of a univariate *self-regular* function.

Definition 3.1 *A univariate function $\psi(t) : \Re_{++} \to \Re_{++}$ is said to be* self-regular *if it satisfies the following two conditions:*

C.1 *$\psi(t)$ is strongly convex with respect to $t > 0$ and vanishes at its global minimal point $t = 1$, i.e., $\psi(1) = \psi'(1) = 0$. Further, there exist positive constants $\nu_1, \nu_2 > 0$ and $p \ge 1$, $q \ge 1$ such that*

$$\nu_1(t^{p-1} + t^{-1-q}) \le \psi''(t) \le \nu_2(t^{p-1} + t^{-1-q}),\ \forall t \in (0, \infty). \tag{6}$$

C.2 *For any $t_1, t_2 > 0$,*

$$\psi(t_1 t_2) \le \frac{1}{2}(\psi(t_1^2) + \psi(t_2^2)). \tag{7}$$

Here the parameters p and q are called the *growth degree* and the *barrier degree* of the function $\psi(t)$, respectively. A typical family of *self-regular* functions is given by

$$\Upsilon_{p,q}(t) = \frac{1}{p(p+1)}\left(t^{p+1}-1\right) + \frac{1}{q(q-1)}\left(t^{1-q}-1\right) + \frac{p-q}{pq}(t-1), \quad p,q \geq 1. \tag{8}$$

It is worth to mention that the function $\Upsilon_{p,q}(t)$ satisfies condition C.1 with $\nu_1 = \nu_2 = 1$.

To facilitate our discussion about new IPMs, we need to introduce more notations. First we remind the reader that whenever no confusion is possible, for convenience we will use capital syllables to denote the diagonal matrix obtained from a vector; for instance $D = \mathrm{diag}\,(d)$. For every $(x,s) > 0$ and $\mu > 0$, we define

$$\begin{cases} v & := \sqrt{\frac{xs}{\mu}}, \\ v_{\min} & := \min\{v_i : i \in \mathcal{I}\}, \\ v_{\max} & := \max\{v_i : i \in \mathcal{I}\}. \end{cases} \tag{9}$$

As in [22], we define the proximity for CP by

$$\Psi(xs,\mu) := \Psi(v) = \sum_{i=1}^{n} \psi(v_i). \tag{10}$$

Correspondingly we say the proximity $\Psi(v)$ is *self-regular* if its kernel function $\psi(t)$ is *self-regular*. Let us define

$$\sigma := \|\nabla\Psi(v)\|. \tag{11}$$

The following proposition collects some basic features of the function $\Psi(v)$ which is a copy of Proposition 3.3 in [22]. For ease of reference, we quote it here without proof.

Proposition 3.2 *Let the proximity $\Psi(v)$ be defined by (10). Then there holds*

$$\Psi(v) \leq \frac{\sigma^2}{2\nu_1}, \tag{12}$$

$$v_{\min} \geq \left(1 + \frac{q\sigma}{\nu_1}\right)^{-\frac{1}{q}}, \tag{13}$$

and

$$v_{\max} \leq \left(1 + \frac{p\sigma}{\nu_1}\right)^{\frac{1}{p}}. \tag{14}$$

If $v_{\max} > 1$ and $v_{\min} < 1$, then

$$\sigma \geq \nu_1 \left(\frac{(v_{\max}^p - 1)^2}{p^2} + \frac{(v_{\min}^{-q} - 1)^2}{q^2}\right)^{\frac{1}{2}}. \tag{15}$$

For any $\vartheta > 1$,

$$\Psi(\vartheta v) \leq \frac{\nu_2}{\nu_1}\left(\vartheta^{p+1}\Psi(v) + \vartheta\Upsilon'_{p,q}(\vartheta)\sqrt{2n\nu_1\Psi(v)} + n\nu_1\Upsilon_{p,q}(\vartheta)\right). \tag{16}$$

Remark 3.4 *In the algorithm we always assume that* $v_{\max} > 1$*. This is because when* $v_{\max} \leq 1$*, we can reduce the value of the proximity in the algorithm (or stay in a certain neighborhood of the central path) by appropriately reducing* μ*. In such case we even do not need to solve the Newton-type system.*

4 Complexity of the algorithm

This section is devoted to estimating the complexity of the algorithm. The section consists of three parts. In the first subsection, we will present some bounds for the norm of the search direction and the maximal feasible step size. In the second subsection we estimate the decrement of the proximity for a feasible step size. Finally, we summarize the complexity of the algorithm in the last subsection.

4.1 Ingredients for estimating the proximity

In this section, we provide certain ingredients that are used for estimating the proximity. We start by introducing some notations. For each $\alpha > 0$ and $\chi \in \Re^n$, let us define

$$\Delta s(\alpha) := \frac{1}{\alpha}(f(x+\alpha\Delta x) - f(x)), \tag{23}$$

$$d_x := \frac{v}{x}\Delta x, \qquad d_s := \frac{v}{s}\Delta s, \tag{24}$$

$$d_s(\alpha) := \frac{v}{s}\Delta s(\alpha), \tag{25}$$

$$\nabla d_s(\alpha) := \frac{1}{\alpha}\left(\frac{v}{s}\nabla f(x+\alpha\Delta x)\Delta x\right), \tag{26}$$

$$\nabla d_s(\chi) := ([\nabla d_s(\chi_1)]_1, [\nabla d_s(\chi_2)]_2, \ldots, [\nabla d_s(\chi_n)]_n)^T \tag{27}$$

$$\text{or equivalently } [\nabla d_s(\chi)]_i := \frac{1}{\chi_i}\left(\frac{v_i}{s_i}\nabla f_i(x+\chi_i\Delta x)\Delta x\right).$$

Note that the functions $\Delta s(\alpha)$ and $d_s(\alpha)$ are not defined at $\alpha = 0$. However, one can easily see that these two definitions can be extended to the case $\alpha = 0$ as

$$\Delta s(0) := \lim_{\alpha\to 0}\Delta s(\alpha) = \nabla f(x)\Delta x; \quad d_s(0) := \lim_{\alpha\to 0} d_s(\alpha) = \frac{v}{s}\nabla f(x)\Delta x. \tag{28}$$

It should be noticed that, by using the notations introduced by (24), we can rewrite the system (20) - (21) as

$$-\Lambda_f d_x + d_s = 0, \qquad d_x + d_s = -\nabla\Psi(v) \tag{29}$$

where $\Lambda_f := \mu V S^{-1}\nabla f(x) V S^{-1}$. Note that the choice of $\Psi(v)$ is completely independent of the mapping $f(x)$. Therefore, in the rest of the paper, we assume that $\Psi(v)$ is *self-regular.* The following lemma about $P_*(\kappa)$ matrices is precisely the same as Lemma 3.4 in [12], we copy it here for purpose of ease reference.

Lemma 4.1 *A matrix* M *is a* $P_*(\kappa)$ *matrix if and only if for any positive definite diagonal matrix* Λ*, and any* $\Delta x, \Delta s, h \in \Re^n$*, the relations*

$$\Lambda^{-1}\Delta x + \Lambda\Delta s = h; \quad \Delta s = M\Delta x$$

always imply

$$\Delta x^T\Delta s \geq -\kappa\|h\|^2.$$

Now let us recall definition (11) of σ. By using the above lemma and following an analogous discussion as that in the proof of Lemma 3.1 in [9], one can readily obtain the following results which present some bounds for the search direction in various scaled spaces.

Lemma 4.2 *Suppose that Assumption A.2 holds. Let $(\Delta x, \Delta s)$ be the unique solution of the system (20)-(21) and (d_x, d_s) be the corresponding solution of the system (29) in the scaled v-space. Then we have*

(i) $-\kappa\sigma^2 \leq \frac{\Delta x^T \Delta s}{\mu} = d_x^T d_s \leq \frac{1}{4}\sigma^2$,

(ii) $\|d_x d_s\|_\infty = \frac{1}{\mu}\|\Delta x \Delta s\|_\infty \leq \frac{1}{4}(1+\kappa)\sigma^2$,

(iii) $\|d_x\|^2 + \|d_s\|^2 = \|d_x + d_s\|^2 - 2d_x^T d_s \leq (1+2\kappa)\sigma^2$,

(iv) $\|x^{-1}\Delta x\| = \|v^{-1}d_x\| \leq \frac{1}{v_{\min}}\|d_x\| \leq \frac{\sqrt{1+2\kappa}}{v_{\min}}\sigma$,

(v) $\|s^{-1}\Delta s\| = \|v^{-1}d_s\| \leq \frac{1}{v_{\min}}\|d_s\| \leq \frac{\sqrt{1+2\kappa}}{v_{\min}}\sigma$.

Let us define

$$\hat{\alpha} := \min\left(1, \frac{v_{\min}}{\sigma\sqrt{1+2\kappa}}\right). \tag{30}$$

It follows from result (iv) of Lemma 4.2 that $x + \alpha\Delta x > 0$ for all $\alpha \in [0, \hat{\alpha})$. In light of the definition of $\nabla d_s(\alpha)$ in (26), $\nabla d_s(\alpha)$ can be represented by

$$\nabla d_s(\alpha) = \frac{1}{\alpha}\frac{\partial}{\partial\alpha}(\alpha d_s(\alpha)) \tag{31}$$

and the equation (18) in Assumption A.3 can be expressed as

$$\|\chi\nabla d_s(\chi) - d_s\| \leq \mathcal{L}\,\|\chi\|_\infty\,\|d_s\|. \tag{32}$$

The following result follows directly from the the above observations.

Lemma 4.3 *Suppose that Assumption A.3 holds. Then*

(i) $\|\chi\nabla d_s(\chi)\| \leq (1+\alpha\mathcal{L})\|d_s\| \leq (1+\mathcal{L})\|d_s\|$;

(ii) $\|\chi d_s(\chi)\| \leq \alpha(1+\alpha\mathcal{L})\|d_s\| \leq \alpha(1+\mathcal{L})\|d_s\|$

for every $\alpha \in [0,\hat{\alpha}), \hat{\alpha} \leq 1$ and every vector χ satisfying $0 \leq \chi \leq \alpha e$,

Proof: Assertion (i) directly follows from (32) and the relation that $\alpha \leq \hat{\alpha} \leq 1$. To confirm assertion (ii), we observe that for any $i \in \mathcal{I}$, via using Taylor's series expansion, one can deduce

$$\begin{aligned}
[\chi d_s(\chi)]_i &= \frac{v_i}{s_i}(f_i(x+\chi_i\Delta x) - f_i(x)) \\
&= \frac{v_i}{s_i}(f_i(x) + \chi_i\nabla f_i(x+\chi_i'\Delta x)\Delta x - f_i(x)) \\
&= \frac{v_i}{s_i}\chi_i\nabla f_i(x+\chi_i'\Delta x)\Delta x \\
&= \chi_i[\chi'\nabla d_s(\chi')]_i,
\end{aligned}$$

where $\chi_i' \in (0, \chi_i)$. Since $\chi_i' < \chi_i \le \alpha$ for all $i \in \mathcal{I}$, it follows from (i) that

$$\begin{aligned}\|\chi d_s(\chi)\| &= \|\chi(\chi'\nabla d_s(\chi'))\| \le \|\chi\|_\infty \|\chi'\nabla d_s(\chi')\| \\ &\le \alpha\|\chi'\nabla d_s(\chi')\| \le \alpha(1+\alpha\mathcal{L})\|d_s\|.\end{aligned}$$

This completes the proof of Assertion (ii). □

Note that the constant $\hat{\alpha}$ has already provided a lower bound for a step size to keep the feasibility of $x(\alpha) = x + \alpha\Delta x$. However, we do not know whether for all $\alpha \in (0, \hat{\alpha})$, the displacement $s(\alpha)$ is strictly feasible as well. In what follows we will estimate the growth behavior of the norm of $s(\alpha)$ for all $\alpha \in (0, \hat{\alpha})$. This further gives a lower bound for a strictly feasible step size for both $x(\alpha)$ and $s(\alpha)$. By combining Lemma 4.2, Lemma 4.3 and Proposition 3.2 together, we obtain the following result:

Lemma 4.4 *Suppose that Assumptions A.2 and A.3 hold and that the function $\Psi(v)$ given by (10) is* self-regular. *Then for any $\alpha \in (0, \hat{\alpha})$, there holds*

$$\|(x^{-1}\Delta x, s^{-1}\Delta s(\alpha))\| = \|(v^{-1}d_x, v^{-1}d_s(\alpha))\| \le \bar{\alpha}^{-1} \le \nu_3\sigma\left(1+\frac{q\sigma}{\nu_1}\right)^{\frac{1}{q}}$$

where

$$\bar{\alpha} := \frac{\hat{\alpha}}{1+\mathcal{L}}; \quad \nu_3 := \sqrt{1+2\kappa}(1+\mathcal{L}). \tag{33}$$

Further, the maximal step size $\alpha_{\max} \ge \bar{\alpha}$.

Proof: By the definition (24), we see that

$$\begin{aligned}\|(x^{-1}\Delta x, s^{-1}\Delta s)\| &= \|(v^{-1}d_x, v^{-1}d_s)\| \\ &\le \frac{1}{v_{\min}}\sqrt{\|d_x\|^2+\|d_s\|^2} \\ &\le \frac{\sigma\sqrt{1+2\kappa}}{v_{\min}} \\ &\le \sigma\left(1+\frac{q\sigma}{\nu_1}\right)^{\frac{1}{q}}\sqrt{1+2\kappa}\end{aligned}$$

where the second and third inequalities follow from (iii) of Lemma 4.2 and Proposition 3.2, respectively. Since

$$\|d_s(\alpha)\| \le (1+\mathcal{L})\|d_s\|$$

for every $\alpha \in (0, \hat{\alpha})$, from result (ii) of Lemma 4.3,Lemma 4.2 and Proposition 3.2 it follows that

$$\begin{aligned}\|(x^{-1}\Delta x, s^{-1}\Delta s(\alpha))\| &= \|(v^{-1}d_x, v^{-1}d_s(\alpha))\| \\ &\le \frac{1}{v_{\min}}\sqrt{\|d_x\|^2+\|d_s(\alpha)\|^2} \\ &\le \frac{1}{v_{\min}}\sqrt{\|d_x\|^2+(1+\mathcal{L})^2\|d_s\|^2} \\ &\le \frac{1}{v_{\min}}(1+\mathcal{L})\sqrt{\|d_x\|^2+\|d_s\|^2} \\ &\le \nu_3\cdot\frac{\sigma}{v_{\min}} \\ &\le \nu_3\,\sigma\left(1+\frac{q\sigma}{\nu_1}\right)^{\frac{1}{q}}\end{aligned}$$

for every $\alpha \in (0, \hat{\alpha})$. Note that a step size α is feasible if and only if both $x + \alpha\Delta x \geq 0$ and $s + \alpha\Delta s(\alpha) \geq 0$ hold. This gives the last statement of the lemma. □

Note that because $\mathcal{L} \geq 0$, obviously $\bar{\alpha} \leq \hat{\alpha}$ holds.

4.2 Estimate of the proximity after a step

We are going to estimate the decrement of the proximity for a feasible step size. First let us define

$$v(\alpha) := \sqrt{\frac{x(\alpha)s(\alpha)}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s(\alpha))} = \sqrt{v^2(e + \alpha v^{-1} d_x)(e + \alpha v^{-1} d_s(\alpha))} \tag{34}$$

Since the proximity after one feasible step is defined by $\Psi(v(\alpha))$, to estimate the decrement of the proximity for a step size α, it suffices to consider the gap of the proximities before and after one step which is defined as a function of the step size α:

$$\Delta\Psi(\alpha) := \Psi(v(\alpha)) - \Psi(v). \tag{35}$$

Because the function $\Psi(v)$ is *self-regular*, from condition C.2 it follows directly

$$\begin{aligned}
\Delta\Psi(\alpha) &= \sum_{i=1}^{n}(\psi(v_i(\alpha)) - \psi(v_i)) \\
&\leq -\Psi(v) + \frac{1}{2}\sum_{i=1}^{n}(\psi(v_i + \alpha[d_x]_i) + \psi(v_i + \alpha[d_s(\alpha)]_i).
\end{aligned}$$

For any $i \in \mathcal{I}$, let us define

$$\eta_i(\alpha) := \psi(v_i + \alpha[d_s(\alpha)]_i).$$

Obviously

$$\eta_i(\alpha) = \eta_i(0) + \alpha\eta_i'(0) + \int_0^\alpha (\eta_i'(\xi) - \eta_i'(0))d\xi$$

holds. Moreover, by simple calculus and using the notations defined by (25),(26) and the relation (28), one can directly check that the terms in the previous equation can be written as

$$\begin{aligned}
\eta_i(0) &= \psi(v_i); \\
\eta_i'(\alpha) &= \psi'(v_i + \alpha[d_s(\alpha)]_i)[\alpha\nabla d_s(\alpha)]_i \\
&= \psi'(v_i + \alpha[d_s(\alpha)]_i)\left(\frac{v}{s}\nabla f(x + \alpha\Delta x)\Delta x\right); \qquad (36) \\
\eta_i'(0) &= \psi'(v_i)\left(\frac{v}{s}\nabla f(x)\Delta x\right) \\
&= \psi'(v_i)[d_s]_i. \qquad (37)
\end{aligned}$$

Thus we obtain that

$$\psi(v_i + \alpha[d_s(\alpha)]_i) = \psi(v_i) + \alpha\psi'(v_i)[d_s]_i + \int_0^\alpha (\eta_i'(\xi) - \eta_i'(0))d\xi.$$

Similarly, we have

$$\psi(v_i + \alpha[d_x]_i) = \psi(v_i) + \alpha\psi'(v_i)[d_x]_i + \int_0^\alpha (r_i'(\xi) - r_i'(0))d\xi$$

where

$$r_i(\alpha) := \psi(v_i + \alpha d_x).$$

Therefore, from (10), (11) and (29), we conclude that the summation is given by

$$\begin{aligned}\sum_{i=1}^{n}(\psi(v_i+\alpha[d_x]_i)+\psi(v_i+\alpha[d_s(\alpha)]_i)) &= \sum_{i=1}^{n}\psi(v_i)+\alpha\sum_{i=1}^{n}\psi'(v_i)[d_x+d_s]_i+\Delta\Psi_1(\alpha)\\ &= \Psi(v)-\alpha\|\nabla\Psi(v)\|^2+\Delta\Psi_1(\alpha)\\ &= \Psi(v)-\alpha\sigma^2+\Delta\Psi_1(\alpha), \qquad (38)\end{aligned}$$

where

$$\begin{aligned}\Delta\Psi_1(\alpha) &:= \sum_{i=1}^{n}\int_0^{\alpha}(r_i'(\xi)-r_i'(0))d\xi+\sum_{i=1}^{n}\int_0^{\alpha}(\eta_i'(\xi)-\eta_i'(0))d\xi\\ &= \int_o^{\alpha}\left(\sum_{i=1}^{n}(r_i'(\xi)-r_i'(0))+\sum_{i=1}^{n}(\eta_i'(\xi)-\eta_i'(0))\right)d\xi\end{aligned}$$

An important step in the estimation of the value $\Delta\Psi_1(\alpha)$ is to estimate the derivatives $r_i'(\alpha)$ and $\eta_i'(\alpha)$. This is done in the following lemma.

Lemma 4.5 *Suppose that Assumptions A.2 and A.3 hold and that the proximity $\Psi(v)$ is* self-regular. *Then for any $\alpha \in (0, \bar{\alpha})$, we have*

(i)

$$\max_{i\in\mathcal{I}}\{\psi''(v_i+\alpha[d_x]_i),\psi''(v_i+\alpha[d_s(\alpha)]_i)\} \le \nu_2\omega(\alpha)$$

where

$$\omega(\alpha) := (v_{\max}+\alpha\nu_3\sigma)^{p-1}+(v_{\min}-\alpha\nu_3\sigma)^{-q-1}; \qquad (39)$$

(ii)

$$\sum_{i=1}^{n}(r_i'(\alpha)-r_i'(0))+\sum_{i=1}^{n}(\eta_i'(\alpha)-\eta_i'(0)) \le \alpha\nu_2\nu_4\sigma^2\omega(\alpha)$$

where

$$\nu_4 := \nu_3^2+\frac{\mathcal{L}\sqrt{1+2\kappa}}{\nu_2}. \qquad (40)$$

Proof: (i): By Lemma 4.4 and the assumption on α we know that the step size in the lemma satisfies $(v+\alpha[d_x], v+\alpha[d_s(\alpha)]) > 0$. Since the proximity $\Psi(v)$ is *self-regular*, from condition C.1 we obtain

$$\begin{aligned}\psi''(v_i+\alpha[d_x]_i) &\le \nu_2\left((v_i+\alpha[d_x]_i)^{p-1}+(v_i+\alpha[d_x]_i)^{-q-1}\right),\\ \psi''(v_i+\alpha[d_s(\alpha)]_i) &\le \nu_2\left((v_i+\alpha[d_s(\alpha)]_i)^{p-1}+(v_i+\alpha[d_s(\alpha)]_i)^{-q-1}\right).\end{aligned}$$

Note that the result (iii) of Lemma 4.2 and result (ii) of Lemma 4.3 ensure that

$$\alpha\|d_x\| \le \alpha\sigma\sqrt{1+2\kappa}, \quad \alpha\|d_s(\alpha)\| \le \alpha\nu_3\sigma.$$

Combining the above relations together we get the desired assertion (i).

We proceed to consider the assertion (ii). For this we first prove the following inequality

$$\sum_{i=1}^{n}(r_i'(\alpha) - r_i'(0) + \eta_i'(\alpha) - \eta_i'(0)) \leq \alpha \left\{\nu_2\nu_3^2\omega(\alpha) + \sqrt{1+2\kappa}\ \mathcal{L}\right\}\sigma^2. \tag{41}$$

By using (36), (37) and the mean-value theorem [23], we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\ (\eta_i'(\alpha) - \eta_i'(0)) &= \sum_{i=1}^{n}\{(\psi'(v_i + \alpha[d_s(\alpha)]_i)(\alpha[\nabla d_s(\alpha)]_i) - \psi'(v_i)[d_s]_i\} \\
&= \sum_{i=1}^{n}\{\{(\psi'(v_i + \alpha[d_s(\alpha)]_i) - \psi'(v_i)\}(\alpha[\nabla d_s(\alpha)]_i) + \psi'(v_i)(\alpha[\nabla d_s(\alpha)]_i - [d_s]_i)\} \\
&= \sum_{i=1}^{n}\{\alpha\psi''(v_i + \bar{\chi}_i[d_s(\bar{\chi}_i)]_i)(\chi_i[\nabla d_s(\chi_i)]_i)(\alpha[\nabla d_s(\alpha)]_i) + \psi'(v_i)(\alpha[\nabla d_s(\alpha)]_i - [d_s]_i)\}
\end{aligned}$$

for some $0 \leq \chi = (\chi_1, \chi_2, \cdots, \chi_n)^T, \bar{\chi} = (\bar{\chi}_1, \cdots, \bar{\chi}_n)^T \leq \alpha e$. Now, by making use of assertion (i) and applying the Cauchy–Schwarz inequality to the vectors $\psi'(v)$ and $\alpha\nabla d_s(\alpha) - d_s$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\{\alpha\psi''(v_i + \bar{\chi}_i[d_s(\bar{\chi}_i)]_i)(\chi_i[\nabla d_s(\chi_i)]_i)(\alpha[\nabla d_s(\alpha)]_i) + \psi'(v_i)(\alpha[\nabla d_s(\alpha)]_i - [d_s]_i)\} \\
&\quad\leq \alpha\nu_2\omega(\alpha)\sum_{i=1}^{n}|\chi_i[\nabla d_s(\chi_i)]_i| \cdot |\alpha[\nabla d_s(\alpha)]_i| + \sum_{i=1}^{n}|\psi'(v_i)| \cdot |\alpha[\nabla d_s(\alpha)]_i - [d_s]_i| \\
&\quad\leq \alpha\nu_2\omega(\alpha)\ \|\chi[\nabla d_s(\chi)]\| \cdot \|\alpha[\nabla d_s(\alpha)]\| + \|\nabla\Psi(v)\| \cdot \|\alpha\nabla d_s(\alpha) - d_s\|.
\end{aligned} \tag{42}$$

From Assumption A.3 and Lemma 4.3 we conclude that

$$\|\chi[\nabla d_s(\chi)]\| \cdot \|\alpha[\nabla d_s(\alpha)]\| \leq (1+\mathcal{L})^2\|d_s\|^2,$$

and

$$\|\nabla\Psi(v)\| \cdot \|\alpha\nabla d_s(\alpha) - d_s\| \leq \sigma \cdot \alpha\mathcal{L}\|d_s\|.$$

The above two inequalities, combined with (42) further imply

$$\sum_{i=1}^{n}(\eta_i'(\alpha) - \eta_i'(0)) \leq \alpha\nu_2(1+\mathcal{L})^2\omega(\alpha)\|d_s\|^2 + \sigma \cdot \alpha\mathcal{L}\|d_s\|. \tag{43}$$

Similarly, we can prove that

$$\sum_{i=1}^{n}(r_i'(\alpha) - r_i'(0)) \leq \alpha\nu_2\omega(\alpha)\|d_x\|^2. \tag{44}$$

Recalling the inequalities

$$\begin{aligned}
&\|d_x\|^2 + \|d_s\|^2 \leq (1+2\kappa)\sigma^2, \\
&\|d_s\| \leq \sqrt{1+2\kappa}\sigma,
\end{aligned}$$

from Lemma 4.2 and substituting them into (43) and (44), we obtain readily the following relation

$$\begin{aligned}
&\sum_{i=1}^{n}\ (r_i'(\alpha) - r_i'(0) + \eta_i'(\alpha) - \eta_i'(0)) \\
&\quad\leq \alpha\nu_2\omega(\alpha)\|d_x\|^2 + \alpha\nu_2(1+\mathcal{L})^2\omega(\alpha)\|d_s\|^2 + \sigma \cdot \alpha\mathcal{L}\|d_s\| \\
&\quad\leq \alpha\nu_2\nu_3^2\omega(\alpha)\sigma^2 + \alpha\mathcal{L}\sqrt{1+2\kappa}\sigma^2,
\end{aligned}$$

which gives (41). Now let us recall the fact $\omega(\alpha) \geq 1$ whenever $v_{\max} \geq 1$. It follows directly from (41)

$$\sum_{i=1}^{n}(r_i'(\alpha) - r_i'(0)) + \sum_{i=1}^{n}(\eta_i'(\alpha) - \eta_i'(0)) \leq \alpha\nu_2\left(\nu_3^2 + \frac{\sqrt{1+2\kappa\mathcal{L}}}{\nu_2}\right)\omega(\alpha)\sigma^2, \tag{45}$$

which gives assertion (ii). □

We progress to discuss the decreasing behavior of $\Psi(\alpha)$ for a strictly feasible step size α. By (38) and Lemma 4.5, we see that

$$\Delta\Psi_1(\alpha) \leq -\sigma^2\alpha + \nu_2\nu_4\sigma^2\int_0^{\alpha}\zeta\omega(\zeta)d\zeta := \Delta\Psi_2(\alpha). \tag{46}$$

Since the function $\Delta\Psi_2(\alpha)$ is strictly convex and twice differentiable for $a \in [0, \bar{\alpha})$. Further it is easy to see that the function $\Delta\Psi_2(\alpha)$ is decreasing at $\alpha = 0$ and increases as α goes to $\bar{\alpha}$. Therefore, it attains its global minimum at the extreme point $\bar{\alpha}$ or at its unique stationary point α^* which is the unique solution of the following system

$$0 = -\sigma + \nu_2\nu_4\alpha\sigma\left((v_{\max} + \alpha\nu_3\sigma)^p + (v_{\min} - \alpha\nu_3\sigma)^{-q}\right). \tag{47}$$

There exist two cases for the solution α^* of the above system. The first case is $\alpha^* \geq \bar{\alpha}$. In this situation we obtain immediately a lower bound for α^* from Lemma 4.4. In the sequel will focus on the second case and discuss the issue how to estimate the value of α^* when $\alpha^* < \bar{\alpha}$. For this we need some technical results. First we quote a technical lemma from [22]. This lemma is very helpful in our later analysis.

Lemma 4.6 *Suppose that $\beta \in [0, 1]$. Then there holds*

$$(1+t)^{\beta} \leq 1 + \beta t, \qquad \forall\, t \geq 0; \tag{48}$$
$$(1-t)^{\beta} \leq 1 - \beta t, \qquad \forall\, t \in [0, 1]. \tag{49}$$

In the sequel we will give some estimations for the roots of some specific equations. These conclusions have a key role in our later estimation about the stationary point of $\Delta\Psi_2(\alpha)$. We have

Lemma 4.7 *Suppose that p and ρ are two constants satisfying $p > 1$ and $\rho > 0$. Then the following two statements are true.*

(i) If t_* is the unique solution of the equation $t\,(1+t)^{p-1} - \rho = 0$, then the relation

$$t_* \geq \frac{\rho}{1 + (p-1)\,\rho}$$

holds.

(ii) If $t_* \in (0, 1)$ is the unique solution of the equation $t\,(1-t)^{-1-p} = \rho$, then

$$t_* \geq \frac{\rho}{1 + \rho\,(p+1)}$$

holds.

Proof: First we observe that in view of the assumption of the lemma, there holds obviously $t_* > 0$. This further implies

$$(1+t_*)^p = \rho\left(1+\frac{1}{t_*}\right) > 1.$$

It follows

$$\begin{aligned} t_* &= \left(\rho + \frac{\rho}{t_*}\right)^{\frac{1}{p}} - 1 = \left(1 - \frac{\rho+\rho t_* - t_*}{\rho+\rho t_*}\right)^{-\frac{1}{p}} - 1 \\ &\geq \left(1 - \frac{\rho+\rho t_* - t_*}{p\rho + p\rho t_*}\right)^{-1} - 1 = \frac{\rho + \rho t_* - t_*}{(p-1)(\rho+\rho t_*) + t_*} > 0, \end{aligned} \tag{50}$$

where the inequality is given by (49) and $\rho + \rho t_* - t_* > 0$. Let us define

$$\hbar(t) = (1+\rho(p-1))t^2 + (1+\rho(p-1)-\rho)t - \rho.$$

One can verify easily that the equation system $\hbar(t) = 0$ has two roots with

$$t_1 = -1; \quad t_2 = \frac{\rho}{1+\rho(p-1)}.$$

Note that from (50) we obtain

$$\hbar(t_*) > 0.$$

Since we already assume $t_* > 0$, the above relation means immediately

$$t_* \geq t_2 = \frac{\rho}{1+\rho(p-1)},$$

which completes the proof of statement (i) of the lemma.

Now let us switch to statement (ii). Let us define $\bar{t} = \frac{t_*}{1-t_*}$. Then the equation in statement (ii) can be written as

$$\bar{t}(1+\bar{t})^p = \rho.$$

It follows directly from the first statement of the lemma that

$$\bar{t} \geq \frac{\rho}{1+p\rho},$$

which further implies

$$t_* \geq \frac{\rho}{1+\rho(p+1)}.$$

The proof of the lemma is completed. □

Now we are ready to present one of our major results in this section which provides us a lower bound for the global minimizer α^* of $\Delta\Psi_2(\alpha)$ which is equivalent to the unique solution of equation (47).

Lemma 4.8 *Let α^* be the solution of (47). Suppose that $\Psi(v) \geq \nu_1^{-1}$ and $v_{\max} > 1$. Then*

$$\alpha^* \geq \nu_5 \sigma^{-\frac{q+1}{q}} \tag{51}$$

where

$$\nu_5 := \min\left(\frac{\nu_1}{2\nu_2\nu_4(\nu_1+p)+\nu_1\nu_3(p-1)}, \frac{\nu_1^2}{(1+\nu_1)\left(2\nu_2\nu_4(\nu_1+q)+\nu_1\nu_3(q+1)\right)}\right) \tag{52}$$

and ν_3 and ν_4 are defined by (33) and (40) respectively.

Some special cases are as follows:

(i) *If $\kappa = \mathcal{L} = 0$, i.e., the CP is linear and monotone, then $\nu_4 = \nu_3 = 1$ and*

$$\alpha^* \geq \min\left(\frac{\nu_1}{2\nu_2(p+\nu_1)+\nu_1(p-1)}, \frac{\nu_1^2}{(1+\nu_1)\left(2\nu_2(q+\nu_1)+\nu_1(q+1)\right)}\right)\sigma^{-\frac{q+1}{q}}.$$

(ii) *If the proximity $\psi(v)$ used in the algorithm is defined by the function $\psi(t) = \Upsilon_{p,q}(t)$ given by (8) with $\nu_1 = \nu_2 = 1$, then*

$$\alpha^* \geq \min\left(\frac{1}{2\nu_4(p+1)+\nu_3(p-1)}, \frac{1}{2(\nu_3+2\nu_4)(q+1)}\right)\sigma^{-\frac{q+1}{q}}.$$

(iii) *Under both of the assumptions in (i) and (ii),*

$$\alpha^* \geq \min\left(\frac{1}{3p+1}, \frac{1}{6q+6}\right)\sigma^{-\frac{q+1}{q}}.$$

Proof: We first mention that the lemma holds trivially if the solution α^* of the equation system (47) satisfies $\alpha^* \geq \bar{\alpha}$. Thus it suffices to consider the case $\alpha^* < \bar{\alpha}$. In such a situation, let us define

$$w_1(\alpha) = -\frac{\sigma}{2} + \nu_2\nu_4\sigma\alpha(v_{\max} + \alpha\nu_3\sigma)^{p-1}$$

and

$$w_2(\alpha) = -\frac{\sigma}{2} + \nu_2\nu_4\sigma\alpha(v_{\min} - \alpha\nu_3\sigma)^{-q-1}.$$

It is easy to see that both $w_1(\alpha)$ and $w_2(\alpha)$ are increasing functions of α for $\alpha \in [0, \bar{\alpha})$. Let us further denote by α_1^* and α_2^* the roots of the equation systems $w_1(\alpha) = 0$ and $w_2(\alpha) = 0$, respectively. One can readily verify that the equation system $w_1(\alpha) = 0$ can be equivalently written as

$$t_1(1+t_1)^{p-1} = \rho_1,$$

where

$$t_1 = \alpha_1^*\nu_3 v_{\max}^{-1}\sigma, \quad \rho_1 = \frac{\nu_3\sigma}{2\nu_2\nu_4 v_{\max}^p}.$$

It follows from statement (i) of Lemma 4.7 that

$$t_1 \geq \frac{\rho_1}{1+\rho_1(p-1)},$$

which yields immediately

$$\begin{aligned}
\alpha_1^* &\geq \frac{\rho_1 v_{\max}\nu_3^{-1}\sigma^{-1}}{1+\rho_1(p-1)} = \frac{v_{\max}}{2\nu_2\nu_4 v_{\max}^p + (p-1)\nu_3\sigma} \\
&\geq \frac{1}{2\nu_2\nu_4 v_{\max}^p + (p-1)\nu_3\sigma} \geq \frac{\nu_1}{2\nu_2\nu_4(\nu_1+p\sigma)+\nu_1\nu_3(p-1)\sigma} \\
&\geq \frac{\nu_1}{2\nu_2\nu_4(\nu_1+p)+\nu_1\nu_3(p-1)}\sigma^{-1},
\end{aligned}$$

where the first inequality is implied by $v_{\max} \geq 1$, the second inequality is given by (14) and the last inequality follows from the assumption in the lemma that $\sigma \geq 1$.

We next progress to estimate the root α_2^* of equation $w_2(\alpha) = 0$. Note that we can rewrite the equation system $w_2(\alpha_2^*) = 0$ as

$$t_2(1-t_2)^{-q-1} = \rho_2,$$

where

$$t_2 = \alpha_2^* \nu_3 v_{\min}^{-1} \sigma, \quad \rho_2 = \frac{\nu_3 \sigma v_{\min}^q}{2\nu_2 \nu_4}.$$

From statement (ii) of Lemma 4.7 we can conclude

$$t_2 \geq \frac{\rho_2}{1 + \rho_2(q+1)},$$

which further gives

$$\begin{aligned} \alpha_2^* &\geq \frac{\rho_2 \nu_3^{-1} v_{\min} \sigma^{-1}}{1 + \rho_2(q+1)} = \frac{v_{\min}^q}{2\nu_2\nu_4 + \nu_3 \sigma v_{\min}^q (q+1)} v_{\min} \\ &\geq \frac{\nu_1}{2\nu_2\nu_4(\nu_1 + q\sigma) + \sigma \nu_1 \nu_3 (q+1)} v_{\min}, \end{aligned} \tag{53}$$

where the last inequality follows from (13) and the fact that the function $\frac{t}{2\nu_2\nu_4 + \nu_3(q+1)t}$ is increasing with respect to $t > 0$. Further, since $\sigma \geq 1$, by making use of (13) and (48) we obtain

$$v_{\min} \geq \left(1 + \frac{q\sigma}{\nu_1}\right)^{-\frac{1}{q}} \geq \left(1 + \frac{q}{\nu_1}\right)^{-\frac{1}{q}} \sigma^{-\frac{1}{q}} \geq \frac{\nu_1}{1+\nu_1} \sigma^{-\frac{1}{q}}.$$

This relation, joint with (53), yields

$$\begin{aligned} \alpha_2^* &\geq \frac{\nu_1^2}{(1+\nu_1)\left(2\nu_2\nu_4(\nu_1 + q\sigma) + \sigma\nu_1\nu_3(q+1)\right)} \sigma^{-\frac{1}{q}} \\ &\geq \frac{\nu_1^2}{(1+\nu_1)\left(2\nu_2\nu_4(\nu_1 + q) + \nu_1\nu_3(q+1)\right)} \sigma^{-\frac{q+1}{q}}, \end{aligned}$$

where the last inequality is implied by the fact $\sigma \geq 1$.

Since equation (47) is equivalent to

$$w_1(\alpha^*) + w_2(\alpha^*) = 0,$$

α^* should satisfy

$$\alpha^* \geq \min\{\alpha_1^*, \ \alpha_2^*\},$$

and thus the relation (51) follows immediately from the fact $\sigma \geq 1$. By specifying the parameter ν_1, ν_2, ν_3 and ν_4 in various special cases, we easily obtain results (i), (ii) and (iii). □

By a similar discussion as in the proof of Theorems 3.8 of [22], we obtain the following theorem:

Theorem 4.9 *Let the function $\Delta\Psi(\alpha)$ be defined by (35) with $\Delta\Psi(v) \geq \nu_1^{-1}$. Then the step size α^* defined by (51) is feasible. Moreover it holds*

$$\Delta\Psi(\alpha^*) \leq -\frac{\nu_5}{4} \sigma^{\frac{q-1}{2q}} \leq -\frac{\nu_5 \nu_1^{\frac{q-1}{2q}}}{4} \Psi(v)^{\frac{q-1}{2q}}.$$

Here ν_5 is defined by (52). Some special cases are as follows:

(i) *If $\kappa = \mathcal{L} = 0$, i.e., the CP is linear and monotone, then $\nu_4 = \nu_3 = 1$ and*

$$\Delta\Psi(\alpha^*) \leq -\frac{1}{4} \min\left(\frac{\nu_1}{2\nu_2(p+\nu_1) + \nu_1(p-1)}, \frac{\nu_1^2}{(1+\nu_1)\left(2\nu_2(q+\nu_1) + \nu_1(q+1)\right)}\right) \sigma^{\frac{q-1}{2q}}.$$

(ii) *If the proximity $\psi(v)$ used in the algorithm is defined by the function $\psi(t) = \Upsilon_{p,q}(t)$ with $\nu_1 = \nu_2 = 1$, then*

$$\Delta\Psi(\alpha^*) \le -\frac{1}{4}\min\left(\frac{1}{2\nu_4(p+1)+\nu_3(p-1)}, \frac{1}{2(\nu_3+2\nu_4)(q+1)}\right)\sigma^{\frac{q-1}{2q}}.$$

(iii) *Under both of the assumptions in (i) and (ii),*

$$\Delta\Psi(\alpha^*) \le -\frac{1}{4}\min\left(\frac{1}{3p+1}, \frac{1}{6q+6}\right)\sigma^{\frac{q-1}{2q}}.$$

We remark that in Theorem 4.9, we only provide a theoretical guarantee for the dercrement of the proximity with a carefully chosen step size. In the practical implementation of IPMs, one might prefer to use various line search techniques to find a more aggressive step size which reduces the proximity much more than our conservative prediction.

4.3 Complexity of the algorithm for CP

We summarize the complexity of the algorithm in this last subsection. Suppose that the present iterate is in the neighborhood $\mathcal{N}(\tau,\mu)$, i.e., $\Psi(v) \le \tau$. Then the algorithm will update the parameter μ by $\mu := (1-\theta)\mu$. Note that after such a update, the proximity $\Psi(v)$ might increase. As we showed in [22] (see also inequality (16)), the proximity after the update is still bounded above by

$$\Psi(v) \le \psi_0(\theta,\tau,n),$$

where

$$\psi_0(\theta,\tau,n) := \frac{\nu_2\tau}{\nu_1(1-\theta)^{\frac{p+1}{2}}} + \nu_2\Upsilon'_{p,q}\left((1-\theta)^{-\frac{1}{2}}\right)\sqrt{\frac{2n\tau}{\nu_1(1-\theta)}} + n\nu_2\Upsilon_{p,q}\left((1-\theta)^{-\frac{1}{2}}\right). \quad (54)$$

Note that for $0 < \theta < 1$, $\psi_0(\theta,\tau,n)$ is exponential in p. Thus we usually choose a constant $p \ge 1$. On the other hand, we can use a relatively large q since the choice of a big q does not increase $\psi_0(\theta,\tau,n)$ so much. By using Theorem 4.9 directly and following a similar procedure as the discussion in Section 3.4 of [22], we can get the following results step by step.

Lemma 4.10 *Let $\Psi(xs,\mu) \le \tau$ and $\tau \ge \nu_1^{-1}$. Then after an update of the barrier parameter no more than*

$$\left\lceil \frac{8q\nu_1^{-\frac{q-1}{2q}}}{\nu_5(q+1)}(\psi_0(\theta,\tau,n))^{\frac{q+1}{2q}} \right\rceil$$

iterations are needed to recenter.

In some special cases, the bounds of the number of iterations can be simplified as follows:

(i) *If $\kappa = \mathcal{L} = 0$, i.e., the CP is linear and monotone, then $\nu_4 = \nu_3 = 1$ and*

$$\left\lceil \frac{8q\nu_1^{-\frac{q-1}{2q}}}{q+1}\max\left(\frac{2p\nu_2}{\nu_1} + 2\nu_2 + p - 1, \left(1+\frac{1}{\nu_1}\right)\left(2\nu_2 + 1 + q + \frac{2q\nu_2}{\nu_1}\right)\right)(\psi_0(\theta,\tau,n))^{\frac{q+1}{2q}} \right\rceil.$$

(ii) *If the proximity* $\psi(v)$ *used in the algorithm is defined by the function* $\psi(t) = \Upsilon_{p,q}(t)$ *with* $\nu_1 = \nu_2 = 1$, *then*

$$\left\lceil \frac{8q}{q+1} \max\left(2\nu_4(p+1) + \nu_3(p-1), 2(\nu_3 1 + 2\nu_4)(q+1)\right) (\psi_0(\theta,\tau,n))^{\frac{q+1}{2q}} \right\rceil.$$

(iii) *Under both of the assumptions in (i) and (ii),*

$$\left\lceil \frac{8q}{q+1} \max\left(3p+1, 6q+6\right) (\psi_0(\theta,\tau,n))^{\frac{q+1}{2q}} \right\rceil.$$

Theorem 4.11 *If* $\tau \geq \nu_1^{-1}$, *the total number of iterations required by the algorithm is not more than*

$$\left\lceil \frac{8q\nu_1^{-\frac{q-1}{2q}}}{\nu_5(q+1)} (\psi_0(\theta,\tau,n))^{\frac{q+1}{2q}} \right\rceil \left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil.$$

In some special cases, bounds of the number of iterations are given as follows:

(i) *If* $\kappa = \mathcal{L} = 0$, *i.e., the CP is linear and monotone, then* $\nu_4 = \nu_3 = 1$ *and*

$$\left\lceil \frac{8q\nu_1^{-\frac{q-1}{2q}}}{q+1} \max\left(\frac{2p\nu_2}{\nu_1} + 2\nu_2 + p - 1, \left(1 + \frac{1}{\nu_1}\right)\left(\frac{2q\nu_2}{\nu_1} + 2\nu_2 + q + 1\right)\right) \psi_0(\theta,\tau,n)^{\frac{q+1}{2q}} \right\rceil \left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil.$$

(ii) *If the proximity* $\psi(v)$ *used in the algorithm is defined by the function* $\psi(t) = \Upsilon_{p,q}(t)$ *with* $\nu_1 = \nu_2 = 1$, *then*

$$\left\lceil \frac{8q}{q+1} \max\left(2\nu_4(p+1) + \nu_3(p-1), 2(\nu_3 + 2\nu_4)(q+1)\right) (\psi_0(\theta,\tau,n))^{\frac{q+1}{2q}} \right\rceil \left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil.$$

(iii) *Under both of the assumptions in (i) and (ii),*

$$\left\lceil \frac{8q}{q+1} \max\left(3p+1, 6q+6\right) (\psi_0(\theta,\tau,n))^{\frac{q+1}{2q}} \right\rceil \left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil.$$

It is worthwhile to mention that in case of monotone LCPs, the specified results in this section are slightly weaker (only with a slightly different constant in the estimation) than the ones in its LO analogue [22]. This is because in the present paper, we are aiming at establishing the iteration bound for large classes of CPs. Due to the appearance of the nonlinearity of the underlying mapping, it is very hard to estimate the second derivative of $\Delta\Psi(\alpha)$. Therefore, in our analysis we deal with only the first derivative $\Delta\Psi'(\alpha)$ (or $\Delta\Psi_1'(\alpha)$) which is relatively easier to bound. However, in the case of LCPs, the second derivative $\Delta\Psi''(\alpha)$ can be estimated in a similar approach as presented in [22]. In this situation, a sharper estimation about the decrement of the proximity after one step can be obtained and then we can show that new IPMs for monotone LCPs enjoy the same complexity results as their LO counterparts in [22]. However, we are not sure whether such analysis can be extended to nonlinear and nonmonotone CPs while the analysis in this paper can be indeed applied to LO case. Finally we remark that from its definition (54) one can conclude that $\psi_0(\theta,\tau,n) \leq \mathcal{O}(n)$ if θ is a constant in $(0,1)$ and $\tau \leq \mathcal{O}(n)$. In such a situation, if we choose the kernel function by $\psi(t) = \Upsilon_{p,q}(t)$ with $q = \log n$ and $p \geq 1$ is a constant, then from Theorem 4.11 we can claim that the algorithm has $\mathcal{O}\left(\sqrt{n}\log n \log \frac{n}{\epsilon}\right)$ iteration bound. This gives the to date best iteration bound of large-update IPMs for $P_*(\kappa)$ CPs.

5 Concluding Remarks

Based on the *self-regular* proximities, a new class of search directions and IPMs for solving CPs have been proposed. The results in this paper extend the ones reported for linear optimization in [22]. Polynomial complexity of the algorithm has been set up for large classes of problems under suitable assumptions. Several elementary results about $P_*(\kappa)$ mappings and $P_*(\kappa)$ CPs have been presented as well.

There are some ways to further improve our results. The first is to consider the issue whether we could build up the complexity of the algorithm where the proximity satisfies only condition C.1. We mention that in [22], an affirmative answer to such a question had been given for linear optimization. By following an analogous recipe as in [22], we think a positive answer can be expected for CPs as well. However, since such a relaxation on the condition of the proximity will not lead to an improvement of the complexity of the algorithm, and the technical proofs in the present paper are already quite involved, we do not include such a discussion in this paper.

The second issue is how to get a strictly feasible starting point for general $P_*(\kappa)$ CPs. As we mentioned early in Section 2, for monotone CPs and $P_*(\kappa)$ LCPs, there exist already certain methods to handle this issue. However, it is still not clear whether we can find easily a strictly feasible starting point for nonlinear $P_*(\kappa)$ CPs. We also observe that, to set up the complexity of the algorithm, the involved mapping is required to satisfy a new smoothness condition. As proved by Andersen and Ye, a very interesting property for their homogeneous model for CP is that, if the involved mapping $f(x)$ is monotone and satisfies the scaled Lipschitz condition, then so is the new mapping in the augmented homogeneous model. It is also of interest to consider whether our new smoothness condition can be preserved while applied to their model.

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