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Epistemic Logic of Shallow Depths and Game Theoretical Applications

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ABSTRACT. This paper presents a multi-modal epistemic logic GL_{EF} of shallow depths. Logic GL_{EF} is a fragment of multi-modal KD^n, and enables us to consider interpersonal inferential epistemic complexities. Epistemic structures E and F restrict interpersonal epistemic depths involved in formulae and proofs. We give its Gentzen-type sequent formulation as well as its semantics. Also, we provide some meta-theorems for the purpose of evaluations of interpersonal inferential epistemic complexities of the provability of a given sequent. As applications, these meta-theorems are applied to considerations of decision making in game theoretical problems.

KEY WORDS: Epistemic Structure, Minimal Interpersonal Inferential Epistemic Structure, Decision Making in a Game

1 Introduction

Decision making in a game needs *interpersonal* epistemic introspections as well as *intrapersonal* ones. Although the latter is certainly important, we focus on the former in this paper. In particular, we discuss epistemic complexities of interpersonal inferences and their applications to game theoretical problems.

Multi-modal epistemic logics have shown to be useful for considerations of game theoretical decision making. In literature, various multi-modal epistemic logics have been developed, including products and fusions (cf. Gabbay and Shehtman 1998 and Wolter 1998) as well as common knowledge extensions (for recent textbooks, see Fagin et al. (1995) and Meyer and van der Hoek (1995)). These extensions already involve common knowledge to a certain degree. That is, arbitrarily finite depths and/or infinite depths of interpersonal introspections are involved. On the other hand, human interpersonal epistemic introspections typically stop at very shallow levels. It is the purpose of our research project (Kaneko and Suzuki 1999a, 1999b) to take this limitation of human reasoning seriously and to develop logical systems so that they reflect limitations on interpersonal epistemic introspections. In this paper, we are interested particularly in limitations of interpersonal inferential complexities.

To make our motivation more specific, we discuss a difficulty involved, by referring to epistemic logic KD^n. Logic KD^n is defined as Classical Logic + Axiom K: $B_i(A \supset C) \supset (B_i(A) \supset B_i(C))$; Axiom D: $\neg B_i(\neg A \bigwedge A)$; and Necessitation Rule: if A is provable, so is $B_i(A)$, where $B_1, ..., B_n$ are the *belief* *operators* of players $1, ..., n$.[1] In KD^n, formulae having nested occurrences of $B_1, ..., B_n$ in any depths are allowed, and Necessitation Rule may be applied arbitrarily many times in proofs. In this sense, KD^n involves already common knowledge, though a common knowledge extension of KD^n is needed to capture the entirety of common knowledge. Rather than considering extensions of KD^n, we are interested in fragments of KD^n. For meaningful considerations of such fragments, we have needed to develop proof-theoretic and model-theoretic apparatuses, which are undertaken in Kaneko and Suzuki (1999a, 1999b). By these developments, we are now able to consider suitable fragments of KD^n having finite epistemic depths of nested occurrences of belief operators.

In this paper, we focus on the distinction between *descriptive epistemic structure* E and *inferential epistemic structure* F. These notions are used to define our logical system GL_{EF}. The former, E, restricts admissible formulae, and the latter, F, does admissible proofs. It may be the case that some provable formula has large depths of nested occurrences of belief operators but needs only shallow depths for its proof. From the viewpoint of considering human reasoning, it is more important to give restrictions on the latter F.

To illustrate the above distinction, consider two provable formulae $B_1(p) \supset B_1(p)$ and $B_1(p \supset p)$ in KD^n; Both include B_1 without nesting. Nevertheless, the first is provable without Necessitation Rule, but a proof of the second requires it. These examples suggest to distinguish between the structure of occurrences of $B_1, ..., B_n$ in formulae and the applications of epistemic inferences for interpersonal inferences. The two epistemic structures E and F enable us to capture the distinction in that E suffices for the description of formulae but their derivations require a smaller inferential F.

Kaneko and Suzuki (1999a, 1999b) have developed logical systems GL_{EF} and their semantics without differentiating F from E. Nevertheless, their developments can be modified without much difficulty to allow F different from E. Therefore, we can borrow their basic results.

The purpose of limiting interpersonal introspections leads us to consider a *minimal* inferential epistemic structure F for a (provable) formula. For example, the outside investigator can derive $B_1(p) \supset B_1(p)$ without reading player 1's mind, while he needs to assume some logical ability for player 1 to derive $B_1(p \supset p)$. The focus of this paper is to give some results to facilitate the considerations of minimal inferential epistemic structures. These results will be applied to some game theoretical problems. Extensive discussions on game theoretical applications are given in Kaneko and Suzuki (2000).

2 Set of Formulae $\mathcal{P}$ and Epistemic Structures

We adopt the following list of primitive symbols:

propositional variables: $p, q, r, ...$;

[1] In the literature of episemic logics, systems $S4^n$ and $S5^n$ are typically discussed (cf., Hintikka (1962), Fagin et al. (1995), Meyer and van der Hoek (1995)). In these systems, Axiom T: $B_i(A) \supset A$ is assumed, i.e., beliefs are true from the viewpoint of the outside thinker. If we replace Axiom T by Axiom D, we obtain system $KD4^n$. Without Axiom T, we could discuss *false* beliefs, which would be important in game theoretical applications. Although we adopt the KD-type logic to avoid some complications caused by Axiom 4, our results hold with some modifications for the KD4-type logic of shallow depths. The S4-type logic can be discussed inside the KD4-type logic.

logical connective symbols: $\neg$ (not), $\supset$ (implies), $\bigwedge$ (and), $\bigvee$ (or) ;

unary belief operator symbols: $\mathrm{B}_1, \mathrm{B}_2, ..., \mathrm{B}_n$;

parentheses: (,); *comma*: , and *braces* {, }.

The set of propositional variables is denoted by PV. Based on this list of symbols, we define *formulae* inductively in the standard manner, except that $\bigwedge$ and $\bigvee$ are applied to a finite nonempty set Φ of formulae.[2] We denote the set of all formulae by $\mathcal{P}$. The subscripts of $\mathrm{B}_1, ..., \mathrm{B}_n$ are the names of players (agents), and the set $\{1, ..., n\}$ is denoted by N. We say that a formula A is *nonepistemic* if no B_i occurs in A for $i \in N$.

The notions of the epistemic depths of formulae and epistemic structures play central roles in our discussion of an epistemic logic of shallow depths. First, let $N^{<\omega} := \{(i_1, ..., i_m) : i_1, ..., i_m \in N \text{ and } m \geq 0\}$, where $N^{<\omega}$ contains the null sequence ϵ, i.e., the sequence of length 0. For $e = (i_1, ..., i_m) \in N^{<\omega}$, $\mathrm{B}_{i_1} ... \mathrm{B}_{i_m}(A)$ is denoted by $\mathrm{B}_e(A)$, and $\mathrm{B}_\epsilon(A)$ is stipulated to be A. We define the concatenation: $e \circ e' = (i_1, ..., i_m, j_1, ..., j_k)$ for $e = (i_1, ..., i_m)$, $e' = (j_1, ..., j_k) \in N^{<\omega}$, and stipulate $e \circ \epsilon = \epsilon \circ e = e$. We write $(i) \circ e$ and $e \circ (i)$ as $i \circ e$ and $e \circ i$, respectively.

We define the *epistemic depth* $\delta(A)$ of a formula A inductively by:

D0: $\delta(p) = \{\epsilon\}$ for any $p \in PV$;

D1: $\delta(\neg C) = \delta(C)$;

D2: $\delta(C \supset D) = \delta(C) \cup \delta(D)$;

D3: $\delta(\bigwedge \Phi) = \delta(\bigvee \Phi) = \bigcup_{C \in \Phi} \delta(C)$;

D4: $\delta(\mathrm{B}_i(C)) = \{i \circ e : e \in \delta(C)\}$.

Then $\delta(A) \subseteq N^{<\omega}$. For example, $\delta(p \supset \mathrm{B}_2\mathrm{B}_3(q)) = \{\epsilon, (2, 3)\}$, where $p, q \in PV$. We define $\delta(\Gamma) = \bigcup_{C \in \Gamma} \delta(C)$ for a set Γ of formulae.

We say that a nonempty subset E of $N^{<\omega}$ is an *epistemic structure* if

$$(1) \qquad (i_1, ..., i_m) \in E \text{ implies } (i_1, ..., i_{m-1}) \in E.$$

When $m = 0$, $(i_1, ..., i_m)$ is the null symbol ϵ. Trivial examples are $N^{<\omega}$ and $\{\epsilon\}$. Less trivial examples are $\{\epsilon, (1), (2)\}$ and $\{\epsilon, (2), (2, 1)\}$.

We define $\mathcal{P}_E = \{A \in \mathcal{P} : \delta(A) \subseteq E\}$. For example, any formula in $\mathcal{P}_{\{\epsilon, (1), (2)\}}$ may have B_1 and B_2 without their nested occurrences. Since the null symbol ϵ always belongs to E, all the nonepistemic formulae are included in $\mathcal{P}_E$. For $e \in E$, we define $\mathcal{P}_E(e) = \{C : e \circ \delta(C) \in E\} = \{C : e \circ e' \in E$ for all $e' \in \delta(C)\}$, which will be used in Section 4. Also, we use the convention to write $\mathrm{B}_i(\Phi)$ for $\{\mathrm{B}_i(A) : A \in \Phi\}$.

3 Epistemic Logic GL_{EF} of Shallow Depths

In this paper, we give our logical system GL_{EF} in the Gentzen-style sequent form. With respect to inferential F, our investigation is slightly depending upon the choice of a system. See Remark 3.4.

[2]Hence, we need commas and braces as primitive symbols. Also, we need a stipulation of identifying a finite set.

This deviates slightly from the standard formulation of formulae, but this makes certain game theoretical considerations simpler.

Let E and F be two epistemic structures with $F \subseteq E$, where E and F will be used as *descriptive* and *inferential* epistemic structures. We give restrictions in terms of E and F, respectively, on admissible formulae and on admissible proofs.

To facilitate considerations of interpersonal inferential interactions, we introduce a thought sequent. Let $e = (i_1, ..., i_m) \in E$, and Γ, Θ finite subsets of $\mathcal{P}_E$. Using auxiliary symbols [,], and $\to$, we introduce a new expression $\mathrm{B}_e[\Gamma \to \Theta] := \mathrm{B}_{i_1}...\mathrm{B}_{i_m}[\Gamma \to \Theta]$, which we call a *thought sequent*.[3] As in the standard sequent calculus, $\Gamma \to \Theta$ is intended to be $\bigwedge \Gamma \supset \bigvee \Theta$, where $\bigwedge \emptyset$ and $\bigvee \emptyset$ are $\neg p \bigvee p$ and $\neg p \bigwedge p$ respectively. We say that a thought sequent $\mathrm{B}_e[\Gamma \to \Theta]$ is *admissible in* E if $e \circ \delta(\Gamma \cup \Theta) := \{e \circ e' : e' \in \delta(\Gamma \cup \Theta)\} \subseteq E$. When $e = \epsilon$, we abbreviate $\mathrm{B}_\epsilon[\Gamma \to \Theta]$ as $\Gamma \to \Theta$. We abbreviate set-theoretical parentheses, for example, $\mathrm{B}_e[\{A\} \cup \Gamma \to \Theta \cup \{B\}]$ is written as $\mathrm{B}_e[A, \Gamma \to \Theta, B]$.

The outer $\mathrm{B}_e[\cdots] = \mathrm{B}_{i_1} \ldots \mathrm{B}_{i_m}[\cdots]$ expresses the viewpoint of the player to conduct logical deductions. For example, when $m = 2$, $\mathrm{B}_{i_1}\mathrm{B}_{i_2}[\cdots]$ expresses the idea that player i_2 in the scope of player i_1 conducts logical deductions. In particular, $\mathrm{B}_\epsilon[\cdots]$ corresponds to the viewpoint of the investigator.

The logical deductions of player i_m in $e = (i_1, ..., i_m)$ are governed by one axiom schema and various inference rules.

Axiom (Initial Sequent): $\mathrm{B}_e[A \to A]$,

Structural Rules:

$$\frac{\mathrm{B}_e[\Gamma \to \Theta]}{\mathrm{B}_e[\Delta, \Gamma \to \Theta, \Lambda]} \text{(Th)} \qquad \frac{\mathrm{B}_e[\Gamma \to \Theta, A] \quad \mathrm{B}_e[A, \Delta \to \Lambda]}{\mathrm{B}_e[\Gamma, \Delta \to \Theta, \Lambda]} \text{(Cut)}$$

Operational Rules:

$$\frac{\mathrm{B}_e[\Gamma \to \Theta, A]}{\mathrm{B}_e[\neg A, \Gamma \to \Theta]} (\neg \to) \qquad \frac{\mathrm{B}_e[A, \Gamma \to \Theta]}{\mathrm{B}_e[\Gamma \to \Theta, \neg A]} (\to \neg)$$

$$\frac{\mathrm{B}_e[\Gamma \to \Theta, A] \quad \mathrm{B}_e[B, \Gamma \to \Theta]}{\mathrm{B}_e[A \supset B, \Gamma \to \Theta]} (\supset \to) \qquad \frac{\mathrm{B}_e[A, \Gamma \to \Theta, B]}{\mathrm{B}_e[\Gamma \to \Theta, A \supset B]} (\to \supset)$$

$$\frac{\mathrm{B}_e[A, \Gamma \to \Theta]}{\mathrm{B}_e[\bigwedge \Phi, \Gamma \to \Theta]} (\bigwedge \to) \qquad \frac{\{\mathrm{B}_e[\Gamma \to \Theta, A] : A \in \Phi\}}{\mathrm{B}_e[\Gamma \to \Theta, \bigwedge \Phi]} (\to \bigwedge)$$

$$\frac{\{\mathrm{B}_e[A, \Gamma \to \Theta] : A \in \Phi\}}{\mathrm{B}_e[\bigvee \Phi, \Gamma \to \Theta]} (\bigvee \to) \qquad \frac{\mathrm{B}_e[\Gamma \to \Theta, A]}{\mathrm{B}_e[\Gamma \to \Theta, \bigvee \Phi]} (\to \bigvee)$$

where $A \in \Phi$ in $(\bigwedge \to)$ and $(\to \bigvee)$.

Epistemic Rule (Distribution Rule):

$$\frac{\mathrm{B}_{e \circ i}[\Gamma \to \Theta]}{\mathrm{B}_e[\mathrm{B}_i(\Gamma) \to \mathrm{B}_i(\Theta)]} (\mathrm{B}_i \to \mathrm{B}_i), \text{ subject to } |\Theta| \leq 1,$$

[3]This notion is first introduced in Kaneko and Nagashima (1997).

where $|\Theta|$ is the cardinality of Θ.

The outer $\mathrm{B}_e[\cdots]$'s of the upper and lower thought sequents in each of the structural and operational rules are identical; and these rules are the same as those in classic logic LK if the outer $\mathrm{B}_e[\cdots]$ is ignored. That is, the innermost player i_m in $e = (i_1, ..., i_m)$ is capable of conducting logical deductions described by classical logic. The outer $\mathrm{B}_e[\cdots]$ changes only at $(\mathrm{B}_{i_m} \to \mathrm{B}_{i_m})$, and the innermost i_m goes into the scope of $\mathrm{B}_{(i_1,...,i_{m-1})}[\cdots]$. The length of $e = (i_1, ..., i_m)$ of the outer $\mathrm{B}_e[\cdots]$ gets shorter at an occurrence of $(\mathrm{B}_{i_m} \to \mathrm{B}_{i_m})$ in a proof.

If we delete the restriction on Θ having at most one formula, then we would have the negative introspection, i.e., the system would be a KD5-type. In this paper, we consider only the KD-type GL_{EF}. In the following, we may abbreviate $(\mathrm{B}_i \to \mathrm{B}_i)$ as $(\mathrm{B} \to \mathrm{B})$ when we need not to specify the subscript i.

We say that $\mathrm{B}_e[\Gamma \to \Theta]$ is *provable in* GL_{EF} if there is a proof P of $\mathrm{B}_e[\Gamma \to \Theta]$ such that

(1): all thought sequents in P are admissible in E;

(2): e' belongs to F for any thought sequent $\mathrm{B}_{e'}[\Delta \to \Lambda]$ in P.

We write $\vdash_{EF} \mathrm{B}_e[\Gamma \to \Theta]$ if it is provable in GL_{EF}. Recall that when $e = \epsilon$, we abbreviate the outer $\mathrm{B}_\epsilon[\cdots]$, that is, $\vdash_{EF} \mathrm{B}_\epsilon[\Gamma \to \Theta]$ is written as $\vdash_{EF} \Gamma \to \Theta$. We say that A is *provable in* GL_{EF} if $\vdash_{EF} \to A$.

We say that an inferential epistemic structure F is *minimal for* $\mathrm{B}_e[\Gamma \to \Theta]$ if $\vdash_{EF} \mathrm{B}_e[\Gamma \to \Theta]$ but $\nvdash_{EF'} \mathrm{B}_e[\Gamma \to \Theta]$ for any epistemic structure $F' \subsetneqq F$. A minimal F may not be uniquely determined.

The examples $\mathrm{B}_1(p) \supset \mathrm{B}_1(p)$ and $\mathrm{B}_1(p \supset p)$ of Section 1 are proved as follows:

$$\frac{\mathrm{B}_1(p) \to \mathrm{B}_1(p)}{\to \mathrm{B}_1(p) \supset \mathrm{B}_1(p)}(\to\supset) \qquad \frac{\dfrac{\mathrm{B}_1[p \to p]}{\mathrm{B}_1[\ \to p \supset p]}(\to\supset)}{\to \mathrm{B}_1(p \supset p)}(\mathrm{B}_1 \to \mathrm{B}_1).$$

The first is a proof in $\mathrm{GL}_{\{\epsilon,(1)\}\{\epsilon\}}$, and the second is in $\mathrm{GL}_{\{\epsilon,(1)\}\{\epsilon,(1)\}}$. In the first proof, the investigator derives $\to \mathrm{B}_1(p) \supset \mathrm{B}_1(p)$ without considering deductions by player 1, while in the second, the investigator once assumes that 1 conducts logical deductions. In the first example, it is easy to see that $F = \{\epsilon\}$ is minimal. However, we need some result to prove the minimality of $F = \{\epsilon, (1)\}$ for $\to \mathrm{B}_1(p \supset p)$. It follows from this minimality, using the cut-elimination theorem given below, that $\to \mathrm{B}_1(p \supset p) \bigvee \mathrm{B}_2(p \supset p)$ has two minimal F's, i.e., $\{\epsilon, (1)\}$ and $\{\epsilon, (2)\}$.

Since these are too simple to see the structure of GL_{EF}, we give slightly more complex examples. The minimality of $F = \{\epsilon, (1)\}$ for $\to \mathrm{B}_1(p \supset p)$ will be discussed as part of Example 3.1.(2).

Example 3.1 (1): The sequent $\mathrm{B}_2\mathrm{B}_1(p), \mathrm{B}_2\mathrm{B}_1(q) \to \mathrm{B}_2(\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q))$ is admissible in $E = \{\epsilon, (2), (2,1)\}$. One possible choice of an inferential epistemic structure is $F = \{\epsilon, (2)\}$. A proof of this sequent is given:

$$\frac{\dfrac{\dfrac{\mathrm{B}_2[\mathrm{B}_1(p) \to \mathrm{B}_1(p)]}{\mathrm{B}_2[\mathrm{B}_1(p), \mathrm{B}_1(q) \to \mathrm{B}_1(p)]}(\mathrm{Th}) \quad \dfrac{\mathrm{B}_2[\mathrm{B}_1(q) \to \mathrm{B}_1(q)]}{\mathrm{B}_2[\mathrm{B}_1(p), \mathrm{B}_1(q) \to \mathrm{B}_1(q)]}(\mathrm{Th})}{\mathrm{B}_2[\mathrm{B}_1(p), \mathrm{B}_1(q) \to \mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q)]}(\to \bigwedge)}{\mathrm{B}_2\mathrm{B}_1(p), \mathrm{B}_2\mathrm{B}_1(q) \to \mathrm{B}_2(\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q))}(\mathrm{B}_1 \to \mathrm{B}_1)$$

This proof can be extended so that F coincides with E, for example, we add $(B_1 \to B_1)$ to the uppermost thought sequent $B_2[B_1(p) \to B_1(p)]$. Nevertheless, we are interested in a minimal inferential epistemic structure F, as mentioned in Section 1.

(2): The sequent $\to B_2B_1(p \supset p)$ needs also the descriptive epistemic structure $E = \{\epsilon, (2), (2,1)\}$. The following is a proof of $\to B_2B_1(p \supset p)$ in GL_{EF} with $F = E$.

$$\dfrac{\dfrac{\dfrac{B_2B_1[p \to p]}{B_2B_1[\ \to (p \supset p)]}(\to\supset)}{B_2[\ \to B_1(p \supset p)]}(B_1 \to B_1)}{\to B_2B_1(p \supset p)}(B_2 \to B_2)$$

In this case, the investigator assumes that player 2 reads the mind of player 1. The provability of this sequent is considerably different from that of $\to B_2B_1(p) \supset B_2B_1(p)$ in that the latter does not need to assume that either player reads the other's mind. This difference explains why we are interested in a minimal inferential epistemic structure. In the case of $\to B_2B_1(p \supset p)$, we will show below in this section that $F = \{\epsilon, (2), (2,1)\}$ is a minimal inferential epistemic structure.

Logic GL_{EF} enjoys cut-elimination, which is a basic proof-theoretic property of GL_{EF}.

Theorem 3.2 (Cut-Elimination) *If $\vdash_{EF} \Gamma \to \Lambda$, then there is a cut-free proof P of $\Gamma \to \Lambda$ in GL_{EF}.*

Allowing the outer $B_e[\cdots]$'s is the main difference from this cut-elimination theorem from the original one for Gentzen's LK. For the treatment of the outer $B_e[\cdots]$'s, see Kaneko and Nagashima (1997).

Sequent calculus KD^n is obtained from the above system by deleting all the outer $B_e[\cdots]$ from the axiom and inference rules, and by eliminating the restrictions on the admissibility of thought sequents in proofs. This sequent calculus, KD^n, is known to have the cut-elimination property. Using this cut-elimination, we can have the following theorem.

Theorem 3.3 (Relation to KD^n) *$\Gamma \to \Lambda$ is provable in KD^n if and only if $\vdash_{EE} \Gamma \to \Lambda$, where E is any epistemic structure including $\delta(\Gamma \cup \Lambda)$.*

Thus, the most relevant E is the epistemic structure generated by $\delta(\Gamma \cup \Lambda)$, and $F = E$ is large enough for $\Gamma \to \Lambda$. The above examples of proofs state that it may be possible to choose a smaller F than E.

Remark 3.4 Kaneko and Suzuki (1999a, 1999b) gave a Hilbert-style formulation of GL_{EE} (of KD4-type). This formulation can be translated into the Gentzen-style formulation GL_{EE}. When we take an inferential epistemic structure F into consideration, the provabilities in those formulations may slightly differ because of the choice of basic axioms. The Gentzen-style formulation fits better to the semantics given in Section 4.

By the cut-elimination theorem, we can prove that $F = \{\epsilon, (2)\}$ and $F = \{\epsilon, (2), (2,1)\}$ are the minimal inferential epistemic structures, respectively, in Example 3.1.(1) and (2). For this kind of purposes, however, it would be useful to develop some meta-theorems.

The following is a key theorem to evaluate an inferential epistemic structure F.

Theorem 3.5 (Epistemic Inferences) *Let* Γ *be a finite set of formulae with* $\mathrm{B}_i(\Gamma) \cup \mathrm{B}_i(A) \subseteq \mathcal{P}_E$.

(1): If $\vdash_{EF} \mathrm{B}_i(\Gamma) \rightarrow$ *or* $\vdash_{EF} \rightarrow \mathrm{B}_i(A)$, *then* $(i) \in F$.

(2): If $\vdash_{EF} \mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)$ *and* $A \notin \Gamma$, *then* $(i) \in F$.

(3): Let F *be a minimal inferential epistemic structure for* $\mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)$. *Then* $A \in \Gamma$ *implies* $F = \{\epsilon\}$.

(4): If $F = \{\epsilon\}$ *and* $\vdash_{EF} \mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)$, *then* $A \in \Gamma$.

Proof. (1): Suppose $\vdash_{EF} \ \rightarrow \mathrm{B}_i(A)$. Then there is a cut-free proof P of $\rightarrow \mathrm{B}_i(A)$ by Theorem 3.2. Then the lowermost inference of P is $(\mathrm{B}_i \rightarrow \mathrm{B}_i)$. The upper thought sequent of this inference is $\mathrm{B}_i[\rightarrow A]$. Hence $(i) \in F$. It is proved in the same way that if $\vdash_{EF} \mathrm{B}_i(\Gamma) \rightarrow$, then $(i) \in F$.
(2): By (1), we can assume that $\nvdash_{EF} \mathrm{B}_i(\Gamma) \rightarrow$ and $\nvdash_{EF} \ \rightarrow \mathrm{B}_i(A)$. Since $\vdash_{EF} \mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)$, there is a cut-free proof P of $\mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)$ in GL_{EF} by Theorem 3.2. Then the lowermost inference must be either (Th) or $(\mathrm{B}_i \rightarrow \mathrm{B}_i)$. If this is $(\mathrm{B}_i \rightarrow \mathrm{B}_i)$, then $(i) \in F$. Consider the former case, i.e.,

$$\frac{\mathrm{B}_i(\Gamma') \rightarrow \mathrm{B}_i(A)}{\mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)} \text{(Th)} \quad \text{or} \quad \frac{\mathrm{B}_i(\Gamma') \rightarrow}{\mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)} \text{(Th)},$$

where Γ' is a nonempty subset of Γ. Then $A \notin \Gamma'$ by $A \notin \Gamma$. The latter case is excluded by the assumption $\nvdash_{EF} \mathrm{B}_i(\Gamma) \rightarrow$. Consider the former case. The second lowermost inference must be again either (Th) or $(\mathrm{B}_i \rightarrow \mathrm{B}_i)$. If this is $(\mathrm{B}_i \rightarrow \mathrm{B}_i)$, then $(i) \in F$. If it is (Th), then we consider the 3rd lowermost inference. We repeat this argument. Then we meet $(\mathrm{B}_i \rightarrow \mathrm{B}_i)$ or an initial sequent. If the uppermost sequent was the initial sequent, it must be $\mathrm{B}_i(A) \rightarrow \mathrm{B}_i(A)$, which is impossible by $A \notin \Gamma$. Hence we meet $(\mathrm{B}_i \rightarrow \mathrm{B}_i)$, which implies $(i) \in F$.

(3): Let $A \in \Gamma$. Then we have

$$\frac{\mathrm{B}_i(A) \rightarrow \mathrm{B}_i(A)}{\mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)} \text{(Th)}.$$

This means $\vdash_{E\{\epsilon\}} \mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)$. Since F is a minimal epistemic structure for $\mathrm{B}_i(\Gamma) \rightarrow \mathrm{B}_i(A)$, we have $F = \{\epsilon\}$.

(4): This follows (2). $\dashv$

Let us see the minimality of $F = \{\epsilon, (2), (2,1)\}$ for $\vdash_{EF} \ \rightarrow \mathrm{B}_2\mathrm{B}_1(p \supset p)$ in Example 3.1.(2), where $E = \{\epsilon, (2), (2,1)\}$. Applying Theorem 3.5.(1) to this sequent, we have $(2) \in F$. It remains to show $(2,1) \in F$. For this step, the following two lemmas are useful.

Lemma 3.6 *Consider a sequent* $\mathrm{B}_i(\Gamma) \to \mathrm{B}_i(\Theta)$ *with* $|\Theta| \leq 1$. *If* $\vdash_{EF} \mathrm{B}_i(\Gamma) \to \mathrm{B}_i(\Theta)$ *and* $(i) \in F$, *then* $\vdash_{EF} \mathrm{B}_i[\Gamma \to \Theta]$.

Proof. Suppose $\vdash_{EF} \mathrm{B}_i(\Gamma) \to \mathrm{B}_i(\Theta)$. If $\Theta = \{A\} \subseteq \Gamma$, then $\vdash_{EF} \mathrm{B}_i[A \to A]$, which implies $\vdash_{EF} \mathrm{B}_i[\Gamma \to \Theta]$. Now, let $\Theta \cap \Gamma = \emptyset$. Then there is a cut-free proof P of $\mathrm{B}_i(\Gamma) \to \mathrm{B}_i(\Theta)$ in GL_{EF}. Then the lowermost inference is either (Th) or ($\mathrm{B}_i \to \mathrm{B}_i$). If it is ($\mathrm{B}_i \to \mathrm{B}_i$), then the upper sequent of ($\mathrm{B}_i \to \mathrm{B}_i$) is $\mathrm{B}_i[\Gamma \to \Theta]$, which implies $\vdash_{EF} \mathrm{B}_i[\Gamma \to \Theta]$. Consider the other case where the lowermost inference is (Th). Then the upper sequent is $\mathrm{B}_i(\Gamma') \to \mathrm{B}_i(\Theta')$ for some $\Gamma' \subseteq \Gamma$ and $\Theta' \subseteq \Theta$. We can apply the same argument to this sequent $\mathrm{B}_i(\Gamma') \to \mathrm{B}_i(\Theta')$. That is, the lowermost inference of a cut-free proof of $\mathrm{B}_i(\Gamma') \to \mathrm{B}_i(\Theta')$ is either (Th) or ($\mathrm{B}_i \to \mathrm{B}_i$). Repeat the same argument, we would meet ($\mathrm{B}_i \to \mathrm{B}_i$). In this case, the upper sequent of ($\mathrm{B}_i \to \mathrm{B}_i$) is $\mathrm{B}_i[\Gamma'' \to \Theta'']$ for some $\Gamma'' \subseteq \Gamma$ and $\Theta'' \subseteq \Theta$. Hence $\vdash_{EF} \mathrm{B}_i[\Gamma'' \to \Theta'']$. By (Th), we have $\vdash_{EF} \mathrm{B}_i[\Gamma \to \Theta]$. ⊣

For an epistemic structure E and $i \in N$, we write $E_{-i} = \{e : i \circ e \in E\}$. Then E_{-i} is also an epistemic structure.

Lemma 3.7 $\vdash_{EF} \mathrm{B}_i[\Gamma \to \Theta]$, *then* $\vdash_{E_{-i}F_{-i}} \Gamma \to \Theta$.

Proof. Suppose $\vdash_{EF} \mathrm{B}_i[\Gamma \to \Theta]$. Then there is a proof P of $\mathrm{B}_i[\Gamma \to \Theta]$. Then i_1 of any thought sequent $\mathrm{B}_{i_1}...\mathrm{B}_{i_m}[\Delta \to \Lambda]$ in P is i. We delete the outermost B_i from each $\mathrm{B}_{i_1}...\mathrm{B}_{i_m}[\Delta \to \Lambda]$ in P. Let P' be a tree obtained in this manner. Then P' is a proof of $\Gamma \to \Theta$ in $\mathrm{GL}_{E_{-i}F_{-i}}$. ⊣

Let us return to the minimality of $F = \{\epsilon, (2), (2,1)\}$ for $\vdash_{EF} \ \to \mathrm{B}_2\mathrm{B}_1(p \supset p)$. By Lemma 3.6, we have $\vdash_{EF} \mathrm{B}_2[\ \to \mathrm{B}_1(p \supset p)]$. By Lemma 3.7, we have $\vdash_{E_{-2}F_{-2}} \ \to \mathrm{B}_1(p \supset p)$. Again, by Theorem 3.5.(1), we have $(1) \in F_{-2}$, which implies $(2,1) \in F$.

4 Semantics for GL_{EF}

In this section, we consider the semantics for GL_{EF}, which is a modification of the one for the logic GL_{EE} of KD4-type given in Kaneko and Suzuki (1999a). We have the completeness result, which is also obtained by modifying the completeness for GL_{EE} taking the inferential epistemic structure F into account. The semantics helps often us evaluate the required inferential epistemic structure F. For the application purpose, we give two more meta-theorems given in Kaneko and Suzuki (1999b).

We modify the standard Kripke semantics to incorporate descriptive and inferential epistemic structures E and F. In the modification, E restricts the set of admissible formulae and F does a structure of a frame. An *F-frame* $\mathcal{F}$ is given as an $n+2$ tuple $(W : w_0 : R_1, ..., R_n)$ satisfying:

F1: W is a nonempty set, and is partitioned into $\{W_e\}_{e \in F}$ with $W_\epsilon = \{w_0\}$;

F2: each R_i is a binary relation over W satisfying

F2a(F-structure): $w \in W_e$ and wR_iu imply $u \in W_{e \circ i}$;

F2b(F-seriality): $w \in W_e$ and $e \circ i \in F$ imply wR_iu for some $u \in W$.

The set W_e is the set of possible epistemic worlds corresponding to the thought sequents $\mathrm{B}_e[\cdots]$. The following $(W : w_0 : R_1, R_2)$ is an example for an F-frame with $F = \{\epsilon, (2), (2,1)\}$, where $W = \{w_0, w_1, w_2\}, W_\epsilon = \{w_0\}, W_{(2)} = \{w_1\}, W_{(2,1)} = \{w_2\}, R_1 = \{(w_1, w_2)\}$ and $R_2 = \{(w_0, w_1)\}$.

$$\begin{array}{l} w_2 : \mathcal{P}_E(2,1) \\ \uparrow_1 \\ w_1 : \mathcal{P}_E(2) \\ \uparrow_2 \\ w_0 : \mathcal{P}_E \end{array}$$

Diagram 4.1.

Let $\mathcal{F}$ be an F-frame, and E a descriptive epistemic structure with $F \subseteq E$. Recall $\mathcal{P}_E(e) = \{C : e \circ \delta(C) \in E\}$ for $e \in F$. Then we associate the set of formulae $\mathcal{P}_E(e)$ with each possible world $w_e \in W_e$ and $e \in F$. The root node w_ϵ is associated with $\mathcal{P}_E(\epsilon) = \mathcal{P}_E$. This association is made, since we like to have the correspondence between the worlds in W_e in $\mathcal{F}$ and the sequents with the outer $\mathrm{B}_e[\cdots]$. The formulae in $\mathcal{P}_E(e)$ are those in the scope of $\mathrm{B}_e[\cdots]$.

When F is a proper subset of E, it may be the case that $w \in W_e$, wR_iu for no $u \in W_e$ but $\mathrm{B}_i(A) \in \mathcal{P}_E(e)$. For example, when $E = \{\epsilon, (1), (2), (2,1)\}$ in the example of Diagram 4.1, $\mathcal{P}_E$ has $\mathrm{B}_1(p)$ but there is no $u \in W$ with w_0R_1u. We cannot refer to any world for the valuation of $\mathrm{B}_1(p)$. In this case, we allow $\mathrm{B}_1(p)$ to take either truth value $\top$ or $\bot$, i.e., we treat it just like a propositional variable.

An *EF-assingment* σ in an F-frame $\mathcal{F}$ is a function from $\bigcup_{e \in F}(W_e \times (PV \cup \{\mathrm{B}_i(C) \in \mathcal{P}_E(e) : e \circ i \notin F\}))$ to $\{\top, \bot\}$. An *EF-model* is a pair $(\mathcal{F}, \sigma)$ of an F-frame and an EF-assignment σ in $\mathcal{F}$. The *valuation relation* $(\mathcal{F}, \sigma, w) \models$ is defined inductively as follows: for any $w \in W_e$ and $e \in F$,

V0: for any $A \in PV \cup \{\mathrm{B}_i(C) \in \mathcal{P}_E(e) : e \circ i \notin F\}$, $(\mathcal{F}, \sigma, w) \models A$ iff $\sigma(w, A) = \top$;

V1: $(\mathcal{F}, \sigma, w) \models \neg A$ iff $(\mathcal{F}, \sigma, w) \not\models A$;

V2: $(\mathcal{F}, \sigma, w) \models A \supset B$ iff $(\mathcal{F}, \sigma, w) \not\models A$ or $(\mathcal{F}, \sigma, w) \models B$;

V3: $(\mathcal{F}, \sigma, w) \models \bigwedge \Phi$ iff $(\mathcal{F}, \sigma, w) \models A$ for all $A \in \Phi$;

V4: $(\mathcal{F}, \sigma, w) \models \bigvee \Phi$ iff $(\mathcal{F}, \sigma, w) \models A$ for some $A \in \Phi$;

V5: for any $\mathrm{B}_i(A) \in \mathcal{P}_E(e)$ with $e \circ i \in F$, $(\mathcal{F}, \sigma, w) \models \mathrm{B}_i(A)$ iff $(\mathcal{F}, \sigma, u) \models A$ for all u with wR_iu.

For finite subsets Γ, Θ of $\mathcal{P}_E(e)$ and $w \in W_e$, we write $(\mathcal{F}, \sigma, w) \models \Gamma \to \Theta$ if $(\mathcal{F}, \sigma, w) \models \bigwedge \Gamma \supset \bigvee \Theta$, where $\bigwedge \emptyset$ and $\bigvee \emptyset$ are $\neg p \bigvee p$ and $\neg p \bigwedge p$, respectively.

Example 4.1 Let $F = \{\epsilon, (2)\}$ and $E = \{\epsilon, (2), (2,1)\}$. One example of an F-frame is given by Diagram 4.2. It is written as $(W : w_0, R_1, R_2)$, where $W = W_\epsilon \cup W_{(2)} = \{w_0\} \cup \{w_1, w_2\}$, $R_1 = \emptyset$ and $R_2 = \{(w_0, w_1), (w_0, w_2)\}$.

$$\begin{array}{ccc} w_1 : \mathcal{P}_E(2) & & w_2 : \mathcal{P}_E(2) \\ & \nwarrow_2 \quad \nearrow_2 & \\ & w_0 : \mathcal{P}_E & \end{array}$$

Diagram 4.2.

Let $PV := \{p, q\}$. Then $\sigma(w_0, \cdot)$ is defined over $\{p, q\}$, but $\sigma(w_1, \cdot)$, $\sigma(w_2, \cdot)$ are defined over $\{p, q\} \cup \{\mathrm{B}_1(A) : \mathrm{B}_1(A) \in \mathcal{P}_E(2)\}$. In world w_0, the valuations of $\mathrm{B}_2\mathrm{B}_1(p)$, $\mathrm{B}_2\mathrm{B}_1(q)$ and $\mathrm{B}_2(\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q))$ are defined by referring to w_1 and w_2. In world w_1 and w_2, however, $\mathrm{B}_1(p)$, $\mathrm{B}_1(q)$ are treated in the same way as propositional variables. Nevertheless, since in w_1 and w_2, $\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q)$ is true if both are true, we have $(\mathcal{F}, \sigma, w_0) \models \mathrm{B}_2\mathrm{B}_1(p), \mathrm{B}_2\mathrm{B}_1(q) \to \mathrm{B}_2(\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q))$ for any assignment σ.

We have the completeness theorem, which is obtained by modifying completeness for the logic GL_{EE} of KD4-type proved in Kaneko and Suzuki (1999a).

Theorem 4.2 (Completeness) *Let Γ, Θ be finite subsets of $\mathcal{P}_E$. Then $\vdash_{EF}$ $\Gamma \to \Theta$ if and only if $(\mathcal{F}, \sigma, w_0) \models \Gamma \to \Theta$ for all EF-models $(\mathcal{F}, \sigma)$.*

To illustrate the usefulness of this completeness theorem, we return to the above example.

Example 4.3 In Example 3.1.(1), we showed that sequent $\mathrm{B}_2\mathrm{B}_1(p), \mathrm{B}_2\mathrm{B}_1(q) \to \mathrm{B}_2(\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q))$ is provable in GL_{EF} with $E = \{\epsilon, (2), (2,1)\}$ and $F = \{\epsilon, (2)\}$. By the completeness theorem, we can show that this inferential F is minimal for this sequent. Suppose $F = \{\epsilon\}$. Then any F-frame has only one possible world w_0. In this case, $\sigma(w_0, \mathrm{B}_2\mathrm{B}_1(p))$, $\sigma(w_0, \mathrm{B}_2\mathrm{B}_1(q))$ and $\sigma(w_0, \mathrm{B}_2(\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q)))$ have independent truth values. Hence we find an EF-model $(\mathcal{F}, \sigma)$ so that $(\mathcal{F}, \sigma, w_0) \not\models \mathrm{B}_2\mathrm{B}_1(p), \mathrm{B}_2\mathrm{B}_1(q) \to \mathrm{B}_2(\mathrm{B}_1(p) \bigwedge \mathrm{B}_1(q))$.

Kaneko and Suzuki (1999b) developed surgical operations of models for GL_{EE} and obtained various meta-theorems. Here, we refer to two theorems for the use in game theoretical applications in Section 5, which are obtained by modifying their results slightly for the present GL_{EF} of KD-type.

Let S be a subset of $N \cup \{0\}$, and A a formula in $\mathcal{P}$. We say that A is an *S-formula* if

(i): $e = (i_1, ..., i_m) \in \delta(A)$ and $m \geq 1$ imply $i_1 \in S$;

(ii): $\epsilon \in \delta(A)$ implies $0 \in S$.

For example, $\mathrm{B}_2\mathrm{B}_1(p \supset p)$ is a $\{2\}$-formula. Any $\{0\}$-formula is nonepistemic.

Theorem 4.4 (Epistemic Separation) *Let $S_1, ..., S_k$ be disjoint nonempty subsets of $N \cup \{0\}$. Let Γ_t be a finite set of S_t-formulae for $t = 1, ..., k$, and A_t an S_t-formula for $t = 1, ..., k$. Also, we let $E^t = \{(i_1, ..., i_m) \in E : i_1 \in S_t\} \cup \{\epsilon\}$ and $F^t = \{(i_1, ..., i_m) \in F : i_1 \in S_t\} \cup \{\epsilon\}$ for $t = 1, ..., k$.*

(1): *$\vdash_{EF} \Gamma_1, ..., \Gamma_k \to A_1 \bigvee ... \bigvee A_k$ if and only if $\vdash_{E^tF^t} \Gamma_t \to A_t$ for some $t = 1, ..., k$.*

(2): *Suppose that $\Gamma_1 \cup \cdots \cup \Gamma_k$ is consistent in GL_{EF}. Then $\vdash_{EF} \Gamma_1, \ldots, \Gamma_k \to A_1 \bigwedge ... \bigwedge A_k$ if and only if $\vdash_{E^tF^t} \Gamma_t \to A_t$ for all $t = 1, ..., k$.*

Theorem 4.5 (Epistemic Disjunction Property) *Let Γ and Φ be a finite sets of formulae with $\Phi \neq \emptyset$. Then $\vdash_{EF} \mathrm{B}_i(\Gamma) \to \bigvee \mathrm{B}_i(\Phi)$ if and only if $\vdash_{EF}$ $\mathrm{B}_i(\Gamma) \to \mathrm{B}_i(A)$ for some $A \in \Phi$.*

The restriction by F on a proof or a frame requires that we treat $\mathrm{B}_i(A)$ in the same way as propositional variable if $\mathrm{B}_i(A) \in \mathcal{P}_E(e)$ and $e \circ i \notin F$. The following remark may help us understand this requirement.

Remark 4.6 Suppose $\vdash_{EF} \Gamma \to \Lambda$. Then there is another sequent $\Gamma' \to \Lambda'$ such that (1) $\delta(\Gamma' \cup \Lambda') \subseteq F$, (2) $\vdash_{FF} \Gamma' \to \Lambda'$ and (3) $\Gamma \to \Lambda$ is obtained by the substitutions of all the occurrences of some propositional variables $p_1, ..., p_m$ in $\Gamma' \to \Lambda'$ with some formulae $\mathrm{B}_{i_1}(A_1), ..., \mathrm{B}_{i_m}(A_m)$.

5 Applications: Prediction-Decision Making in Games

In this section, we give applications of GL_{EF} and the meta-theorems given in the previous sections to game theoretical problems. More extensive considerations of such game theoretical problems are given in Kaneko and Suzuki (2000).

Consider a 2-person finite noncooperative game $g = (g_1, g_2)$. Each player $i = 1, 2$ has ℓ_i (pure) strategies ($\ell_i \geq 2$). The set of strategies for player i is denoted by $S_i := \{\mathrm{s}_{i1}, ..., \mathrm{s}_{i\ell_i}\}$. His *payoff function* is a real valued function g_i on $S := S_1 \times S_2$. A strategy $s_1 \in S_1$ is a *best response to* s_2 if $g_1(s_1, s_2) \geq g_1(t_1, s_2)$ for all $t_1 \in S_1$. We say that s_1 is a *dominant strategy* if s_1 is a best response to s_2 for any $s_2 \in S_2$. A dominant strategy for player 2 is defined in the parallel manner. See Myerson (1991) for these basic notions.

In the game g^1 of Table 5.1 (Prisoner's Dilemma), the second strategy s_{i2} for each player i is a dominant strategy. In the game g^2 of Table 5.2 which is obtained from g^1 by changing payoff 6 to 2 in the northeast corner, player 1 has the same dominant strategy in g^1, but 2 has no dominant ones.

	s_{21}	s_{22}
s_{11}	$(5,5)$	$(1,6)$
s_{12}	$(6,1)$	$(3,3)$

Table 5.1: $g^1 = (g_1^1, g_2^1)$

	s_{21}	s_{22}
s_{11}	$(5,5)$	$(1,\tilde{2})$
s_{12}	$(6,1)$	$(3,3)$

Table 5.2: $g^2 = (g_1^2, g_2^2)$

First, consider the following prediction-decision criteria.

DC1: Player i makes a decision by choosing a dominant strategy.

In game g^1, DC1 recommends each player i to choose s_{i2}. However, in game g^2, DC1 is applied only to player 1 but not to player 2, since 2 has no dominant strategies. We consider another decision criterion for player 2.

DC21: Player $j \neq i$ first predicts that player i would choose a strategy following DC1, and then j would choose a best strategy to the predicted strategy.

Let us apply this criterion to game g^2 for $j = 2$: player 2 predicts that 1 would choose s_{12} following DC1, and then 2 would choose s_{22} as the best strategy to s_{12}.

The above two criteria are related to the *procedure of iterated elimination of dominated strategies* (cf., Moulin (1982) and Myerson (1991)). This is more or less a standard example of a prediction-decision criterion in the game theory literature. However, we can consider a lot of other prediction-decision criteria which have never been discussed in literature. Here, we show that our logical approach enables us to take such prediction-decision criteria in the scope of our research. In particular, we discuss the subtlety of inferential epistemic structures required for decision making with such criteria.

We give the following somewhat extreme criterion, which suggests that there are a lot of variants of prediction-decision criteria.

DC0: Player i has the predetermined strategy s_{i1} as his decision, which we call the *pure default decision.*

This may sound too trivial if it is adopted for player i's decision making. However, it would not be so if this is adopted for player j's prediction on i's decision making. That is, the prediction on the other player's decision making is made without much deliberations on the other's subjective elements. We modify criterion DC21 into the following.

DC20: Player j first predicts that player i would choose strategy s_{i1} following DC0, and then j would choose a best strategy to the predicted strategy s_{i1}.

This criterion is free from the symmetric assumption that a decision maker assumes payoff maximization for himself as well as for the other player in his prediction. In this sense, DC20 differs considerably from DC21. In game g^2, this criterion states that $j = 2$ chooses s_{21} as the best strategy to the default prediction s_{11}.

Now, we look at how these criteria can be discussed in our logical approach. First, we replace the set of propositional variables by the set of atomic formulae defined by

strategy symbols: $\mathrm{s}_{11}, ..., \mathrm{s}_{1\ell_1}$ and $\mathrm{s}_{21}, ..., \mathrm{s}_{2\ell_2}$;

4-ary preference symbols: P_1 and P_2;

unary default decision symbols: d_1 and d_2.

That is, for $s, t \in S = S_1 \times S_2, s_i \in S_i$ and $i = 1, 2$, we call the expression $P_i(s : t)$ and $d_i(s_i)$ *atomic* formulae. Let AF be the set of all atomic formulae, and the set of formulae $\mathcal{P}$ is defined based on AF instead of PV. The expressions $P_i(s : t)$ and $d_i(s_i)$ are intended to mean, respectively, that player i weakly prefers strategy pair s to t and that s_i is a "default" decision of player i.

In the present language, the statement "s_1 is a best response to s_2 for player 1" is expressed by the formula $\bigwedge\{P_1(s_1, s_2 : t_1, s_2) : t_1 \in S_1\}$, which we denote by $\mathrm{Best}_1(s_1 \mid s_2)$. The statement "$s_1$ is a dominant strategy for 1" is expressed by $\bigwedge\{\mathrm{Best}_1(s_1 \mid s_2) : s_2 \in S_2\}$. This is equivalent to $\bigwedge\{P_1(s_1, s_2 : t_1, s_2) : t_1 \in S_1$ and $s_2 \in S_2\}$, which we denote by $\mathrm{Dom}_1(s_1)$. In the parallel manner, we define $\mathrm{Best}_2(s_2 \mid s_1)$ and $\mathrm{Dom}_2(s_2)$.

We express the payoff function g_i by the set of preferences (formulae):

$$\text{(2)} \qquad \hat{g}_i := \{P_i(s : t) : g_i(s) \geq g_i(t)\} \cup \{\neg P_i(s : t) : g_i(s) < g_i(t)\}.$$

Hence game $g = (g_1, g_2)$ is expressed as $\hat{g}_1 \cup \hat{g}_2$.

Now, we are in the state to formulate the above prediction-decision criteria. One possible formulation of a prediction-decision criterion for player i is given as a set of formulae indexed by $s_i \in S_i$:

$$\text{(3)} \qquad \mathcal{D}_i = \{D_i(s_i) : s_i \in S_i\}.$$

This means that if $D_i(s_i)$ holds, then s_i would be a decision. The capability of player i's decision making is expressed as the following sequent:

$$\text{(4)} \qquad \mathrm{B}_i(\Gamma_i) \to \bigvee_{s_i} \mathrm{B}_i(D_i(s_i)),$$

where Γ_i is the finite set of player i's basic beliefs. That is, the existence of some decision s_i is derived from player i's beliefs Γ_i. This statement itself is made from the viewpoint of the investigator. A general statement by the investigator is expressed as the provability of the following:

$$\text{(5)} \qquad \Gamma, \mathrm{B}_1(\Gamma_1), \mathrm{B}_2(\Gamma_2) \to \bigvee_{s_1} \mathrm{B}_1(D_1(s_1)) \bigwedge \bigvee_{s_2} \mathrm{B}_2(D_2(s_2)),$$

where Γ is a finite set of nonepistemic formulae, which expresses the objective situation such as $\hat{g}_1 \cup \hat{g}_2$.

The ultimate goal is to consider the provability of (5) from the viewpoint of the investigator. However, the following theorem states that for each player's decision making, it suffices to consider (4) separately and that we can ignore the objective part Γ.

Theorem 5.1 (Decomposition) *Suppose that* $\Gamma \cup \mathrm{B}_1(\Gamma_1) \cup \mathrm{B}_2(\Gamma_2)$ *is consistent in* GL_{EF}. *Then the following two statements are equivalent:*

(1): $\vdash_{EF} \Gamma, \mathrm{B}_1(\Gamma_1), \mathrm{B}_2(\Gamma_2) \to \bigvee_{s_1} \mathrm{B}_1(D_1(s_1)) \bigwedge \bigvee_{s_2} \mathrm{B}_2(D_2(s_2))$;

(2): *for* $i = 1, 2$, $\vdash_{E_i F_i} \mathrm{B}_i(\Gamma_i) \to \mathrm{B}_i(D_i(s_i))$ *for some* s_i,

where $E_i = \{(i_1, ..., i_m) \in E : i_1 = i\} \cup \{\epsilon\}$ *and* $F_i = \{(i_1, ..., i_m) \in F : i_1 = i\} \cup \{\epsilon\}$.

Proof. The derivation of (1) from (2) is straightforward. Suppose (1). We apply Theorem 4.4 to (1), and have $\vdash_{E_i F_i} \mathrm{B}_i(\Gamma_i) \to \bigvee_{s_i} \mathrm{B}_i(D_i(s_i))$ for $i = 1, 2$. Then, by Theorem 4.5, we have (2). $\dashv$

Now, we focus on the statement on each player's decision making.

DC1: This is formulated as $\mathcal{D}_i^1 = \{\mathrm{Dom}_i(s_i) : s_i \in S_i\}$. Applying this criterion to game g^1, we have the following derivation:

$$\cfrac{\cfrac{\left\{ \cfrac{\mathrm{B}_1[P_1(\mathrm{s}_{12}, s_2 : s_1, s_2) \to P_1(\mathrm{s}_{12}, s_2 : s_1, s_2)]}{\mathrm{B}_1[\hat{g}_1^1 \to P_1(\mathrm{s}_{12}, s_2 : s_1, s_2)]}\,(\mathrm{Th}) \right\}_{s \in S}}{\mathrm{B}_1[\hat{g}_1^1 \to \mathrm{Dom}_1(\mathrm{s}_{12})]}\,(\to \bigwedge)}{\mathrm{B}_1(\hat{g}_1^1) \to \mathrm{B}_1(\mathrm{Dom}_1(\mathrm{s}_{12})).}\,(\mathrm{B}_1 \to \mathrm{B}_1)$$

That is, player 1 derives s_{12} as his decision from his basic beliefs $\hat{g}_1^1$. This is a proof in $\mathrm{GL}_{E_1 F_1}$, where $E_1 = F_1 = \{\epsilon, (1)\}$. The parallel assertion holds for 2. In the game g^2 of Table 5.2, the above result holds only for player 1.

In fact, we can show that $F_1 = \{\epsilon, (1)\}$ is minimal for the above sequent. However, we will discuss the minimality of the suggested F_i only for DC20.

DC21: This criterion for player 2 is formulated $\mathcal{D}_2^{21} = \{D_2^{21}(s_2) : s_2 \in S_2\}$, where each $D_2^{21}(s_2)$ is given as

$$(6) \qquad \bigvee_{s_1} \mathrm{B}_1(\mathrm{Dom}_1(s_1)) \bigwedge \bigwedge_{s_1} (\mathrm{B}_1(\mathrm{Dom}_1(s_1)) \supset \mathrm{Best}_2(s_2 \mid s_1)).$$

In game g^2, we can prove

$$(7) \qquad \vdash_{E_2F_2} \mathrm{B}_2(\hat{g}_2^2), \mathrm{B}_2\mathrm{B}_1(\hat{g}_1^2) \rightarrow \mathrm{B}_2(D_2^{21}(\mathrm{s}_{22})),$$

where $E_2 = F_2 = \{\epsilon, (2), (2,1)\}$. The set $\mathrm{B}_2(\hat{g}_2^2) \cup \mathrm{B}_2\mathrm{B}_1(\hat{g}_1^2)$ means that 2 has the beliefs on his own payoff function g_2^2 and believes that 1 has the beliefs on 1' payoff function g_1^2. The latter is needed to derive 2's prediction on 1's decision. Here, $F_2 = \{\epsilon, (2), (2,1)\}$ is minimal.

DC0: The default decision criterion $\mathcal{D}_i^0$ is given as

$$(8) \qquad \mathcal{D}_i^0 = \{d_i(s_i) : s_i \in S_i\}.$$

To derive a default decision, player i needs to assume that some is a default decision. Let $\Gamma_i^0 = \{d_i(\mathrm{s}_{i1})\} \cup \{\neg d_i(\mathrm{s}_{it}) : t = 2, ..., \ell_i\}$. Then

$$(9) \qquad \vdash_{E_iF_i} \mathrm{B}_i(\Gamma_i^0) \rightarrow \mathrm{B}_i(d_i(\mathrm{s}_{i1})),$$

$$(10) \qquad \vdash_{E_iF_i} \mathrm{B}_i(\Gamma_i^0) \rightarrow \mathrm{B}_i(\neg d_i(\mathrm{s}_{it})) \text{ for } t = 2, ..., \ell_i,$$

where $E_i = \{\epsilon, (i)\}$ and $F_i = \{\epsilon\}$. Here player i chooses s_{i1} *dogmatically* and is *conscious of* the other strategies not to be default decisions. Here, player i needs no beliefs on the game but does his dogmatic beliefs Γ_i^0. He conducts no logical reasonings to have his decision.

DC20: This decision criterion for player 2 is expressed as $\mathcal{D}_2^{20} = \{D_2^{20}(s_2) : s_2 \in S_2\}$, where each $D_2^{20}(s_2)$ is given as

$$(11) \qquad \bigvee_{s_1} \mathrm{B}_1(d_1(s_1)) \bigwedge \bigwedge_{s_1} (\mathrm{B}_1(d_1(s_1)) \supset \mathrm{Best}_2(s_2 \mid s_1)).$$

For this, 2 predicts that 1 would choose a default decision. In the game g^2 of Table 5.2, the following holds:

$$(12) \qquad \vdash_{E_2F_2} \mathrm{B}_2(\hat{g}_2^2), \mathrm{B}_2\mathrm{B}_1(\Gamma_1^0) \rightarrow \mathrm{B}_2(D_2^{20}(\mathrm{s}_{21})),$$

where $E_2 = F_2 = \{\epsilon, (2), (2,1)\}$.

For (12), $F_2 = \{\epsilon, (2), (2,1)\}$ is minimal. The reader may wonder why this minimal F_2 includes $(2,1)$, since 2 believes that 1 simply uses his dogmatic decision s_{11}. To verify $D_2^{20}(\mathrm{s}_{21})$, however, 2 derives $\neg\mathrm{B}_1(d_1(\mathrm{s}_{12}))$ from $\mathrm{B}_1(\neg d_1(\mathrm{s}_{12}))$. Here, 2 needs to think about the relationship between $\neg\mathrm{B}_1(d_1(\mathrm{s}_{12}))$ and $\mathrm{B}_1(\neg d_1(\mathrm{s}_{12}))$.

Theorem 5.2 $F_2 = \{\epsilon, (2), (2,1)\}$ *is a unique minimal inferential epistemic structure for (12).*

Proof. It suffices to show $(2) \in F_2$ and $(2,1) \in F_2$. Suppose $\vdash_{E_2F_2} \mathrm{B}_2(\hat{g}_2^2), \mathrm{B}_2\mathrm{B}_1(\Gamma_1^0) \to \mathrm{B}_2(D_2^{20}(\mathrm{s}_{21}))$. By Theorem 3.5.(2), we have $(2) \in F_2$. By Lemma 3.6, we have $\vdash_{E_2F_2} \mathrm{B}_2[\hat{g}_2^2, \mathrm{B}_1(\Gamma_1^0) \to D_2^{20}(\mathrm{s}_{21})]$. Here, we denote $(E_2)_{-2}$ and $(F_2)_{-2}$ by E_{-2} and F_{-2}, respectively. Hence $\vdash_{E_{-2}F_{-2}} \hat{g}_2^2, \mathrm{B}_1(\Gamma_1^0) \to D_2^{20}(\mathrm{s}_{21})$ by Lemma 3.7. Since $D_2^{20}(\mathrm{s}_{21})$ is given as (11), we have $\vdash_{E_{-2}F_{-2}} \hat{g}_2^2, \mathrm{B}_1(\Gamma_1^0) \to \neg\mathrm{B}_1(d_1(\mathrm{s}_{12})) \bigvee \mathrm{Best}_2(\mathrm{s}_{21} \mid \mathrm{s}_{12})$. By Theorem 4.4,

$$\vdash_{E_{-2}F_{-2}} \hat{g}_2^2 \to \mathrm{Best}_2(\mathrm{s}_{21} \mid \mathrm{s}_{12}) \text{ or } \vdash_{E_{-2}F_{-2}} \mathrm{B}_1(\Gamma_1^0) \to \neg\mathrm{B}_1(d_1(\mathrm{s}_{12})).$$

Due to Table 5.2, the first is not the case. From the second, we have $\vdash_{E_{-2}F_{-2}} \mathrm{B}_1(d_1(\mathrm{s}_{12})), \mathrm{B}_1(\Gamma_1^0) \to$. By Theorem 3.5.(1), we have $(1) \in F_{-2}$, which implies $(2,1) \in F_2$. $\dashv$

Simple Dogmatic Belief: Consider the following belief set Γ_i^{00} rather than $\Gamma_i^0 = \{d_i(\mathrm{s}_{i1})\} \cup \{\neg d_i(\mathrm{s}_{it}) : t = 2, ..., \ell_i\}$:

$$\Gamma_i^{00} = \{d_i(\mathrm{s}_{i1})\}. \tag{13}$$

In (13), player i is conscious only of his first strategy s_{i1} and ignores the other possibilities. In this case, the result for player i's own decision making is more or less the same as (9), i.e., $\vdash_{E_iF_i} \mathrm{B}_i(\Gamma_i^{00}) \to \mathrm{B}_i(d_i(\mathrm{s}_{i1}))$, where $E_i = \{\epsilon, (i)\}$ and $F_i = \{\epsilon\}$.

However, if $\mathrm{B}_1(\Gamma_1^{00})$ is used as a belief set for player 2, the result would be different from (12). First, we give our result in a sightly general manner. After the theorem, we discuss specific cases.

Theorem 5.3 *Let $g = (g_1, g_2)$ be a game where 2 has no dominant strategy, and Γ_2 a set of $\{1\}$-formulae. Then $\vdash_{E_2F_2} \mathrm{B}_2(\hat{g}_2), \mathrm{B}_2(\Gamma_2) \to \mathrm{B}_2(D_2^{20}(s_2))$ if and only if*

(1): $\vdash_{E_2F_2} \mathrm{B}_2[\Gamma_2 \to \mathrm{B}_1(d_1(s_1))]$ *for some* s_1;

(2): *for any* $s_1 \in S_1$, $\vdash_{E_2F_2} \mathrm{B}_2[\hat{g}_2 \to \mathrm{Best}_2(s_2 \mid s_1)]$ *or* $\vdash_{E_2F_2} \mathrm{B}_2[\Gamma_2 \to \neg\mathrm{B}_1(d_1(s_1))]$.

Remark that (2) is equivalent to

(2*): *for any* $s_1 \in S_1$, $\vdash_{\{\epsilon\}\{\epsilon\}} \hat{g}_2 \to \mathrm{Best}_2(s_2 \mid s_1)$ *or* $\vdash_{E_{-2}F_{-2}} \Gamma_2 \to \neg\mathrm{B}_1(d_1(s_1))$.

Proof. We prove the *only-if* part. Suppose that $\vdash_{EF} \mathrm{B}_2(\hat{g}_2), \mathrm{B}_2(\Gamma_2) \to \mathrm{B}_2(D_2^{20}(s_2))$. Since $D_2^{20}(s_2) \notin \hat{g}_2 \cup \Gamma_2$ by the assumption of the theorem, we have $(2) \in F_2$ by Theorem 3.5.(2). Then $\vdash_{E_2F_2} \mathrm{B}_2[\hat{g}_2, \Gamma_2 \to D_2^{20}(s_2)]$ by Lemma 3.6. Since $D_2^{20}(s_2)$ is given as (11), we have

$$\vdash_{E_2F_2} \mathrm{B}_2[\hat{g}_2, \Gamma_2 \to \bigvee_{s_1} \mathrm{B}_1(d_1(s_1))] \tag{14}$$

$$\vdash_{E_2F_2} \mathrm{B}_2[\hat{g}_2, \Gamma_2 \to \mathrm{B}_1(d_1(s_1)) \supset \mathrm{Best}_2(s_2 \mid s_1)] \text{ for all } s_1 \in S_1. \tag{15}$$

First, consider (15). This is equivalent to that $\vdash_{E_2F_2} \mathrm{B}_2[\hat{g}_2, \Gamma_2 \to \neg\mathrm{B}_1(d_1(s_1)) \bigvee \mathrm{Best}_2(s_2 \mid s_1)]$ for all $s_1 \in S_1$. Let s_1 be an arbitrary one in S_1. By Lemma 3.7, we have $\vdash_{E_{-2}F_{-2}} \hat{g}_2, \Gamma_2 \to \neg\mathrm{B}_1(d_1(s_1)) \bigvee \mathrm{Best}_2(s_2 \mid s_1)$. By Theorem 4.4, we have

$$\vdash_{E_{-2}F_{-2}} \hat{g}_2 \to \mathrm{Best}_2(s_2 \mid s_1) \text{ or } \vdash_{E_{-2}F_{-2}} \Gamma_2 \to \neg\mathrm{B}_1(d_1(s_1)).$$

This implies the statement (2).

Second, consider (14). By Lemma 3.7, we have $\vdash_{E_{-2}F_{-2}} \hat{g}_2, \Gamma_2 \to \bigvee_{s_1} \mathrm{B}_1(d_1(s_1))$. Since Γ_2 is a set of $\{1\}$-formulae, we have $\vdash_{E_{-2}F_{-2}} \Gamma_2 \to \bigvee_{s_1} \mathrm{B}_1(d_1(s_1))$ or $\vdash_{E_{-2}F_{-2}} \hat{g}_2 \to$ by Theorem 4.4. Since $\hat{g}_2$ is a consistent set, the former is the case. Suppose $\nvdash_{E_{-2}F_{-2}} \Gamma_2 \to \mathrm{B}_1(d_1(s_1))$ for any s_1. By (2), $\vdash_{E_{-2}F_{-2}} \hat{g}_2 \to \mathrm{Best}_2(s_2 \mid s_1)$ for any $s_1 \in S_1$, which implies $\vdash_{E_{-2}F_{-2}} \hat{g}_2 \to \bigwedge_{s_1} \mathrm{Best}_2(s_2 \mid s_1)$. This means that 2 has a dominant strategy in g, which contradicts the assumption of the theorem. Thus $\vdash_{E_{-2}F_{-2}} \Gamma_2 \to \mathrm{B}_1(d_1(s_1))$ for some s_1. Hence $\vdash_{E_2F_2} \mathrm{B}_2[\Gamma_2 \to \mathrm{B}_1(d_1(s_1))]$ for some s_1. $\dashv$

Let us return to the simple dogmatic default belief set $\Gamma_1^{00} = \{d_1(\mathrm{s}_{11})\}$ in the game g^2 of Table 5.2. In contrast to (12), we have

$$\nvdash_{E_2F_2} \mathrm{B}_2(\hat{g}_2^2), \mathrm{B}_2\mathrm{B}_1(\Gamma_1^{00}) \to \mathrm{B}_2(D_2^{20}(\mathrm{s}_{21})) \text{ for } E_2 = F_2 = \{\epsilon, (2), (2,1)\}.$$

This unprovability is a consequence of Theorem 5.2: Suppose, on the contrary, that this sequent is provable. Then $\vdash_{E_2F_2} \mathrm{B}_2[\hat{g}_2^2 \to \mathrm{Best}_2(\mathrm{s}_{21} \mid \mathrm{s}_{12})]$ or $\vdash_{E_2F_2} \mathrm{B}_2[\mathrm{B}_1(\Gamma_1^{00}) \to \neg\mathrm{B}_1(d_1(\mathrm{s}_{12}))]$. However, neither is the case because of Table 5.2 and $\Gamma_1^{00} = \{d_1(\mathrm{s}_{11})\}$.

One possible way out from the above unprovability is to expand $\mathrm{B}_1(\Gamma_1^{00})$ into

(16) $$\Gamma_2^0 = \{\mathrm{B}_1(d_1(\mathrm{s}_{11})), \neg\mathrm{B}_1(d_1(\mathrm{s}_{12})), ..., \neg\mathrm{B}_1(d_1(\mathrm{s}_{1\ell_1}))\}.$$

Then we have the following:

(17) $$\vdash_{E_iF_i} \mathrm{B}_2(\hat{g}_2^2), \mathrm{B}_2(\Gamma_2^0) \to \mathrm{B}_2(D_2^{20}(\mathrm{s}_{21})),$$

where $E_2 = \{\epsilon, (2), (2,1)\}$ and $F_2 = \{\epsilon, (2)\}$. This F_2 is the minimal inferential epistemic structure. Here, player 2 takes into account that player 1 does not think about the second strategy s_{12} in the simple dogmatic default belief Γ_1^{00}.

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