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PRICING PATH DEPENDENT SECURITIES BY THE EXTENDED TREE METHOD

by

Naoki Kishimoto

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* University of Tsukuba, Institute of Policy and Planning Sciences, Tsukuba, Ibaraki 305-8573, Japan, (0298) 53-5096. The author thanks Giovanni Barone-Adesi, Douglas Breeden, Douglas Foster, Chi-fu Huang, Andrew Lo, Kevin McCardle, and seminar participants at Duke University, MIT, and European Finance Association Meetings for helpful comments on earlier drafts. In addition, Thomas Ho's excellent introduction to the U.S. mortgage-backed securities market and financial support from Japan Productivity Center are gratefully acknowledged. Part of this paper was circulated in earlier papers, titled "Pricing Path Dependent Securities by the Extended Binomial Tree" and "A Simplified Approach to Pricing Path Dependent Securities." The usual disclaimer applies.

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Abstract

This paper presents a discrete time method (ET method) for pricing path dependent securities (PDSs) by the supplementary variable technique and examines the ET method rigorously from the point of view of Arrow-Debreu's event tree. If supplementary variables satisfy certain conditions, the ET method always yields the same PDS price as a valuation method based on Arrow-Debreu's event tree. In addition, the ET method compares favorably with competing methods in computational size. Two applications of the ET method are presented: collateralized mortgage obligations and American average options where the average is computed over a moving period with a fixed length.

PRICING PATH DEPENDENT SECURITIES BY THE EXTENDED TREE METHOD

I. Introduction

Since the seminal work of Goldman, Sosin, and Gatto (1979) on lookback options, extensive research has been carried out on various types of path dependent securities (PDSs), such as average options, barrier options, and mortgage-backed securities. The purposes of this paper are to present a discrete time method for pricing PDSs where the path dependence is dealt with by the supplementary variable technique and to examine this method rigorously from the point of view of Arrow-Debreu's event tree.[1] This valuation method is referred to as the extended tree (ET) method. It is developed in discrete time for two reasons. First, the discrete time setup makes the method directly applicable to PDSs of American type and PDSs that contain embedded options of American type. Second, the discrete time setup allows us to easily account for time lags of specified lengths, which are found in typical floating rate notes, adjustable rate mortgages, and average options.

A number of researchers valued PDSs by the supplementary variable technique in discrete time. Earlier examples include McDermott, Hakanoglu, and Roman (1988) on corporate bonds with optional sinking fund provisions, Kishimoto (1989) on collateralized mortgage obligations (CMOs) and floating rate notes, Kau, Keenan, Muller, and Epperson (1990) on adjustable rate mortgages, Hull and White (1993) (HW) on lookback options, average options, and mortgage-backed securities, and Thompson (1995) on take-or-pay contracts, which are commonly employed in the natural gas industry. In addition, the forward shooting grid method proposed by Barraquand and Pudet (1996) can be viewed as an implementation of the supplementary variable technique for computing partial differential equations numerically.

The ET method builds on the work of Cox, Ross, and Rubinstein (1979) (CRR) in specifying the uncertainty model for an underlying state variable and Goldman, Sosin, and Gatto (1979) in dealing with path dependence by the supplementary variable technique. In particular, we assume that a state variable follows a binomial process, which is a generalization of CRR's binomial process. In addition, we assume that for a PDS

[1] See, for example, Cox and Miller (1965) and Bertzekas (1987) for expositions of the supplementary variable technique in general.

in question, there exists a set of supplementary variables that summarizes past realizations of the state variable and satisfies five conditions.[2] First, the set of values each supplementary variable may take on is finite. Second, if the PDS pays cash flows other than those that are specified by boundary and terminal conditions, they are correctly specified for each combination of time, the state variable, and the supplementary variables. Third, if the PDS has boundary conditions, they are correctly specified for each combination of time, the state variable, and the supplementary variables. Fourth, the terminal condition of the PDS is correctly specified for each combination of the state variable and the supplementary variables at the maturity of the PDS. Fifth, conditional on time, the state variable, and the supplementary variables, the ensuing values of the supplementary variables are correctly determined by the ensuing value of the state variable. Under these conditions, the PDS admits a one-period risk-neutral valuation relationship (RNVR) conditional on the supplementary variables. Therefore, with boundary and terminal conditions, the PDS can be valued by applying the one-period RNVR recursively.

We develop our valuation method in a general and formal setting for three reasons. The first reason is to examine this method rigorously. In particular, we prove that the conditions listed above are sufficient to ensure that the ET method always yields the same PDS price as a valuation method based on a comparable Arrow-Debreu's event tree; a proof requires a formal setup. The second reason is that once we have a general and formal framework, we can reduce the problem of valuing a complex PDS to that of finding appropriate supplementary variables. In other words, we can concentrate on a smaller and well-defined problem. The third reason is that we can address the computational size of the ET method in general terms. In particular, we present a measure for the computational size of the ET method, and show that the ET method yields exact model prices with manageable amounts of computation.

This paper is organized as follows. Section II sets up a valuation framework for PDSs and derives a one-period RNVR conditional on supplementary variables. In addition, this section presents a valuation procedure based on the one-period RNVR. The next two sections present applications of the ET method to two types of PDSs. In the first application, the CMO model by Kishimoto (1989) is extended to CMOs where a CMO collateral is modeled as a pool of mortgage loans with heterogeneous prepayment costs. In the second application, we value American average options where

[2] Some, but not all, of these conditions were pointed out by Kishimoto (1989) and HW. A precise statement of these conditions is given in Assumption 6.

the average is computed over a moving period with a fixed length. We call such options American moving average options to distinguish them from ordinary average options studied in earlier papers. Section V discusses the valuation of PDSs on Arrow-Debreu's event tree and examines the ET method from the point of view of the event tree. In particular, this section shows that the ET method yields the same PDS price as a valuation method based on a comparable event tree. In addition, Section V presents a measure for the computational size of the ET method and shows that the ET method requires much less computation than the method based on a comparable event tree. Finally, Section VI summarizes the paper and provides additional discussion.

II. Extended Tree Method

A. The Basic Framework

These are the basic assumptions of the ET method.[3]

Assumption 1. The market opens at $N+1$ discrete points in time, which are indexed by $n = 0, \ldots, N$.

The discrete point indexed by n is called *time* n, and the interval from time $n-1$ to time n is referred to as *period* n.

Assumption 2. The market is frictionless unless otherwise noted.

Assumption 3. At time $n = 1, \ldots, N$, one of two events, an *up event* and a *down event*, occurs. The event at time n is independent of events at time $1, \ldots, n-1$.

The number of up events that occur from time 1 to time n is denoted by $i_n = 0, \ldots, n$ and referred to as the *underlying state variable.*[4] Clearly, each pair of (n, i_n) corresponds to a node of a binomial tree, which is labeled as a *binomial node.*

Assumption 4. There exists a security such that its price is uniquely determined at

[3] These assumptions can be relaxed to extend the ET method to other discrete time, discrete state frameworks, including discretized versions of continuous time models as in Boyle (1988).

[4] In addition, we include time 0 in the time domain of the underlying state variable, where we set $i_0 = 0$.

each binomial node (n, i), $i = 0, \ldots, n$, $n = 0, \ldots, N$, and its price at (n, i) differs from that at $(n, i-1)$, $i = 1, \ldots, n$, $n = 1, \ldots, N$.

In CRR, for example, an underlying stock satisfies Assumption 4. In Ho and Lee (1986) (HL), any default-free bond satisfies it.

Assumption 5. There exists a risk-free asset at time $0, \ldots, N-1$, where the risk-free rate is specified at each binomial node (n, i), $i = 0, \ldots, n$, $n = 0, \ldots, N-1$.

Let $r(n, i)$ denote one plus the risk-free rate prevailing at a binomial node (n, i). The risk-free rate can be either stochastic or deterministic, depending on the specification of the model. If it is stochastic, $r(n, i)$ differs across $i = 0, \ldots, n$ for a given value of n. If it is deterministic, $r(n, i)$ is identical across $i = 0, \ldots, n$ for a given value of n. Clearly, the framework set up by Assumptions 1-5 contains CRR and HL as special cases.

Next, we define terms and notations for securities where the cash flow depends on both current and past realizations of an underlying state variable. Set $e_n = (i_0, \ldots, i_n)$. In addition, let E_n denote the set of values e_n may take on at time n. Each element of E_n is referred to as a *path* through time n. The structure of paths can be represented by a tree as in Debreu ((1959), p.99), where each node represents a particular path, i.e., a particular value of e_n. We call this tree a *path tree* (PT) and each node a *path tree* (PT) *node*. An example of the path tree is illustrated in Figure 1. A security is labeled as *path dependent*, if the determination of its cash flow requires more information on the path than the current value of the underlying state variable. In addition, a security is labeled as *path independent*, if the determination of its cash flow requires no more information on the path than the current value of the underlying state variable.

For some PDSs, we can find a variable or a set of variables that summarizes information on the path and, with the current value of an underlying state variable, provides sufficient information to determine cash flows. We label these variables as *supplementary variables* and develop a valuation method for PDSs based on them. Specifically, for a PDS in question, we first identify supplementary variables at each time $n = 0, \ldots, N$. Let $h_n^{(j)}$ denote the j^{th} supplementary variable at time n and $\mathbf{h}_n$ the vector of all, say, m supplementary variables at time n, $(h_n^{(1)}, \ldots, h_n^{(m)})$.[5] Second, we

[5] In general, m depends on n. Note also that m can be set at 0 initially, because there is no need to define supplementary variables at time 0.

identify a set that contains the set of all values each supplementary variable may take on at time n.[6] Third, we define the *state space* at time n of $\mathbf{h}_n$ as Cartesian product of the sets that are identified above for the individual supplementary variables.[7] Fourth, we define Z_n as the set of points in Euclidean $(m+2)$-space that $(n, i_n, \mathbf{h}_n)$ may take on at time n. Frequently, $(n, i_n, \mathbf{h}_n)$ is denoted by z_n. Given these terms and notations, we impose five conditions on PDSs.[8]

Assumption 6. For a PDS with a finite life through time N, there exist a set of supplementary variables and its state spaces such that:

i) At each $n = 0, \ldots, N$ when the supplementary variables are defined, the state space is finite and contains all values the set of supplementary variables, $\mathbf{h}_n$, may take on.

ii) If the PDS pays cash flows other than those that are specified by boundary and terminal conditions, they are correctly specified at all $z \in Z_n$, $n = 1, \ldots, N$.

iii) If the PDS has boundary conditions, they are correctly specified at all $z \in Z_n$, $n = 0, \ldots, N-1$.

iv) The terminal condition of the PDS is correctly specified at every $z \in Z_N$.

v) Given $z \in Z_n$, $\mathbf{h}_{n+1}$ is correctly determined by $i_{n+1}, n = 0, \ldots, N-1$. Furthermore, this value of $\mathbf{h}_{n+1}$ is contained by the state space at time $n+1$ of $\mathbf{h}_{n+1}$.

The condition ii) is not imposed at time 0, because we solve for the initial price, exclusive of the cash flow at time 0. ii)-iv) jointly require that the $(m+2)$-tuple $z_n = (n, i_n, \mathbf{h}_n)$ provides sufficient information to determine the cash flow to the PDS correctly. v) requires that the ensuing value of the set of supplementary variables, $\mathbf{h}_{n+1}$, is correctly determined by its current value, the current time, and the current and ensuing values of the underlying state variable.[9] In addition, v) requires that

[6] In general, identifying the exact set of all values each supplementary variable may take on at time n requires more information on the path than the supplementary variables provide. This is why we identify a set that is sufficiently large to contain the exact set. Clearly, the set we identify is not unique.

[7] We make state spaces to be specific to time n to keep them relatively small and to accommodate the situation where different sets of supplementary variables are defined at different points in time. Incidentally, for some PDSs, such as the one discussed in Section IV, state spaces can be made even smaller if they are made specific to each binomial node.

[8] In Section V, we elaborate on the term "correctly" used in Assumption 6.

[9] This part of v) holds if and only if $(i_n, \mathbf{h}_n)$ is Markov. The proof is available upon

this value at time $n+1$ of the set of supplementary variables is contained by the state space at time $n+1$. In other words, we must construct state spaces such that each of them satisfies this condition in relation to the immediately preceding one. As these conditions suggest, the choice of supplementary variables and the construction of their state spaces are the most important considerations in implementing our method. Note also that the structure of $(m+2)$-tuples $Z_0, \ldots, Z_N$ can be illustrated by a tree diagram, as in Figure 2, where each node represents a particular value of a $(m+2)$-tuple $z_n = (n, i_n, \mathbf{h}_n)$. We label this tree as the *extended tree* (ET) and each node as an *extended tree* (ET) *node.*[10] Clearly, the ET provides more coarse information than the path tree, but finer information than the binomial tree of the underlying state variable.

B. Valuation of Path Dependent Securities on the Extended Tree

We turn to the valuation of a generic PDS that satisfies Assumption 6. Let $S(n,i)$ denote the price at a binomial node (n,i) of a security that satisfies Assumption 4, and $C(n,i,\mathbf{h})$ the price at an ET node $(n,i,\mathbf{h})$ of the PDS.[11] In addition, let $X(n,i,\mathbf{h})$ denote the cash flow at an ET node $(n,i,\mathbf{h})$ to the PDS, as specified by ii) of Assumption 6. Now, v) of Assumption 6 implies that given $(n,i,\mathbf{h}) \in Z_n$, the value of $\mathbf{h}_{n+1}$ is determined for each of $i_{n+1} = i+1$ and $i_{n+1} = i$. Let $\mathbf{h}^u$ and $\mathbf{h}^d$ denote the value of $\mathbf{h}_{n+1}$ when $i_{n+1} = i+1$ and when $i_{n+1} = i$, respectively. Using these notations, we present one-period RNVR for the PDS.

Proposition 1. Under Assumptions 1-5, if a PDS satisfies Assumption 6, its arbitrage-free price at an ET node $(n,i,\mathbf{h}) \in Z_n$, $n = 0, \ldots, N-1$, is given by $C_{AF}(n,i,\mathbf{h})$, where

$$(1) \qquad C_{AF}(n,i,\mathbf{h}) = \frac{1}{r(n,i)}[p(n,i)(C(n+1,i+1,\mathbf{h}^u) + X(n+1,i+1,\mathbf{h}^u)) \\ + (1-p(n,i))(C(n+1,i,\mathbf{h}^d) + X(n+1,i,\mathbf{h}^d))],$$

and

$$(2) \qquad p(n,i) = \frac{r(n,i)S(n,i) - S(n+1,i)}{S(n+1,i+1) - S(n+1,i)}.$$

request.

[10] A binomial node is viewed as an ET node when supplementary variables are not defined.

[11] $\mathbf{h}$ drops as an argument in these and all other expressions in the paper whenever supplementary variables are not defined.

Proof. See Appendix.

$p(n,i)$ is the risk-neutral probability that an up event occurs at time $n+1$ following the binomial node (n,i). Therefore, (1) implies that the arbitrage-free price of the PDS is given by the expected value of the ensuing price plus the cash flow of the PDS that is discounted by the current risk-free rate. Certainly, this expectation is computed with respect to the risk-neutral probabilities.[12]

Next, we specify boundary and terminal conditions for the generic PDS. First, the price at time N of the PDS is determined by the terminal condition.

$$C(N,i,\mathbf{h}) = H(i,\mathbf{h}), \quad \forall (N,i,\mathbf{h}) \in Z_N. \tag{3}$$

In addition, the PDS may have upper and/or lower bounds, which are specified by iii) of Assumption 6. Let $U(n,i,\mathbf{h})$ and $L(n,i,\mathbf{h})$ denote the upper and lower bounds at an ET node $(n,i,\mathbf{h})$, respectively. Then, we have

$$L(n,i,\mathbf{h}) \leq C(n,i,\mathbf{h}) \leq U(n,i,\mathbf{h}), \quad \forall (n;i,\mathbf{h}) \in Z_n, \quad n = 0,\ldots;N-1. \tag{4}$$

Given these conditions, the PDS can be valued as follows. First, (3) determines the PDS price at time N. Second, (1) is used to compute the arbitrage-free price at time $N-1$ of the PDS. Third, the PDS price at time $N-1$ is determined by combining (1) and (4).

$$C(N-1,i,\mathbf{h}) = \max\left[\, L(N-1,i,\mathbf{h}), \min(C_{AF}(N-1,i,\mathbf{h}), U(N-1,i,\mathbf{h}))\right]. \tag{5}$$

Similarly, the PDS price at time $n = N-2, N-3, \ldots, 0$ is computed by (1) and (4) recursively. To be precise, the *extended tree* (ET) *method* refers to this valuation procedure.

III. Application of the ET Method to CMOs

In this section, we present an application of the ET method to CMOs. The primary purpose of this presentation is to illustrate how the ET method can be applied to CMOs. Therefore, for ease of exposition, we take a simplified view on underlying mortgage loans and consider a CMO with very simple structure. Note, however, that

[12] (1) would reduce to the one-period RNVR of CRR (or HL), if the uncertainty were modeled as in CRR (or HL), and if the security to be valued were path independent.

with slight modifications, our CMO model can be applied to various CMOs with more complex structures.

In general, a CMO is a set of multiple classes of bonds (or tranches) collateralized by a pool of mortgage loans where the cash flow from the collateral is allocated among the bond classes by prespecified rules. Each mortgage loan is accompanied with a prepayment option, which allows its holder, i.e., the mortgagor, to repay a part or the entirety of the loan any time before the stated maturity. It is widely observed in the U.S. mortgage market that bulks of loans are prepaid when market interest rates fall below contract interest rates and that loans differ in interest rate differential that triggers an interest-rate-related prepayment. As a result, prepayments are heavily dependent on the interest rate path, which leads to the strong path dependence of CMO bonds.

Our CMO model has two features, which are designed to address the path dependence of CMOs. First, following the studies on pass-throughs by Timmis (1985) and Johnston and Van Drunen (1988) (JV), we model the collateral of a CMO as a pool of mortgage loans that differ in costs of prepayment. This heterogeneity in prepayment costs is intended as a mechanism for generating the path dependence of prepayments. According to the study of pass-throughs by Stanton ((1995), p. 677), this way of modeling collaterals "captures many of the empirical features of mortgage prepayments."[13] Note, however, that adopting this collateral model for CMOs entails the difficulty that did not exist in Timmis (1985) and JV. Namely, because a pass-through represents a claim to the entire collateral, it can be valued by summing the values of underlying mortgage loans, as in Timmis (1985) and JV. By contrast, a CMO bond does not have a claim to any fixed portion of the collateral. Rather, its relation to each portion of the collateral changes, depending on the past prepayment history. Therefore, CMO bonds cannot be valued by the same method as in Timmis (1985) and JV.

Fortunately, this seemingly complex path dependence of CMOs can be dealt with by the ET method, if the collateral is modeled as in Timmis (1985) and JV. Therefore, we value CMOs by the ET method, which is the second feature of our CMO model. As discussed in Section V, the ET method has clear numerical advantages over Monte Carlo simulation, which is the standard method for CMO valuation.[14] One advantage

[13] To be precise, Stanton (1995) extended Timmis (1985) and JV in several ways, where the most important extension was to allow prepayment decisions to occur at random discrete intervals.

[14] McConnell and Singh (1994) and Singh and McConnell (1996) used Monte Carlo

is that the ET method yields model prices free of statistical errors, which are inherent in Monte Carlo simulation. Another advantage is that the ET method compares favorably with Monte Carlo simulation in computational size.

A. Model for Collaterals

For ease of exposition, we take a very simple view on CMO collaterals. First, we assume that all loans in a collateral are default-free and identical in contract interest rate, stated maturity, and scheduled payments for each dollar of the initial principal.[15] In particular, we set

$N =$ number of periods from time 0 to the stated maturity.

$c =$ contract interest rate per period.

$X_n =$ scheduled payment at time $n = 1, \ldots, N$ for each dollar of the initial principal.

$F_n =$ principal remaining in the n^{th} period, $n = 1, \ldots, N + 1$, for each dollar of the initial principal, given that the loan has not been prepaid. Clearly $F_1 = 1$. We assume that the loans are fully amortizing; namely, $F_{N+1} = 0$.

Each mortgagor is given a prepayment option. Prepayment options are exercised for various reasons, including refinancing, moving, default, deaths, and divorce of mortgagors. Clearly, prepayments for refinancing are driven by interest rate movements. In addition, it is likely that prepayments due to moving and default are partially correlated with general economic conditions and as a result, with interest rates. To account for these interest-rate-related prepayments, we assume that each mortgagor prepays if an arbitrage profit from prepayment exceeds costs of prepayment. Furthermore, to generate prepayment heterogeneity within a collateral, we assume that mortgagors in the collateral differ in prepayment costs. Note that prepayment costs in our model include not only monetary costs but also non-monetary costs, such as the opportunity costs of keeping track of interest rate movements and executing transactions for prepayment.

simulation to value CMOs, where the collateral of a CMO was modeled as in Stanton (1995). Childs, Ott and Riddiough (1996) extended the valuation method of McConnell and Singh (1994) to CMOs backed by commercial mortgages.

[15] Mortgage loans underlying CMOs are typically collateralized through agency pass-through securities. Therefore, they can be viewed as default-free from the standpoint of CMO investors.

Therefore, prepayment costs may vary substantially across mortgagors. Specifically, we assume that the costs of prepaying a loan are proportional to the remaining principal of the loan and that mortgagors differ in proportionality factor. Let G_k denote the k^{th} lowest proportionality factor, where $k = 1, 2, \ldots, \bar{k}$. A loan with the proportionality factor of G_k is labeled as a loan of type k. B_k denotes the total principal at time 0 of type-k loans. Then, the initial principal B of the entire collateral satisfies the relation

$$B = \sum_{k=1}^{\bar{k}} B_k. \tag{6}$$

In addition, some prepayments occur independent of interest rate movements. Such prepayments can be attributed to events such as moving, default, deaths, and divorce of mortgagors. We refer to these interest-rate-unrelated prepayments as exogenous prepayments.[16] Now, suppose that interest rate uncertainty constitutes the only systematic source for prepayments and that exogenous prepayments are completely independent of the interest rate uncertainty. Then, because there are several thousands of mortgage loans in a typical CMO collateral, the law of large numbers implies that the exogenous prepayment rate of the entire collateral is roughly nonrandom. Therefore, it is plausible that the market expects some known proportion of the collateral to be prepaid for exogenous reasons at each time n. To implement this idea in a tractable manner, we assume that all mortgagors in the collateral make partial prepayments gradually, such that the sum of these partial prepayments at time n coincides with the market expectation of exogenous prepayments in the collateral at time n. In particular, we assume that a known proportion δ_n of the remaining principal of each loan is prepaid for exogenous reasons at time $n = 1, \ldots, N-1$ and that mortgagors do not incur costs in these partial prepayments. In addition, we define

$$\alpha_n = \begin{cases} 1, & \text{if } n = 1; \\ \prod_{j=1}^{n-1}(1-\delta_j), & \text{if } n = 2, \ldots, N. \end{cases} \tag{7}$$

Then, given that a loan has not been prepaid for refinancing, the partial prepayments reduce the period-n principal of the loan for each dollar of the initial principal to

$$F'_n = \begin{cases} \alpha_n F_n, & \text{if } n = 1, \ldots, N; \\ 0, & \text{if } n = N+1. \end{cases} \tag{8}$$

[16] To be precise, we should exclude the interest-rate-related component of prepayments due to moving, default, deaths, and divorce from exogenous prepayments.

Furthermore, these partial prepayments alter the time-n cash flow of the loan for each dollar of the initial principal to

$$X'_n = \begin{cases} \alpha_n X_n + \alpha_n \delta_n F_{n+1}, & \text{if } n = 1, \ldots, N-1; \\ \alpha_n X_n, & \text{if } n = N. \end{cases} \tag{9}$$

The term $\alpha_n \delta_n F_{n+1}$ accounts for the partial prepayment at time n, exclusive of the scheduled principal payment.[17] Note that if we allow δ_n to change over time, we can easily incorporate stylized facts about prepayments, such as the seasonality of prepayments and low prepayment rates in the early stage of the collateral life, into our collateral model.

We now turn to the valuation of a loan of type k. First, we specify a one-factor interest rate model that is constructed on a binomial tree. Among such models, we choose the one by HL because it is a simple and well-known binomial model for interest rate uncertainty. Next, let $V_k(n,i)$ denote the value at a binomial node (n,i) of the type-k loan to the capital market for each dollar of the initial principal. Also, let $W_k(n,i)$ denote the value at (n,i) of the type-k loan to the mortgagor for each dollar of the initial principal. Both $V_k(n,i)$ and $W_k(n,i)$ are exclusive of the cash flow at time n that are specified by ii) of Assumption 6. As Dunn and Spatt (1986) have shown, $V_k(n,i)$ is less than $W_k(n,i)$ because the capital market does not receive servicing fees and prepayment costs the mortgagor incurs. In our model, a servicing fee equal to s times the principal balance is payable at the end of every period.

In essence, $V_k(n,i)$ and $W_k(n,i)$ are computed by the standard backward recursive procedure used by HL and others. That is, our first step is to determine their terminal values, assuming that the loan has not been prepaid for refinancing. Because $F'_{N+1} = 0$, we have

$$W_k(N,i) = 0, \quad i = 0, \ldots, N. \tag{10}$$

$$V_k(N,i) = 0, \quad i = 0, \ldots, N. \tag{11}$$

Our second step is to determine the arbitrage-free value at time $N-1$ of the loan to the mortgagor, $W_k^{AF}(N-1,i)$, and that to the capital market, $V_k^{AF}(N-1,i)$. They are given by

$$W_k^{AF}(N-1,i) = P(N-1,i)X'_N, \quad i = 0, \ldots, N-1. \tag{12}$$

[17] For instance, the principal balance in period 2 is $\alpha_2 F_2$. Hence, the partial prepayment at time 2 is given by $\alpha_2 \delta_2 F_2$. This amount $\alpha_2 \delta_2 F_2$, however, includes part of the scheduled principal payment equal to $\alpha_2 \delta_2 (F_2 - F_3)$, which is also included in $\alpha_2 X_2$. To avoid double counting, $\alpha_2 \delta_2 (F_2 - F_3)$ is subtracted from $\alpha_2 \delta_2 F_2$, which yields $\alpha_2 \delta_2 F_3$.

$$(13) \qquad V_k^{AF}(N-1,i) = P(N-1,i)(X'_N - sF'_N), \quad i = 0,\ldots,N-1.$$

$P(N-1,i)$ represents the price at a binomial node $(N-1,i)$ of a one-period discount bond with \$1 face value. The term sF'_N represents the servicing fee the mortgagor pays but the capital market does not receive. Now, if $W_k^{AF}(N-1,i)$ is greater than or equal to the sum of the remaining principal and the prepayment costs at time $N-1$, the mortgagor prepays the loan to make an arbitrage-free profit.[18] By contrast, if $W_k^{AF}(N-1,i)$ is less than this sum, the mortgagor holds the loan. Therefore, the value at a binomial node $(N-1,i)$ of the loan to the mortgagor is given by

$$(14) \qquad W_k(N-1,i) = \begin{cases} (1+G_k)F'_N, & \text{if } W_k^{AF}(N-1,i) \geq (1+G_k)F'_N; \\ W_k^{AF}(N-1,i), & \text{if } W_k^{AF}(N-1,i) < (1+G_k)F'_N, \end{cases}$$

for $i = 0,\ldots,N-1$. Similarly, the value at a binomial node $(N-1,i)$ of the loan to the capital market is given by

$$(15) \qquad V_k(N-1,i) = \begin{cases} F'_N, & \text{if } W_k^{AF}(N-1,i) \geq (1+G_k)F'_N; \\ V_k^{AF}(N-1,i), & \text{if } W_k^{AF}(N-1,i) < (1+G_k)F'_N, \end{cases}$$

for $i = 0,\ldots,N-1$.

At time $n = N-2, N-3, \ldots, 0$, the arbitrage-free value of the loan to the mortgagor and that to the capital market are given by

$$(16) \qquad W_k^{AF}(n,i) = P(n,i)[\pi W_k(n+1,i+1) + (1-\pi)W_k(n+1,i) + X'_{n+1}];$$

$$(17) \quad V_k^{AF}(n,i) = P(n,i)[\pi V_k(n+1,i+1) + (1-\pi)V_k(n+1,i) + X'_{n+1} - sF'_{n+1}],$$

for $i = 0,\ldots,n$. In these equations π is the risk-neutral probability of an up event in HL. The term sF'_{n+1} represents the servicing fee, which reduces the interest the capital market receives. In short, (16), (17), and analogues of (14) and (15) are used recursively to obtain the initial value of the loan to the mortgagor and that to the capital market.

B. Model for CMOs

This section presents a model for CMOs based on the collateral model in Section III.A. CMOs vary widely in rule for allocating cash flows among CMO bonds. Most

[18] This implies that in this model all mortgagors of the same type prepay entire remaining principals for refinancing at the same time.

CMOs, however, have the common feature that the allocation is based on the principal balance of the collateral. Therefore, in our CMO model, the allocation rule is based solely on the principal balance of the collateral. Furthermore, for ease of exposition, we consider a CMO with extremely simple structure. Specifically, we assume that a CMO consists of two classes of securities, Bond A and a residual.[19] Bond A has principal B_A at time 0. At the end of every period, if Bond A is not fully retired, it receives all principal payments made by the mortgagors in the collateral. In addition, Bond A receives an interest equal to its interest rate c_A times its remaining principal. By contrast, the residual receives the difference between the interest payments made by the mortgagors, exclusive of servicing fees, and the interest paid to Bond A. In addition, if Bond A is fully retired, the residual receives all principal payments made by the mortgagors in the collateral.

As pointed out earlier, CMOs exhibit complex path dependence, which cannot be dealt with by standard backward recursive methods in option pricing. Therefore, we use the ET method to value CMOs. In particular, we define a supplementary variable $h_n = 0, 1, \ldots, \bar{k}$ as the number of types of loans that have been prepaid for refinancing before time n, that is, at time $1, \ldots, n-1$. Note that if it is optimal to prepay a loan of type h, it is optimal to prepay all loans with proportionality factors lower than or equal to G_h. In other words, if $h_n = h$, loans of type 1 to type h must have been prepaid for refinancing previously, and loans of type $h+1$ to type $\bar{k}$ must be outstanding immediately before time n. Hence, the value of h_n implies the amount $Q_n(h)$ of cumulative principal payments made at time $1, \ldots, n-1$. In particular, if $h < \bar{k}$, $Q_n(h)$ is given by the initial collateral principal B minus the sum of the principal balances immediately before time n of type-$(h+1)$ to type-$\bar{k}$ loans. By contrast, if $h = \bar{k}$, all types of loans have been prepaid, and hence, $Q_n(h)$ is equal to B. Combining these two cases, we have

$$Q_n(h) = \begin{cases} B - \sum_{k=h+1}^{\bar{k}} F_n' B_k, & \text{if } h = 0, \ldots, \bar{k}-1; \\ B, & \text{if } h = \bar{k}, \end{cases} \tag{18}$$

for $i = 0, \ldots, n,\ n = 1, \ldots, N$.

Next, let $\eta(n, i, h)$ denote the number of types of loans that are prepaid before and at an ET node (n, i, h). Clearly, if $h = \bar{k}$, $\eta(n, i, h) = \bar{k}$. If $h < \bar{k}$, $\eta(n, i, h)$ is given by h plus the number of types of loans that are prepaid at the ET node $\eta(n, i, h)$.

[19] At the end of this section, we show how our model for two-class CMOs can be used to value CMOs with more than two classes.

As discussed in regard to (14), loans of type k are prepaid at time n, if the loans are outstanding immediately before time n, and if the condition $W_k^{AF}(n,i) \geq (1+G_k)F'_{n+1}$ is satisfied. Hence, we have

$$(19)\quad \eta(n,i,h) = \begin{cases} h + \sum_{k=h+1}^{\bar{k}} \mathrm{I}[W_k^{AF}(n,i) \geq (1+G_k)F'_{n+1}], & \text{if } h = 0,\dots,\bar{k}-1; \\ \bar{k}, & \text{if } h = \bar{k}, \end{cases}$$

for $i = 0,\dots,n$, $n = 1,\dots,N$. In (19), $\mathrm{I}[\cdot]$ denotes the indicator function that takes on one if the condition in the bracket is satisfied and zero if it is not. Note that the value of $\eta(n,i,h)$ implies the amount $\bar{Q}(n,i,h)$ of cumulative principal payments made before and at the ET node (n,i,h). In particular, if $h < \bar{k}$ and $n < N$, $\bar{Q}(n,i,h)$ is given by the initial collateral principal B minus the sum of principal balances immediately after time n of type-$(\eta+1)$ to type-$\bar{k}$ loans, where η denotes the value of $\eta(n,i,h)$. By contrast, if $h = \bar{k}$ or $n = N$, $\bar{Q}(n,i,h)$ is equal to B. Combining these two cases, we have

$$(20)\quad \bar{Q}(n,i,h) = \begin{cases} B - \sum_{k=\eta+1}^{\bar{k}} F'_{n+1}B_k, & \text{if } h = 0,\dots,\bar{k}-1 \text{ and } n = 1,\dots,N-1; \\ B, & \text{if } h = \bar{k} \text{ or } n = N, \end{cases}$$

for $i = 0,\dots,n$.

Now, we show that the supplementary variable h_n and its state space $\{0,\dots,\bar{k}\}$ at $n = 1,\dots,N$ satisfy Assumption 6 with respect to Bond A. First, it is clear that this state space satisfies i). Second, we consider ii). If the cumulative principal payments made before time n are greater than or equal to the initial principal of Bond A ($Q_n(h) \geq B_A$), Bond A must have been fully retired before time n. Hence, the cash flow $X(n,i,h)$ to Bond A is 0.

$$(21)\qquad X(n,i,h) = 0.$$

If the cumulative principal payments are less than B_A before time n but greater than or equal to B_A immediately after time n ($Q_n(h) < B_A \leq \bar{Q}(n,i,h)$), Bond A is fully retired at time n, and its cash flow is equal to the remaining principal balance of Bond A plus interest.

$$(22)\qquad X(n,i,h) = (1+c_A)(B_A - Q_n(h)).$$

If Bond A is still outstanding immediately after time n ($B_A > \bar{Q}(n,i,h)$), its cash flow is equal to the interest on the Bond A principal remaining in the n^{th} period plus all current principal payments made by the mortgagors in the collateral.

$$(23)\qquad X(n,i,h) = c_A(B_A - Q_n(h)) + (\bar{Q}(n,i,h) - Q_n(h)).$$

Combining (21)-(23), we have

$$X(n,i,h) = (1+c_A)(B_A - Q_n(h))^+ - (B_A - \bar{Q}(n,i,h))^+, \tag{24}$$

at each ET node $(n,i,h) \in Z_n, n = 1,\ldots,N$, where $(\cdot)^+$ takes on the value of the expression in the parenthesis if that value is nonnegative and 0 if that value is negative. Since (24) is well defined, ii) of Assumption 6 is satisfied with respect to Bond A. Third, Bond A has no boundary condition to consider in regard to iii). Fourth, the terminal condition is $X(N,i,h) = 0$ for all $(N,i,h) \in Z_N$. Fifth, (19) implies that the value of h_{n+1} is correctly determined by the value of (n, i_n, h_n).[20] In addition, the value of h_{n+1} is contained by the state space at time $n+1$. Therefore, v) is satisfied with respect to Bond A. To sum up, the supplementary variable h_n and its state space satisfy all the conditions in Assumption 6 with respect to Bond A, and as a result, the ET method is applicable to Bond A.

Next, we turn to the valuation of Bond A. Let $T(n,i,h)$ denote the price at an ET node (n,i,h) of Bond A, exclusive of its cash flow at (n,i,h). Because the PDS has no boundary condition, $T(n,i,h)$ directly satisfies the one-period RNVR of Proposition 1. Namely,

$$\begin{aligned} T(n,i,h) = P(n,i)[&\pi(T(n+1,i+1,\eta) + X(n+1,i+1,\eta)) \\ &+ (1-\pi)(T(n+1,i,\eta) + X(n+1,i,\eta))], \end{aligned} \tag{25}$$

where $\eta = \eta(n,i,h)$ and π is the risk-neutral probability of an up event in HL. Therefore, starting with the terminal condition $T(N,i,h) = 0$ for all $(N,i,h) \in Z_N$, (25) is used recursively to obtain the initial price of Bond A.[21]

To complete the valuation of the CMO in question, we consider the valuation of the residual. As shown in Section III.A, the initial value $V_k(0,0)$ of the type-k loan to the capital market for each dollar of the initial principal can be computed by the backward recursive procedure. Therefore, the initial value V_c of the entire collateral to the capital market is given by summing the initial values of all loans in the collateral.

$$V_c = \sum_{k=1}^{\bar{k}} V_k(0,0) B_k. \tag{26}$$

Next, considering that cash flows from the collateral are divided between Bond A and the residual, the initial price V_r of the residual is given by the initial value V_c of the

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[20] This is a trivial case of h_{n+1} being a function of i_{n+1}.

[21] To be precise, h drops as an argument of $T(n,i,h)$ when $n = 0$.

collateral minus the initial price $T(0,0,0)$ of Bond A.

$$V_r = V_c - T(0,0,0). \tag{27}$$

Before closing this section, we illustrate how this model for two-class CMOs can be used to value CMOs with more than two classes. In particular, consider a CMO that consists of Tranches A, B, and C and a residual with sequential-pay structure, where the initial principals are equal to four, two, three, and one tenths of the entire collateral, respectively. To value Tranche B, for example, we consider a hypothetical portfolio of a long position in Bond A of the two-class CMO model with principal equal to six tenths of the principal of the entire collateral and a short position in Bond A of the two-class CMO model with principal equal to four tenths of the principal of the entire collateral. Clearly, the cash flows to Tranche B are identical to those to this portfolio. Therefore, the price of Tranche B is given by the price of the first Bond A minus the price of the second Bond A.[22] This example shows that the two-class CMO model of this section provides a way to value sequential-pay CMOs with more than two classes.

IV. Application of the ET Method to Moving Average Options

Average options (or Asian options) are options where the payoff is based on the average price of an underlying asset over a prespecified period. They were studied by a number of researchers, including Kemna and Vorst (1990), Turnbull and Wakeman (1991), Levy (1992), HW, Geman and Yor (1993), Cho and Lee (1997), Milevsky and Posner (1998), and He and Takahashi (2000). Note, however, that in these studies, the average was computed over a period whose beginning and end were both fixed. In other words, none of them addressed the valuation of average options where the average was computed over a moving period with a fixed length. As HW (p. 30) pointed out, "The pricing of these types of options presents ongoing challenges for analysts." We call such options moving average options (MAOs) and value them by the ET method.

First, we specify a model for asset price uncertainty. Following CRR, we assume that the underlying asset price at a binomial node (n,i) is given by

$$S(n,i) = u^i\, d^{n-i}\, S, \qquad i = 0,\dots,n,\ n = 0,\dots,N, \tag{28}$$

22 Alternatively, Tranche B can be valued by the direct application of the ET method.

where u and d are constant and S is the underlying asset price at time 0. In addition, we assume that the one-period interest rate is constant through time $N-1$.

Next, we consider a MAO with very simple structure. Namely, the MAO gives its holder the right to buy one unit of the underlying asset for the arithmetic average of the asset price over two prior calendar periods. Specifically, a *calendar period*, such as a week and a month, consists of two unit periods of the model, and the average is computed based on the price at the end of each of two prior calendar periods.[23] The MAO is exercisable at any time $n = 0, \ldots, N$. Furthermore, for ease of exposition, we assume that time 0 is at the end of a calendar period and so is time N.

Clearly, this MAO is path dependent, and we value it by the ET method with two supplementary variables. The first supplementary variable $h_n^{(1)}$ represents the binomial node that the path has passed at the end of the last calendar period in relation to the current binomial node at time n. In particular, we define $h_n^{(1)}$, $n = 1, \ldots, N$, depending on whether time n is at the end or in the middle of a calendar period. If time n is at the end of a calendar period, the end of the last calendar period is time $n-2$, and $h_n^{(1)}$ is defined as

$$h_n^{(1)} = i_n - i_{n-2}, \qquad n = 2k,\ k = 1, 2, \ldots, \frac{N}{2}. \tag{29}$$

Therefore, $h_n^{(1)}$ takes on 0, 1, or 2, depending on the number of up events that may occur at time $n-1$ and time n. By contrast, if time n is in the middle of a calendar period, the end of the last calendar period is time $n-1$, and $h_n^{(1)}$ is defined as

$$h_n^{(1)} = i_n - i_{n-1}, \qquad n = 2k-1,\ k = 1, 2, \ldots, \frac{N}{2}. \tag{30}$$

Clearly, $h_n^{(1)}$ takes on 0 or 1 in this case.

In addition, we define a supplementary variable $h_n^{(2)}$, which represents the binomial node that the path has passed at the end of the calendar period immediately preceding the last calendar period in relation to the binomial node that the path has passed at the end of the last calendar period. In particular, we define $h_n^{(2)}$, $n = 3, \ldots, N$, depending on whether time n is at the end or in the middle of a calendar period. If time n is at the end of a calendar period, time $n-2$ is at the end of the last calendar period, and time $n-4$ is at the end of the calendar period immediately preceding the last calendar

[23] Average options are typically based on averages of prices at specified points in time.

period. Therefore, $h_n^{(2)}$ is defined as

$$h_n^{(2)} = i_{n-2} - i_{n-4}, \qquad n = 2k,\ k = 2, 3, \ldots, \frac{N}{2}. \tag{31}$$

By contrast, if time n is in the middle of a calendar period, time $n-1$ is at the end of the last calendar period, and time $n-3$ is at the end of the calendar period immediately preceding the last calendar period. Therefore, $h_n^{(2)}$ is defined as

$$h_n^{(2)} = i_{n-1} - i_{n-3}, \qquad n = 2k-1,\ k = 2, 3, \ldots, \frac{N}{2}. \tag{32}$$

$h_n^{(2)}$ takes on 0, 1, or 2 in either (31) or (32).

Next, we set

$$\mathbf{h}_n = \left(h_n^{(1)}, h_n^{(2)}\right), \tag{33}$$

and define its state spaces as[24]

$$\begin{cases} \{0,1\}, & \text{for } h_1^{(1)}; \\ \{0,1,2\}, & \text{for } h_2^{(1)}; \\ \{0,1\} \times \{0,1,2\}, & \text{for } \mathbf{h}_n,\ n = 2k-1,\ k = 2,3,\ldots,\frac{N}{2}; \\ \{0,1,2\} \times \{0,1,2\}, & \text{for } \mathbf{h}_n,\ n = 2k,\ k = 2,3,\ldots,\frac{N}{2}. \end{cases}$$

Let us examine whether Assumption 6 is satisfied by $\mathbf{h}_n$ and its state spaces. First, it is clear that i) is satisfied. Second, this MAO pays no cash flow other than those that are specified by a boundary condition. Hence, ii) is not applicable to this MAO. Third, we consider iii). Let $A(n, i, h^{(1)}, h^{(2)})$ denote the average at an ET node $(n, i, h^{(1)}, h^{(2)})$ of the underlying asset price over the last two calendar periods. If time $n \geq 3$ is at the end of a calendar period ($n = 2k, k = 2, 3, \ldots, \frac{N}{2}$), (29) and (31) imply

$$\begin{aligned} A(n, i, h^{(1)}, h^{(2)}) &= \frac{1}{2}\left(S(n-2, i-h^{(1)}) + S(n-4, i-h^{(1)}-h^{(2)})\right) \\ &= \frac{1}{2}\left(u^{i-h^{(1)}} d^{n-2-i+h^{(1)}} S + u^{i-h^{(1)}-h^{(2)}} d^{n-4-i+h^{(1)}+h^{(2)}} S\right). \end{aligned} \tag{34}$$

[24] With these state spaces, we need to allow the argument i of $S(n,i)$ to take on $-4, \ldots, -1$ and $n+1, \ldots, n+4$ because of (34). Note also that the values of $S(n,i)$ with these values of i do not contribute to the initial MAO price, because ET nodes with these values of i are not accessible from the initial ET node $(0,0)$. Incidentally, if we allow state spaces to be specific to each binomial node, we can avoid having i of $S(n,i)$ take on these values.

If time $n \geq 3$ is in the middle of a calendar period ($n = 2k-1, k = 2, 3, \ldots, \frac{N}{2}$), (30) and (32) imply

$$(35)\quad A(n,i,h^{(1)},h^{(2)}) = \frac{1}{2}\big(S(n-1,i-h^{(1)}) + S(n-3,i-h^{(1)}-h^{(2)})\big)$$
$$= \frac{1}{2}\big(u^{i-h^{(1)}}d^{n-1-i+h^{(1)}}S + u^{i-h^{(1)}-h^{(2)}}d^{n-3-i+h^{(1)}+h^{(2)}}S\big).$$

At time 1 and time 2, the average is computed by

$$(36)\qquad A(n,i,h^{(1)}) = \frac{1}{2}(S(0,i-h^{(1)}) + S_{-2}),$$

where S_{-2} denotes the underlying asset price two unit periods before the initial time. At time 0, the average is computed by

$$(37)\qquad A(0,0) = \frac{1}{2}(S_{-2} + S_{-4}),$$

where S_{-4} denotes the underlying asset price four unit periods before the initial time. Given (34)-(37), the boundary condition of the MAO is

$$(38)\qquad C(n,i,h^{(1)},h^{(2)}) \geq \max\big(S(n,i) - A(n,i,h^{(1)},h^{(2)}),0\big),$$

for $n = 0, \ldots, N-1$, where $h^{(1)}$ drops as an argument when $n = 0$, and $h^{(2)}$ drops as an argument when $n = 0, 1, 2$. (38) clearly shows that this boundary condition is well defined at each ET node. Fourth, the terminal condition is given by

$$(39)\qquad C(N,i,h^{(1)},h^{(2)}) = \max\big(S(N,i) - A(N,i,h^{(1)},h^{(2)}),0\big).$$

Fifth, we consider v) of Assumption 6 in two cases. The first case is where time n is at the end of a calendar period, and the second case is where time n is in the middle of a calendar period. In the first case, the first supplementary variable at time $n+1$ counts an occurrence of an up event at time $n+1$. Therefore, we have

$$(40)\qquad h^{(1)}_{n+1} = i_{n+1} - i_n, \qquad n = 2k,\ k = 0, 1, \ldots, \frac{N}{2} - 2.$$

The second supplementary variable at time $n+1$, in this case, represents the relation of the time-n binomial node to the time-$(n-2)$ binomial node that the path has passed through, which is what the first supplementary variable represents at time n. Therefore, we have

$$(41)\qquad h^{(2)}_{n+1} = h^{(1)}_n, \qquad n = 2k,\ k = 1, 2, \ldots, \frac{N}{2} - 2.$$

In the second case, where time n is in the middle of a calendar period, the first supplementary variable at time $n+1$ counts the number of up events that may occur at time n and time $n+1$, which is given by $h_n^{(1)}$ plus $(i_{n+1} - i_n)$.

$$(42) \qquad h_{n+1}^{(1)} = h_n^{(1)} + (i_{n+1} - i_n), \qquad n = 2k-1,\ k = 1, 2, \ldots, \frac{N}{2}.$$

The second supplementary variable, in this case, does not change value from time n to time $n+1$, because both binomial nodes that the path has passed at the end of the last calendar period and at the end of the calendar period immediately preceding the last calendar period are unchanged from time n to time $n+1$.

$$(43) \qquad h_{n+1}^{(2)} = h_n^{(2)}, \qquad n = 2k-1,\ k = 2, 3, \ldots, \frac{N}{2}.$$

(40)-(43) imply that v) of Assumption 6 is satisfied. To sum up, $\mathbf{h}_n$ and its state spaces satisfy all the conditions in Assumption 6, and as a result, the RNVR in Proposition 1 is applicable to this MAO. In particular, if time n is at the end of a calendar period $(n = 2k,\ k = 0, 1, \ldots, N/2-2)$, we have

$$(44) \quad C_{AF}(n, i, h^{(1)}, h^{(2)}) = \frac{1}{r}\left[p\ C(n+1, i+1, 1, h^{(1)}) + (1-p)\ C(n+1, i, 0, h^{(1)})\right],$$

where r is one plus the one-period interest rate and p is the risk-neutral probability of an up event in CRR.[25] If time n is in the middle of a calendar period $(n = 2k-1,\ k = 1, 2, \ldots, N/2)$, we have[26]

$$(45) \qquad C_{AF}(n, i, h^{(1)}, h^{(2)}) = \frac{1}{r}\left[p\ C(n+1, i+1, h^{(1)}+1, h^{(2)}) + (1-p)\ C(n+1, i, h^{(1)}, h^{(2)})\right].$$

Therefore, starting with (39), we use (38), (44), and (45) recursively to obtain the initial price of the MAO.

To conclude this section, we point out that this MAO model can be easily extended to a more general setup, where the moving average is computed based on the asset prices at the ends of the last k calendar periods and a calendar period consists of j unit periods of the model. In such a setup, we define k supplementary variables and use a one-period RNVR that is conditional on these supplementary variables.

[25] To be precise, if $n = 2$, $h^{(2)}$ drops as an argument of C_{AF}. If $n = 0$, both $h^{(1)}$ and $h^{(2)}$ drop as arguments of C_{AF}, and $h^{(1)}$ drops as an argument of C.

[26] If $n = 1$, $h^{(2)}$ drops as an argument of C_{AF} and C.

V. The Extended Tree and The Path Tree

In Assumption 6, we have imposed five conditions on PDSs. Clearly, these conditions are necessary to apply the ET method to PDSs. A question remains, however, whether these conditions are sufficient to ensure that the ET method yields correct prices of PDSs at all times. In this section, we address this question from the point of view of the PT. In particular, given that PDSs are valued correctly on the PT, we examine whether the conditions in Assumption 6 ensure that the ET method yields the same initial prices of PDSs as a valuation method based on the PT.

A. Valuation of Path Dependent Securities on the Path Tree

First, we describe a method for pricing a generic PDS on the PT. The definition of PDSs implies that the cash flow to the PDS is correctly specified at each PT node. Let $X^*(e)$ denote the cash flow at a PT node e, exclusive of the cash flows that are specified by boundary and/or terminal conditions. In addition, let $C^*(e)$ denote the PDS price at the PT node e, exclusive of the cash flow $X^*(e)$. By definition, the PT node e contains the current, say, time-n value of the underlying state variable. Denote this value by i. Then, we can represent the ensuing PT node e_{n+1} by $(e, i+1)$ if an up event occurs at time $n+1$ and by (e, i) if a down event occurs at time $n+1$. Under Assumptions 1-5, the usual no-arbitrage condition yields the arbitrage-free price $C^*_{AF}(e)$ of the PDS at a PT node $e \in E_n$, $n = 0, \ldots, N-1$.

$$
\begin{aligned}
\text{(46)} \qquad C^*_{AF}(e) = \frac{1}{r(n,i)} \big[& p(n,i)(C^*(e,i+1) + X^*(e,i+1)) \\
& + (1 - p(n,i))(C^*(e,i) + X^*(e,i)) \big],
\end{aligned}
$$

where $p(n, i)$ is the risk-neutral probability of an up event given by (2). In addition, the PDS has a terminal condition and may have boundary conditions.

$$\text{(47)} \qquad C^*(e) = H^*(e), \quad \forall e \in E_N;$$

$$\text{(48)} \qquad L^*(e) \le C^*(e) \le U^*(e), \quad \forall e \in E_n, n = 0, \ldots, N-1.$$

Given this setup, we can value the PDS by starting with (47) and proceeding backward, where (46) and (48) are incorporated into the PDS price by

$$\text{(49)} \qquad C^*(e) = \max\big[L^*(e), \min(C^*_{AF}(e), U^*(e))\big], \quad \forall e \in E_n, n = 0, \ldots, N-1.$$

We call this valuation method the *path tree* (PT) *method.* It is clear that (46)-(49) resemble (1) and (3)-(5), respectively. In fact, the only difference is found in the arguments the prices, cash flows, and bounds take on. On the ET, an ET node provides available information. Hence, the ET node enters as an argument. By contrast, on the PT, a PT node provides available information. Hence, the PT node enters as an argument.

B. The Extended Tree and the Path Tree

Next, we examine the ET method from the point of view of the PT method. For this purpose, we introduce additional notations. First, we assume that if the values of z_n and i_{n+1} are given, the value of $\mathbf{h}_{n+1}$ is determined uniquely.[27] Let $f(z_n, i_{n+1})$ denote this value of $\mathbf{h}_{n+1}$ for given values of z_n and i_{n+1}. In other words, $f(\cdot,\cdot)$ specifies transitions of z_n among ET nodes. Second, the definitions of $\mathbf{h}_n$ and e_n imply that the values of n, i_n, and $\mathbf{h}_n$ can be determined by e_n. Namely, the number of elements of e_n determines n, the last element of e_n gives i_n, and the definition of supplementary variables specifies the mapping of E_n to Z_n. Therefore, let $g(e_n)$ denote the value of $z_n = (n, i_n, \mathbf{h}_n)$ for a given value of e_n.

In addition, to examine the ET method from the point of view of the PT method, we rephrase Assumption 6 in relation to the PT.

Assumption 6'. For a PDS with a finite life through time N, there exist a set of supplementary variables and its state spaces such that:

i) At each $n = 0, \ldots, N$ when the supplementary variables are defined, the state space is finite and contains all values the set of supplementary variables, $\mathbf{h}_n$, may take on.

ii) If a pair of an ET node $z \in Z_n$ and a PT node $e \in E_n$ satisfies the relation $z = g(e)$, the cash flow other than those that are specified by boundary and terminal conditions at z is identical to that at e. That is, $\forall z \in Z_n, \forall e \in E_n, n = 1, \ldots, N$,

$$\text{if } z = g(e), \quad X(z) = X^*(e). \tag{50}$$

iii) Given that the PDS has boundary conditions, if a pair of an ET node $z \in Z_n$ and a PT node $e \in E_n$ satisfies the relation $z = g(e)$, the boundary conditions at z are

[27] In this section, we do not take Assumption 6 as given. Instead, we will shortly make Assumption 6', which sets the condition that the value of $\mathbf{h}_{n+1}$ is determined *correctly.*

identical to the boundary conditions at e. That is, $\forall z \in Z_n, \forall e \in E_n, n = 0, \ldots, N-1$,

$$\text{if } z = g(e), \quad L(z) = L^*(e) \text{ and } U(z) = U^*(e). \tag{51}$$

iv) If a pair of an ET node $z \in Z_N$ and a PT node $e \in E_N$ satisfies the relation $z = g(e)$, the terminal condition at z is identical to that at e. That is, $\forall z \in Z_N, \forall e \in E_N$,

$$\text{if } z = g(e), \quad H(z) = H^*(e). \tag{52}$$

v) If a pair of an ET node $z \in Z_n$ and a PT node $e \in E_n$ satisfies the relation $z = g(e)$, the ET node specified by $f(z,i)$ is identical to the ET node specified by $g(e,i)$ for every $i \in \{i_n, \ldots, n+1\}$ where i_n denotes the value of the second element of z. That is, $\forall z \in Z_n, \forall e \in E_n$, and $\forall i \in \{i_n, \ldots, n+1\}$, $n = 0, \ldots, N-1$,

$$\text{if } z = g(e), \quad f(z,i) = g(e,i). \tag{53}$$

Furthermore, $f(z,i)$ is contained by the state space at time $n+1$.

As stated at the beginning of this section, we take it given that PDSs are well defined on the PT. Therefore, in the case of ii), for instance, the question of whether the cash flow $X(z)$ at an ET node z is correctly specified or not can be reduced to the question of whether $X(z)$ is identical to the cash flow $X^*(e)$ at the PT node e that has correspondence $g(\cdot)$ with the ET node z. This is why we can rephrase Assumption 6 as in Assumption 6'.

Now, if a PDS satisfies Assumption $6'$, the ET method yields the same PDS price one period before maturity as the PT method.

Proposition 2. Under Assumptions 1-5, if a PDS with a finite life through time N satisfies Assumption $6'$, then $z = g(e)$ for $z \in Z_{N-1}$ and $e \in E_{N-1}$ implies $C(z) = C^*(e)$.

Proof. See Appendix.

In words, under Assumptions 1-5 and 6', if an ET node $z \in Z_{N-1}$ and a PT node $e \in E_{N-1}$ satisfy the relation $z = g(e)$, the PDS price at the ET node z is identical to the PDS price at the PT node e. This proposition is used to prove our main result in this section.

Proposition 3. Under Assumptions 1-5, if a PDS satisfies Assumption 6′, the ET method yields the same initial price of the PDS as the PT method.

Proof. See Appendix.

Therefore, Assumption 6′ sets sufficient conditions for the ET method to yield correct prices of PDSs at all times.

Next, we elaborate on the relation of Assumption 6' to Assumption 6. Clearly, these assumptions are concerned with the same conditions. They, however, differ in ET nodes they impose the conditions on. Namely, Assumption 6' imposes the conditions on only those ET nodes that have corresponding PT nodes.[28] By contrast, Assumption 6 imposes the conditions on all ET nodes of the ET, including those ET nodes that do not have corresponding PT nodes. We make four points in regard to this difference. First, in general, an ET node does not contain sufficient information to discern whether it has a corresponding PT node or not. As a result, when we construct ET nodes, we need to define a sufficiently large state space, say, at time n of m supplementary variables to ensure that the state space contains all points in Euclidean m-space the supplementary variables may take on at time n. Therefore, this difference of Assumption 6 from Assumption 6' is absolutely necessary. Second, PDSs we apply the ET method to are only those that satisfy Assumption 6. In other words, we deal with PDSs whose contractual terms allow us to examine Assumption 6 at every ET node, independent of whether that ET node is ever accessible from the initial ET node.[29] Third, the fact that Assumption 6' imposes the conditions on less ET nodes than Assumption 6 implies that if a PDS satisfies Assumption 6, it always satisfies Assumption 6'. Fourth, ET nodes that do not have corresponding PT nodes are not accessible from the initial ET node. As a result, the prices of a PDS calculated at such ET nodes do not contribute to the initial price of the PDS. Therefore, there is no harm in including these nodes in the ET.

To conclude this section, we compare the ET method with the PT method in terms

[28] If an ET node $z \in Z_n$ has a PT node $e \in E_n$ that satisfies $z = g(e)$, we say that the ET node z has a corresponding PT node. Clearly, such an ET node is accessible from the initial ET node $(0, 0)$.

[29] Recall that when American put options, which can be viewed as PDSs as in Merton ((1992), p. 446), are valued by an explicit finite difference method, correct boundary conditions can be set at an arbitrary node, independent of whether that node is ever accessible from the initial node.

of computational size. With the PT method, we set up 2^n nodes at time n. Therefore, when we apply the PT method to a PDS with N periods to maturity, we must set up all of 1, $2^1, \ldots, 2^N$ nodes, which sum to $(2^{N+1} - 1)$ nodes. By contrast, with the ET method, we need to set up considerably less nodes to value PDSs. To show this point, we establish an upper bound to the total number of ET nodes.

Proposition 4. Let

$$\bar{h}^{(j)} = \max_{0 \leq n \leq N} \bar{h}_n^{(j)}, \quad j = 1, \ldots, m, \tag{54}$$

where $\bar{h}_n^{(j)}$ denotes the number of values the j^{th} supplementary variable may take on at time n, and m denotes the total number of supplementary variables defined for a PDS in question. Then the total number of ET nodes that are required by the ET method is bounded above by

$$\frac{1}{2}(N+1)(N+2) \prod_{j=1}^{m} \bar{h}^{(j)}. \tag{55}$$

Proof: See Appendix.

Using these results, we examine the computational sizes of the two methods for a few examples. Our first example is a CMO collateralized by 30-year mortgage loans. Let the unit period be a month. Then, with the PT method, we must deal with $(2^{361} - 1)$ nodes, which are approximately 4.7×10^{108} nodes. By contrast, if the ET method is applied to the CMO model in Section III, and if the CMO collateral consists of 5 types of mortgage loans as in McConnell and Singh (1994), then we need at most (1/2)(361)(362)(6) nodes, which are roughly 3.9×10^5 nodes. Therefore, when N is large, the implementation of the PT method is practically infeasible, but that of the ET method is not.[30]

Our second example is the MAO model presented in Section IV. In particular, consider a MAO that expires in 5 years, where a calendar period is a month, and a

[30] Certainly, we can use Monte Carlo simulation to obtain an approximate solution for the PT method. For example, if 10,000 paths are generated to value the CMO as in Childs, Ott and Riddiough (1996), Monte Carlo simulation involves $(360)(10,000) = 3.6 \times 10^6$ nodes. Therefore, in this particular example, Monte Carlo simulation requires about ten times as many nodes to obtain approximate solutions as the ET method requires to obtain exact solutions.

month consists of two unit periods of the model. Then, the number of unit periods to expiration, N, is 120. Therefore, with the PT method, we must consider $(2^{121} - 1)$ PT nodes, which are roughly 2.7×10^{36} nodes. Next, consider applying the ET method to this MAO, where $h_n^{(1)}$ and $h_n^{(2)}$ of Section IV are used as supplementary variables. Recall that at a generic binomial node, each of the supplementary variables takes on a value among three or less possible values. Hence, $\bar{h}^{(1)}$ and $\bar{h}^{(2)}$ in Proposition 4 are both 3, which implies that the upper bound for the ET nodes is $(1/2)(121)(122)(3)(3) = 66,429$.

Our third example is the extension of the MAO model discussed at the end of Section IV, where the moving average is computed based on the asset prices at the ends of the last k calendar periods and a calendar period consists of j unit periods of the model. In particular, we assume that the MAO expires in 5 years, that the moving average is based on the asset prices at the ends of the last 6 months, and that a month consists of 4 unit periods of the model. Then, if we apply the PT method to this MAO, we must consider $(2^{241} - 1)$ PT nodes, which are approximately 3.5×10^{72} nodes. By contrast, if we apply the ET method to this MAO, we define six supplementary variables, each of which takes on a value among no more than five possible values at each binomial node. Therefore, the upper bound to the number of ET nodes is $(1/2)(241)(242)(5^6)$, which is roughly 4.6×10^8. Admittedly, this figure is large, even though it is a major improvement over that of the PT method. Therefore, when j or k or both are relatively large, users of the ET method must reset the values of j and k considering the trade-off between accuracy and computational costs.

VI. Conclusions

This paper presents a discrete time method, called the extended tree (ET) method, for pricing path dependent securities (PDSs) by the supplementary variable technique. In particular, we develop a general and formal framework for the ET method, where the most important considerations are the choice of supplementary variables and the construction of their state spaces. We impose five conditions on the supplementary variables and their state spaces. These conditions are obvious requisites for the ET method to be operational. Yet, possibly because these conditions seem so apparent, they have never been stated explicitly in the prior literature on PDSs, except Kishimoto (1989) and Hull and White (1993), who set forth some, but not all, of these conditions.

The advantages of presenting the general and formal framework are threefold. First, it enables us to examine the ET method rigorously. In particular, we can prove that the five conditions are sufficient to ensure that the ET method always yields the same PDS price as a valuation method based on a comparable Arrow-Debreu's event tree: a proof requires a formal framework. Second, given the general framework, the problem of valuing a complex PDS can be reduced to that of identifying supplementary variables that satisfy the five conditions. Third, the general framework makes it possible to address the computational size of the ET method in general terms. In particular, we establish an upper bound to the computational size of the ET method and provide a few examples of PDSs that can be valued by the ET method with manageable amounts of computation.

We show two applications of the ET method to PDSs. In the first application, we value collateralized mortgage obligations (CMOs) where the collateral of a CMO is modeled as a pool of mortgage loans with heterogeneous prepayment costs. Such a collateral model generates prepayments based on the distribution of mortgagors' prepayment costs, which is probably fairly stable, as Stanton ((1995), p. 707) pointed out. Therefore, this way of modeling prepayments makes CMO models more robust to shifts in economic environments than the alternative way of modeling prepayments based on purely empirical prepayment functions. In addition, this model of collaterals makes it possible to apply the ET method to CMOs. As discussed in Section IV, CMOs are typically valued by Monte Carlo simulation, where the sampling error is related to the requisite observations by a square law: for instance, the reduction of the error tenfold requires a hundredfold increase in observations. By contrast, the ET method yields exact model prices with manageable amounts of computation. Therefore, our CMO model has distinct numerical advantages over Monte Carlo simulation. Furthermore, in a typical CMO, cash flows are allocated among multiple classes of bonds based on the remaining principal of the CMO collateral. Therefore, the CMO model presented in Section III should be applicable to most CMOs if it is modified appropriately.

It should be added that the ET method is complementary to Monte Carlo simulation. This is because the ET method can be used in a control variate technique to reduce sampling errors in Monte Carlo simulation. Consider, for example, applying Monte Carlo simulation to a CMO model where prepayments are generated by a purely empirical prepayment function, as in McConnell and Singh (1993). Y denotes the value of a CMO bond that is obtained by applying crude Monte Carlo simulation to this CMO model. Next, consider another CMO model where the collateral is modeled as in Section III. As shown by McConnell and Singh (1994) and Singh and McConnell (1996), we can first determine a prepayment boundary for each type of mortgagors and second use these boundaries, in conjunction with Monte Carlo simulation, to value the same CMO bond as above. Let C denote this value of the CMO bond. Then, following Rubinstein ((1981), p. 127), we can use C as a control variate to construct a new estimator $Y(\beta)$ for the CMO bond value that is likely to have a smaller variance than the original estimator Y.

$$Y(\beta) = Y - \beta(C - \mu_C),$$

where μ_C is the expectation of C, which is obtained by the ET method. A simple computation shows that the variance of $Y(\beta)$ is minimized at

$$\beta^* = \frac{\text{cov}(Y, C)}{\text{var}(C)}.$$

The variance of $Y(\beta^*)$ is given by

$$\text{var}(Y(\beta^*)) = (1 - \rho^2)\text{var}(Y),$$

where ρ denotes the correlation coefficient between Y and C. Since Y and C are likely to be highly correlated, the ET method provides a promising way to reduce sampling errors in Monte Carlo simulation.

In the second application of the ET method, we value average options where the average is computed over a moving period with a fixed length. We refer to such options as moving average options (MAOs). This application implies that the ET method is applicable to other types of PDSs where path dependence arises from the direct dependence of cash flows on lagged state variables. Examples of such PDSs are floating rate notes, adjustable rate mortgages (ARMs), and swaps.[31] PDSs of this type were studied by Ramaswamy and Sundaresan (1986), Sundaresan (1991), and others. Note, however, that valuing them by the ET method has two major advantages over these studies: the ability to account for time lags of specified lengths and the applicability to PDSs with American option features. Take, for instance, Sundaresan (1991) to understand the latter point. He developed an ingenious model for interest rate swaps where each discrete payment was based on the interest rate that preceded the payment by a specified length of time. His model was, however, inapplicable to an interest rate

[31] Therefore, the ET method is applicable to CMOs backed by ARMs.

swap with an embedded option, if the option was exercisable anytime between the payment date and the reset date. Because with such an option it was impossible to obtain a closed form solution for the value of each floating-rate payment.

Kau, Keenan, Muller, and Epperson (1990) (KKME) were among the first who valued a type of PDSs where cash flows directly depended on lagged state variables by the supplementary variable technique in discrete time. In particular, they valued ARMs by solving a partial differential equation by an explicit finite difference method (EFDM) where the dependence of ARM payments on lagged interest rates was handled by treating the interest rate on the most recent adjustment date as a supplementary variable.[32] Note that their model can be couched in the framework of the ET, if the uncertainty structure of the ET is modified accordingly. Therefore, naturally, their model has the strengths of the ET method. There is, however, a subtle but nontrivial difference between KKME's ARM model and the MAO model presented in this paper. In the extended MAO model described at the end of Section IV, the supplementary variables are defined in relation to the current binomial node or another supplementary variable. Therefore, there are at most $j + 1$ values each supplementary variable may take on, and as a result, the state spaces are kept relatively small. By contrast, KKME constructed the state space for their supplementary variable without utilizing information on the underlying state variable.[33] Therefore, their state space is far larger than those of the MAO model, and so is their overall computational size.

To conclude this paper, we stress that the ET method is not limited to the uncertainty models employed in this paper. First, because the uncertainty structure set up in this paper admits any binomial model, the ET method is directly applicable to other binomial models, such as Black, Derman, and Toy (1990). Second, the ET method can be modified to accommodate any Markov chain, as long as the Markov chain is accompanied with a one-period valuation relationship. For instance, it is a simple exercise to modify the ET method to accommodate a trinomial or multinomial model of uncertainty. Furthermore, as mentioned above, the ET method can be implemented in the framework of an EFDM, if Assumption 6 is modified accordingly.

[32] The valuation procedure employed by KKME satisfies the EFDM analogue of Assumption 6 and is free from the possible numerical problem of EFDM pointed out by Barraquand and Pudet (1996).

[33] This point is apparent in their statement on page 1420: "This entire procedure must be done over the year for each possible value $0 \leq a(n-1) \leq a(0) + c$," where $a(n-1)$ denotes the interest rate set on the most recent adjustment date.

Appendix

A. Proof of Proposition 1

To simplify expressions, we define a function

$$G(n,i,\mathbf{h}) = C(n,i,\mathbf{h}) + X(n,i,\mathbf{h}),$$

on all $(n,i,\mathbf{h}) \in Z_n, n = 1,\ldots,N$. At a generic ET node $(n,i,\mathbf{h}) \in Z_n, n = 0,\ldots,N-1$, form a portfolio of Δ units of a security that satisfies Assumption 4 and B dollars of default-free one-period discount bond such that the value of the portfolio is equal to the price plus payoff of the PDS at each ensuing ET node.

$$\begin{aligned} \Delta S(n+1,i+1) + r(n,i)B &= G(n+1,i+1,\mathbf{h}^u), \\ \Delta S(n+1,i) + r(n,i)B &= G(n+1,i,\mathbf{h}^d). \end{aligned}$$

Solving these equations, we have

$$\begin{aligned} \Delta &= \frac{G(n+1,i+1,\mathbf{h}^u) - G(n+1,i,\mathbf{h}^d)}{S(n+1,i+1) - S(n+1,i)}, \\ B &= \frac{1}{r(n,i)} \frac{S(n+1,i+1)G(n+1,i,\mathbf{h}^d) - S(n+1,i)G(n+1,i+1,\mathbf{h}^u)}{S(n+1,i+1) - S(n+1,i)}. \end{aligned}$$

Then, to preclude an arbitrage opportunity, the arbitrage-free price $C_{AF}(n,i,\mathbf{h})$ of the PDS must be equal to

$$\begin{aligned} C_{AF}(n,i,\mathbf{h}) &= \Delta S(n,i) + B \\ &= \frac{1}{r(n,i)}\Big[\frac{r(n,i)S(n,i) - S(n+1,i)}{S(n+1,i+1) - S(n+1,i)} G(n+1,i+1,\mathbf{h}^u) \\ &\qquad + \frac{S(n+1,i+1) - r(n,i)S(n,i)}{S(n+1,i+1) - S(n+1,i)} G(n+1,i,\mathbf{h}^d)\Big] \\ &= \frac{1}{r(n,i)}[p(n,i)G(n+1,i+1,\mathbf{h}^u) + (1-p(n,i))G(n+1,i,\mathbf{h}^d)]. \end{aligned}$$

Therefore, the arbitrage-free price of the PDS is given by (1) and (2).

B. Proof of Proposition 2[34]

First, fix a pair of an ET node $z \in Z_{N-1}$ and a PT node $e \in E_{N-1}$ that satisfies the relation $z = g(e)$. In addition, let $z^u \in Z_N$ and $z^d \in Z_N$ denote the ET nodes that

[34] This proof is valid when no supplementary variable is defined at time n and/or time $n+1$.

follow the ET node z if an up event and a down event occur at time N, respectively. Similarly, let $e^u \in E_N$ and $e^d \in E_N$ denote the PT nodes that follow the PT node e if an up event and a down event occur at time N, respectively. Then, by v) of Assumption $6'$, $z^u = g(e^u)$ and $z^d = g(e^d)$, which imply $H(z^u) = H^*(e^u)$ and $H(z^d) = H^*(e^d)$ by iv) of Assumption $6'$. By (3) and (47), these equalities imply equalities in price.

$$C(z^u) = C^*(e^u) \text{ and } C(z^d) = C^*(e^d).$$

In addition, $z^u = g(e^u)$ and $z^d = g(e^d)$ imply $X(z^u) = X^*(e^u)$ and $X(z^d) = X^*(e^d)$ by ii) of Assumption $6'$. Furthermore, $z = g(e)$ implies that the value of the underlying state variable at the ET node z is identical to that at the PT node e. Let i denote this value of the underlying state variable. Then, the ET node z and the PT node e share the same values of one plus the one-period interest rate, $r(N-1,i)$, and the risk-neutral probability $p(N-1,i)$ of an up event. Therefore, comparing (1) with (46), we can see that the arbitrage-free price of the PDS at the ET node z, $C_{AF}(z)$, is identical to that at the PT node e, $C^*_{AF}(e)$. Finally, iii) of Assumption $6'$ ensures that the PDS price at the ET node z, $C(z)$, is identical to that at the PT node e, $C^*(e)$, when the PDS has boundary conditions at z and e.

C. Proof of Proposition 3

This proposition is proved by mathematical induction. First, consider a generic PDS with one period to maturity that satisfies Assumptions 1-5 and $6'$. Clearly, the value of the function $g(\cdot)$ for $e_0 = (0)$ is $z_0 = (0,0)$. In other words, the initial ET node z_0 and the initial PT node e_0 trivially satisfy the relation $z_0 = g(e_0)$. In addition, the valuation problem of this one-period PDS is a special case of the valuation problem at time $N-1$ of a generic N-period PDS. Hence, invoking Proposition 2, we have

$$C(z_0) = C^*(e_0).$$

Namely, Proposition 3 holds for $N = 1$.

Next, we prove Proposition 3 for a generic PDS that has $N \geq 2$ periods to maturity, assuming that Proposition 3 holds for PDSs that have $N-1$ periods to maturity. For this purpose, consider a hypothetical $(N-1)$-period PDS that resembles the N-period PDS in question in the following way.

i) The cash flow at an ET node $z \in Z_{N-1}$ to the hypothetical $(N-1)$-period PDS is equal to the cash flow at z of the N-period PDS in question, where the cash flows refer to those that are specified by ii) of Assumption 6'. In addition, the terminal condition at z of the hypothetical $(N-1)$-period PDS is given by the price at z of the N-period PDS in question that is obtained by the ET method.

ii) The cash flow at a PT node $e \in E_{N-1}$ to the hypothetical $(N-1)$-period PDS is equal to the cash flow at e of the N-period PDS in question, where the cash flows refer to those that are specified by ii) of Assumption 6'. In addition, the terminal condition at e of the hypothetical $(N-1)$-period PDS is given by the price at e of the N-period PDS in question that is obtained by the PT method.

iii) The hypothetical $(N-1)$-period PDS is identical to the N-period PDS in question in all respects at time $n = 0, \ldots, N-2$.

Clearly, the initial price of the hypothetical $(N-1)$-period PDS agrees with the initial price of the N-period PDS in question, when these prices are computed by the ET method. Similarly, the same statement holds true, when the prices are computed by the PT method. Therefore, we can conclude that Proposition 3 holds for the N-period PDS in question, if we show that the two methods yield the same initial price of the hypothetical $(N-1)$-period PDS.

Now, to show that this if-statement holds, we invoke the hypothesis of mathematical induction: Proposition 3 holds for any $(N-1)$-period PDS, if the PDS satisfies Assumption $6'$. Hence, we examine the hypothetical $(N-1)$-period PDS with respect to Assumption $6'$. First, it is obvious that this PDS satisfies i) for $n = 0, \ldots, N-1$. Second, the PDS clearly satisfies ii) for $n = 1, \ldots, N-1$, and iii) and v) for $n = 0, \ldots, N-2$. Hence, the only condition that remains to be examined is iv) for time $N-1$. To see that this condition is satisfied, note that Proposition 2 holds for the N-period PDS in question. That is, if a pair of $z \in Z_{N-1}$ and $e \in E_{N-1}$ satisfies the relation $z = g(e)$, then

$$C(z) = C^*(e),$$

for the N-period PDS. Hence, by the construction of the hypothetical $(N-1)$-period PDS, this equality implies that the hypothetical $(N-1)$-period PDS satisfies iv) of Assumption $6'$. To sum up, the hypothetical $(N-1)$-period PDS satisfies all the conditions in Assumption $6'$. Hence, by the hypothesis of mathematical induction, Proposition 3 applies to this $(N-1)$-period PDS, which implies that Proposition 3 holds for the N-period PDS in question.

D. Proof of Proposition 4

The ET method requires computation of the PDS price at each ET node. At time n, there are $n+1$ values the underlying state variable i_n may take on, and for each pair of (n, i) where i denotes a particular value of i_n, there are at most $\bar{h}^{(j)}$ values the j^{th} supplementary variable $h_n^{(j)}$ may take on. Hence, the number of ET nodes at time n is bounded above by

$$(n+1)\prod_{j=1}^{m}\bar{h}^{(j)}.$$

Therefore, the number of ET nodes for all $n = 0, \dots, N$ has an upper bound of

$$\sum_{n=0}^{N}(n+1)\left(\prod_{j=1}^{m}\bar{h}^{(j)}\right).$$

This expression reduces to (55).

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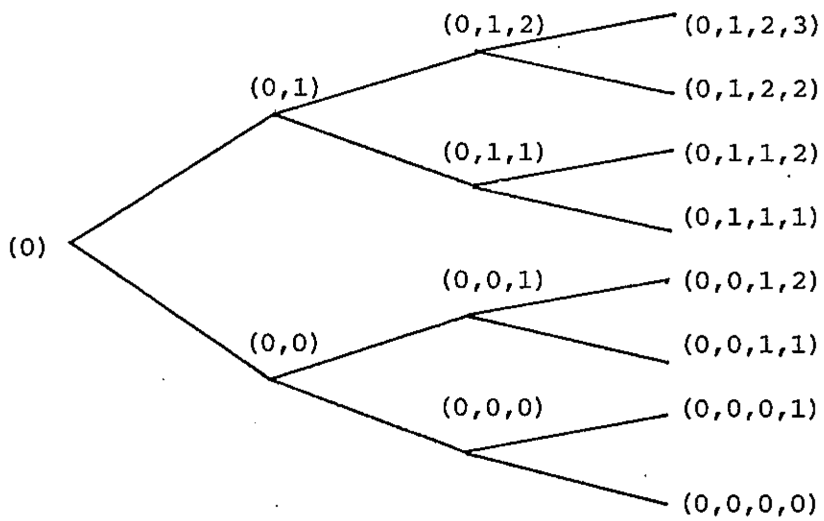

Figure 1. Path Tree. This is a path tree when $N = 3$. Each node of the tree represents a sequence of realizations of an underlying state variable, i.e. $e_n = (i_0, i_1, \ldots, i_n)$.

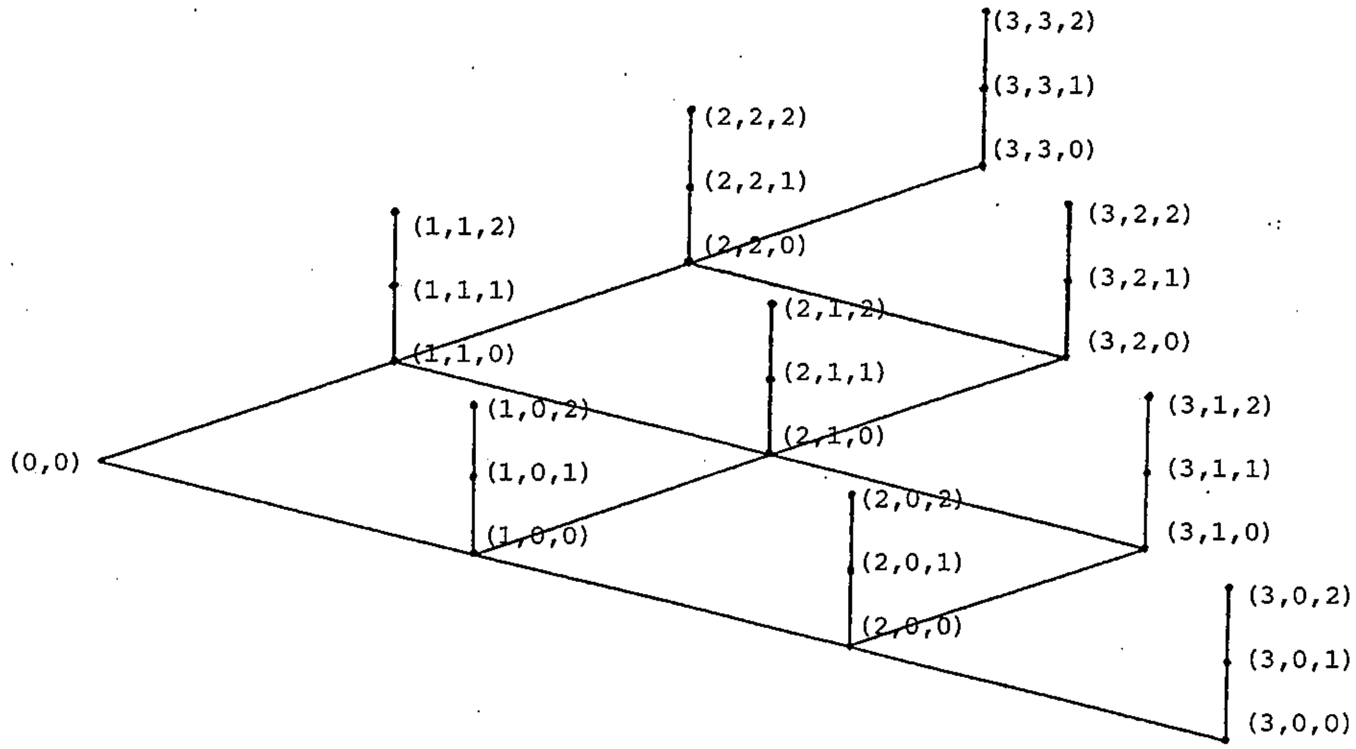

Figure 2. Extended Tree. This is an extended tree where $N = 3$ and $m = 1$ for $n = 1, 2, 3$. The state space of the supplementary variable h_n is $\{0, 1, 2\}$ for all $n = 1, 2, 3$. Each node at time 1, 2 and 3 of the tree represents the value of the triplet (n, i_n, h_n).