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AN OPTIMAL HOSTAGE RESCUE PROBLEM

by

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Abstract

We propose the following mathematical model of optimal rescue problems concerning hostages. Suppose i persons are taken as hostages at any given point in time t, and we have to make a decision on attempting either rescue or no rescue. Let p be the probability of a hostage being killed in a rescue attempt, and let s be the probability of criminal(s) surrendering up to the following point in time if no rescue attempt is made. Further, let q and r be the probabilities of a hostage being, respectively, killed or set free if no rescue attempt and criminal(s) not surrendering up to the following point in time. The objective here is to maximize the probability of no hostage being killed, starting from time t. Several properties of an optimal rescuing rule in the model are clarified and prescribed.

Keywords: Dynamic programming; Stochastic processes; Hostage rescue

1 Introduction

Throughout history acts involving, hostage taking have occurred for different reasons, e.g., social inequality, poverty, religious problems, racial problems, and so on. The problem has become an urgent issue to be tackled worldwide. Typical examples in recent years include:

Case 1. A 17-year-old youth wielding a knife hijacked a bus on the Sanyo Expressway and killed a 68-year-old hostage. After 15 hours, the police stormed the bus, the other hostages were rescued, and the hijacker was arrested (May 4, 2000).

Case 2. An armed man took a Finance Ministry official hostage in the Tokyo Stock Exchange building and demanded a meeting with the Finance Minister. He surrendered to the police after a tense, five and half hour standoff (January 12, 1998).

Case 3. Fourteen guerrillas stormed the home of the Japanese ambassador to Peru and took about three hundred people hostage, including diplomats and government officials attending the emperor's birthday party. All but one of the hostages were rescued while all the rebels were killed when special forces stormed the building (December 17, 1996).

Case 4. A man with a knife broke into a house and took a 2-year-old boy hostage. The police finally rushed into the house, set the uninjured boy free, and arrested the criminal (December 1, 1995).

Although not being available for accurate statistics, it could be said that different scenarios of similar plots are always occurring all over the world. The most important decision for the person in charge of crisis settlement is the timing to enact rescue of the hostages, especially after all possible negotiations have broken down. Wrestling with the problem, needless to say, involves many factors, political, economical, sociological, psychological, and so on, and they must all be taken into acount, together with the safety of hostages, the demands of criminals, the repercussions of success or failure in a rescue attempt, and so on. The purpose of this paper is to propose a mathematical model of an optimal hostage rescue problem by using the concept of a sequential stochastic decision processes and examine properties of an optimal rescuing rule. Unfortunately, for our problem we were unable to find any reference material based on a mathematical approach. Accordingly, we can not list any references to be directly cited.

2 Model

Consider the following sequential stochastic decision process with a finite planning horizon. Here, for convenience, let points in time be numbered backward from the final point in time of the planning horizon, time 0, as $0, 1, \cdots$, and so on. Let the time interval between two successive points, say times t and $t-1$, be called the period t. Here, assume that time 0 is the deadline at which a rescue attempt is considered as the only course of action for some reason, say, the hostage's health condition, the degree of criminal desperation, and so on.

Suppose $i \geq 1$ persons are taken as hostages at any given point in time t, and we have to make a decision on attempting either rescue or no rescue. Let x denote a decision variable of a certain point in time t where $x = 0$ if no rescue attempt and $x = 1$ if rescue attempt, and X_t denote the set of possible decisions of time t, i.e., $X_t = \{0, 1\}$ for $t \geq 1$ and $X_0 = \{1\}$.

Let p $(0 < p < 1)$ be the probability of a hostage being killed if $x = 1$ (Case 3), and let s $(0 \leq s < 1)$ be the probability of criminal(s) surrendering up to the next point in time, i.e., time $t-1$ if $x = 0$ (Case 2), so $1-s$ is the probability of criminal(s) not surrendering. Further, let q and r $(0 < q < 1,\ 0 \leq r < 1$, and $0 < q+r < 1)$ be the probabilities of a hostage being, respectively, killed (Case 1) or set free (Case 3) up to the next point in time if $x = 0$ and criminal(s) not surrendering; accordingly, $1-q-r$ is the probability of the hostage being neither killed nor set free. The objective here is to maximize the probability of no hostage being killed. Here, the case of $p = 0$, $p = 1$, $s = 1$, $q = 0$, $q = 1$, $r = 1$, and $q+r = 1$ makes the problem trivial; accordingly, all of which are excluded in the definition of the model.

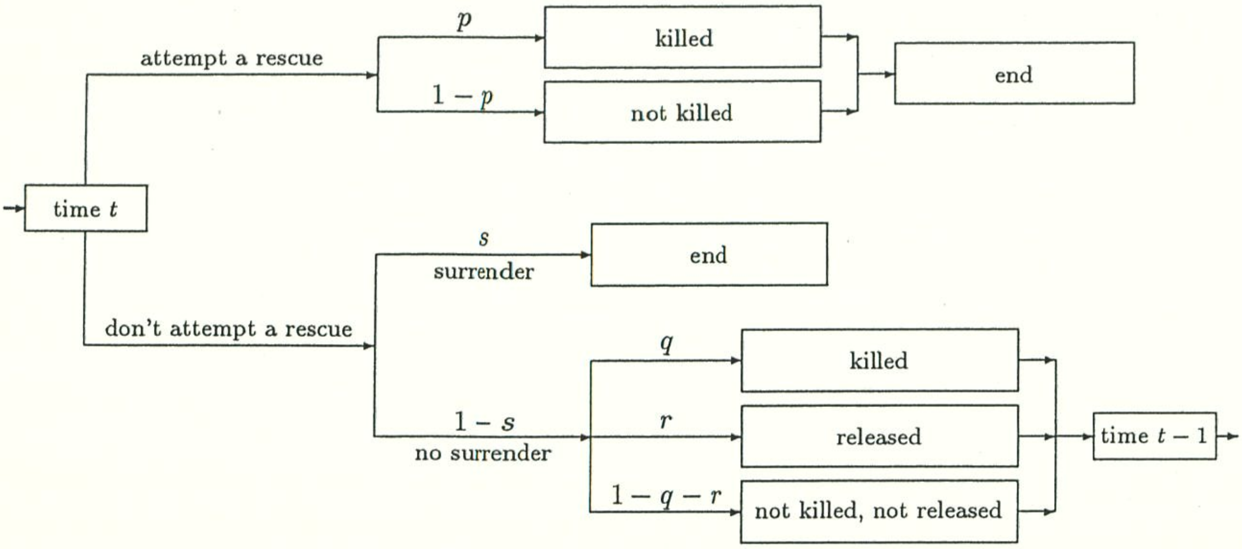

Figure 2.1: Decision Tree

3 Optimal Equation

Let $f_p(m|i)$ be the probability of m hostages being killed among i hostages if rescue attempt is made ($x=1$), which is given by

$$f_p(m|i) = \binom{i}{m} p^m (1-p)^{i-m}, \qquad i \geq 1, \quad 0 \leq m \leq i. \tag{3.1}$$

Let the probability of no hostage being killed if a rescue attempt is made ($x=1$) at any time t be denoted by $P(i)$. Then

$$P(i) = f_p(0|i) = (1-p)^i, \qquad i \geq 1. \tag{3.2}$$

Further, let $f_{qr}(k,\ell|i)$ be the probability of k hostages being killed and ℓ hostages being set free among i hostages if no rescue attempt ($x=0$) and criminal(s) not surrendering up to the next point in time, which is given by

$$f_{qr}(k,\ell|i) = \frac{i!}{k!\ell!(i-k-\ell)!} q^k r^\ell (1-q-r)^{i-k-\ell}, \qquad i \geq 1, \quad 0 \leq k+\ell \leq i. \tag{3.3}$$

Now, let $v_t(i)$ be the maximum probability of no hostage being killed, starting from time $t \geq 0$ with i hostages, expressed as

$$v_0(i) = P(i), \qquad i \geq 1, \tag{3.4}$$

$$v_t(i) = \max\{P(i), W_t(i)\}, \qquad i \geq 1, \quad t \geq 1, \tag{3.5}$$

where $W_t(i)$ is the probability of no hostage being killed over the period from time t to 0 (deadline) if no rescue attempt is made ($x = 0$). Noting $k + \ell \leq i$ and $k = 0$, we can express $W_t(i)$ as

$$W_t(i) = (1-s)\left(\sum_{\ell \leq i-1} f_{qr}(0,\ell|i)v_{t-1}(i-\ell) + f_{qr}(0,i|i) \times 1\right) + s \times 1, \quad i \geq 1, \quad t \geq 1, (3.6)$$

the right hand side of which implies the following:

At any given point in time t,

(a) Suppose the criminal(s) surrender with probability s. All the hostages are released, in other words, no hostage is killed; accordingly, the probability of no hostage being killed is equal to 1.

(b) Suppose criminal(s) do not surrender with probability $1 - s$, Then, the probability of no hostage being killed is $f_{qr}(0, \ell|i)$ if ℓ hostages are released.

1. If i hostages are released, then clearly $\ell = i$, hence the probability of no hostage being killed is equal to 1.
2. If $i - 1$ or less hostages are released ($\ell \leq i - 1$), then the number of remaining hostages becomes $i - \ell$, implying that the probability of no hostage being killed over the period from time $t - 1$ to 0 is equal to $v_{t-1}(i - \ell)$ by definition.

Eq. (3.6) can be easily rearranged as follows.

$$W_t(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)v_{t-1}(i-\ell) + r^i\right) + s, \quad i \geq 1, \quad t \geq 1, \tag{3.7}$$

where

$$W_1(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell) + r^i\right) + s, \quad i \geq 1. \tag{3.8}$$

Now, let

$$V_t(i) = W_t(i) - P(i), \quad i \geq 1, \quad t \geq 1, \tag{3.9}$$

where clearly $-1 \leq V_t(i) \leq 1$ for all i and t. Then, Eq. (3.5) can be rewritten as follows.

$$v_t(i) = \max\{0, V_t(i)\} + P(i), \quad i \geq 1, \quad t \geq 1. \tag{3.10}$$

Hence, the optimal decision rule can be stated as follows: If $V_t(i) < 0$, attempt a rescue, or else, do not attempt a rescue. Furthermore, noting Eqs. (3.7) and (3.10), we can rewrite Eq. (3.9) for $t \geq 2$ as follows.

$$
\begin{aligned}
V_t(i) &= (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(\max\{0,V_{t-1}(i-\ell)\}+P(i-\ell)\Big)+r^i\right)+s-P(i) \\
&= (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell)+r^i\right)+s-P(i) \\
&\quad +(1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0,V_{t-1}(i-\ell)\}, \quad i\ge 1, \quad t\ge 2,
\end{aligned} \tag{3.11}
$$

and noting Eq. (3.8), we can rearrange Eq. (3.9) for $t=1$ as follows.

$$
V_1(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell)+r^i\right)+s-P(i), \quad i\ge 1. \tag{3.12}
$$

Accordingly, Eq. (3.11) becomes as follows.

$$
V_t(i) = V_1(i)+(1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0,V_{t-1}(i-\ell)\}, \quad i\ge 1, \quad t\ge 2. \tag{3.13}
$$

4 Preliminaries

Let

$$
z = q+(1-q-r)p, \tag{4.1}
$$

where $0<z<1$ due to the assumptions of p, q and r.

Lemma 4.1 *For all $i\ge 1$ we have*

$$
\lim_{i\to\infty}(1-p)^i = \lim_{i\to\infty}(1-z)^i = 0, \tag{4.2}
$$

$$
\lim_{i\to\infty} i f_{qr}(k,\ell|i) = 0. \tag{4.3}
$$

PROOF. Using the Stirling asymptotic formula $i! \sim \sqrt{2\pi}\, i^{i+0.5}e^{-i}$, we obtain

$$
\begin{aligned}
i f_{qr}(k,\ell|i) &= i\frac{i!}{k!\ell!(i-k-\ell)!}q^k r^\ell (1-q-r)^{i-k-\ell} \\
&\sim \frac{q^k r^\ell}{k!\ell!(1-q-r)^{k+\ell}} \frac{\sqrt{2\pi} i i^{i+0.5}e^{-i}}{\sqrt{2\pi}(i-k-\ell)^{i-k-\ell+0.5}e^{-(i-k-\ell)}}(1-q-r)^i \\
&= \frac{q^k r^\ell e^{-(k+\ell)}}{k!\ell!(1-q-r)^{k+\ell}}\left(\frac{i}{i-k-\ell}\right)^{0.5}\left(\frac{i}{i-k-\ell}\right)^{i-k-\ell} i^{k+\ell+1}(1-q-r)^i.
\end{aligned}
$$

Now, for convenience, let $u=1-p$ and $w=1-q-r$ where $0<u<1$ and $0<w<1$ due to the assumptions of p, q and r. Then, consider $\delta>0$ and $\gamma>0$ such that

$$u = 1/(1+\delta),$$
$$w = 1/(1+\gamma).$$

Hence, for any sufficiently large i we have

$$\begin{aligned}(1+\delta)^i &= 1 + i\delta + \cdots > i\delta,\\ (1+\gamma)^i &= 1 + i\gamma + \cdots + \binom{i}{k+\ell+2}\gamma^{k+\ell+2} + \cdots\\ &> \binom{i}{k+\ell+2}\gamma^{k+\ell+2} = \frac{i!}{(k+\ell+2)!(i-k-\ell-2)!}\gamma^{k+\ell+2},\end{aligned}$$

from which we get

$$\begin{aligned}u^i &= 1/(1+\delta)^i < \frac{1}{i\delta},\\ w^i &= 1/(1+\gamma)^i < \frac{(k+\ell+2)!(i-k-\ell-2)!}{i!\gamma^{k+\ell+2}} = \frac{(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\gamma^{k+\ell+2}}.\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}(1-p)^i &= u^i < \frac{1}{i\delta},\\ i^{k+\ell+1}(1-q-r)^i &= i^{k+\ell+1}w^i < \frac{i^{k+\ell+1}(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\gamma^{k+\ell+2}}\\ &= \frac{(k+\ell+2)!}{\gamma^{k+\ell+2}}\frac{1}{i(1-\frac{1}{i})(1-\frac{2}{i})\cdots(1-\frac{k+\ell}{i})(1-\frac{k+\ell+1}{i})},\end{aligned}$$

both of which converge to 0 as $i \to \infty$. Noting,

$$\begin{aligned}\lim_{i\to\infty}\Big(\frac{i}{i-k-\ell}\Big)^{0.5} &= \lim_{i\to\infty}\Big(\frac{1}{1-\frac{k+\ell}{i}}\Big)^{0.5} = 1,\\ \lim_{i\to\infty}\Big(\frac{i}{i-k-\ell}\Big)^{i-k-\ell} &= \lim_{i\to\infty}\Big(1+\frac{k+\ell}{i-k-\ell}\Big)^{i-k-\ell} = e^{k+\ell}.\end{aligned}$$

Accordingly, $(1-p)^i$ and $if_{qr}(k,\ell\ i)$ converge to 0 as $i \to \infty$. In quite a similar way, we can show $\lim_{i\to\infty}(1-z)^i = 0$. ∎

5 Analysis

5.1 General Properties of $V_t(i)$

Lemma 5.1 *For all $i \geq 1$ we have*

$$V_1(i) = (1-s)(1-z)^i - (1-p)^i + s, \tag{5.1}$$

$$V_1(1) = p - (1-s)z. \tag{5.2}$$

PROOF. Noting Eqs. (3.2) and (4.1), we can write Eq. (3.12) as follows.

$$\begin{aligned} V_1(i) &= (1-s)\left(\sum_{\ell=0}^{i-1}\binom{i}{\ell}r^\ell(1-q-r)^{i-\ell}(1-p)^{i-\ell} + r^i\right) + s - (1-p)^i \\ &= (1-s)\left(\sum_{\ell=0}^{i}\binom{i}{\ell}r^\ell\Big((1-q-r)(1-p)\Big)^{i-\ell} - r^i + r^i\right) + s - (1-p)^i \\ &= (1-s)\Big(r + (1-q-r)(1-p)\Big)^i - (1-p)^i + s \\ &= (1-s)\Big(1 - (q + (1-q-r)p)\Big)^i - (1-p)^i + s \\ &= (1-s)(1-z)^i - (1-p)^i + s, \end{aligned}$$

from which we immediately have Eq. (5.2). ∎

Lemma 5.2 *For any given $i_2 > i_1 \geq 1$ we have*

(a) *If $p \geq z$, the following two inequalities can not coincide.*

$$V_1(i_1) > V_1(i_1+1), \tag{5.3}$$

$$V_1(i_2) < V_1(i_2+1). \tag{5.4}$$

(b) *If $p \leq z$, the following two inequalities can not coincide.*

$$V_1(i_1) < V_1(i_1+1), \tag{5.5}$$

$$V_1(i_2) > V_1(i_2+1). \tag{5.6}$$

PROOF.

(a) Assume $p \geq z$, and let $b(i) = V_1(i+1) - V_1(i)$. Then, from Eq. (5.1) we have

$$\begin{aligned} b(i) &= (1-s)(1-z)^{i+1} - (1-p)^{i+1} + s - (1-s)(1-z)^i + (1-p)^i - s \\ &= (1-s)(1-z)^i(1-z-1) - (1-p)^i(1-p-1) \\ &= p(1-p)^i - (1-s)z(1-z)^i. \end{aligned} \tag{5.7}$$

Since $V_1(i+1) = V_1(i) + b(i)$, if Eqs. (5.3) and (5.4) are both satisfied, then

$$V_1(i_1) > V_1(i_1) + b(i_1),$$
$$V_1(i_2) < V_1(i_2) + b(i_2),$$

from which $b(i_1) < 0$ and $b(i_2) > 0$; equivalently,

$$p(1-p)^{i_1} < (1-s)z(1-z)^{i_1},$$
$$p(1-p)^{i_2} > (1-s)z(1-z)^{i_2}.$$

Consquently, we obtain

$$(1-s)z\Big(\frac{1-z}{1-p}\Big)^{i_2} < p < (1-s)z\Big(\frac{1-z}{1-p}\Big)^{i_1}.$$

Thus, we get

$$\Big(\frac{1-z}{1-p}\Big)^{i_2} < \Big(\frac{1-z}{1-p}\Big)^{i_1}.$$

Accordingly, since $i_2 > i_1 \geq 1$ by assumption, it must be that $(1-z)/(1-p) < 1$, i.e., $p < z$, which is a contradiction, hence, it follows that Eqs. (5.3) and (5.4) can not coincide.
(b) Almost the same as the proof of (a). ∎

From Lemma 5.2 we immediately get the following corollary.

Corollary 5.1 *For any given $i_2 > i_1 \geq 1$ we have*

(a) *If $p \geq z$, the following two inequalities can not coincide.*

$$V_1(i_1) > 0 \geq V_1(i_1+1), \tag{5.8}$$
$$V_1(i_2) \leq 0 < V_1(i_2+1). \tag{5.9}$$

(b) *If $p \leq z$, the following two inequalities can not coincide.*

$$V_1(i_1) < 0 \leq V_1(i_1+1), \tag{5.10}$$
$$V_1(i_2) \geq 0 > V_1(i_2+1). \tag{5.11}$$

Lemma 5.3 *If $V_1(i) < (=)\, 0$ for all $i \geq 1$, then $V_t(i) = V_1(i) < (=)\, 0$ for all $i \geq 1$ and $t \geq 1$.*

Proof. It is evident for $t = 1$. Suppose $V_{t-1}(i) = V_1(i) < (=)\, 0$ for all $i \geq 1$. Then, since $\max\{0, V_{t-1}(i-\ell)\} = 0$ for all i and ℓ, we immediately get $V_t(i) = V_1(i) < (=)\, 0$ from Eq. (3.13). This completes the induction. ∎

Lemma 5.4 *$V_t(i)$ is nondecreasing in t for all $i \geq 1$.*

PROOF. Since $\max\{0, V_{t-1}(i-\ell, 1)\} \geq 0$, from Eq. (3.13) we have $V_t(i) \geq V_1(i)$ for all $t \geq 1$ and $i \geq 1$, hence $V_2(i) \geq V_1(i)$ for all $i \geq 1$. Suppose $V_{t-1}(i) \geq V_{t-2}(i)$ for all $i \geq 1$. Then, from Eq. (3.13) we have

$$\begin{aligned} V_t(i) &= V_1(i) + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\} \\ &\geq V_1(i) + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-2}(i-\ell)\} = V_{t-1}(i). \end{aligned}$$

This completes the induction. ∎

Lemma 5.5 $\lim_{i\to\infty} V_t(i) = s$ *for all $i \geq 1$ and $t \geq 1$.*

PROOF. Let

$$A_t(i) = \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\}.$$

Then, Eq. (3.13) can be rewritten as follows.

$$V_t(i) = V_1(i) + (1-s)A_t(i).$$

Now, since clearly $|\max\{0, V_t(i)\}| \leq 1$ for all $i \geq 1$ and $t \geq 1$, we obtain

$$0 \leq |A_t(i)| \leq \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)|\max\{0, V_{t-1}(i-\ell)\}| \leq \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i).$$

Now, it follows from Eq. (4.3) of Lemma 4.1 that for any infinitesimal number $\varepsilon > 0$ there exists a certain $I(\varepsilon|\ell)$ such that $if_{qr}(0,\ell|i) < \varepsilon$, or $f_{qr}(0,\ell|i) < \varepsilon/i$ for all $i > I(\varepsilon|\ell)$. Hence,

$$\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i) < \sum_{\ell=0}^{i-1} \varepsilon/i = \varepsilon, \qquad i > I(\varepsilon|\ell).$$

Thus, $\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)$ converges to 0 as $i \to \infty$. Consequently, it follows that $\lim_{i\to\infty} |A_t(i)| = 0$, so $\lim_{i\to\infty} A_t(i) = 0$. From Eq. (5.1) and Eq. (4.2) of Lemma 4.1 we immediately obtain $\lim_{i\to\infty} V_1(i) = s$. Eventually it follows that

$$\lim_{i\to\infty} V_t(i) = \lim_{i\to\infty} V_1(i) + (1-s)\lim_{i\to\infty} A_t(i) = s. \quad ∎$$

5.2 Case of $s = 0$

Since $p - z = (q+r)p - q$ due to Eq. (4.1), noting Eq. (5.2), we have

$$p > (= (<))\, z \iff p > (= (<))\, q/(q+r) \iff V_1(1) > (= (<))\, 0. \tag{5.12}$$

Lemma 5.6 *Let $s = 0$. If $p \geq (<)\, q/(q+r)$, then $V_t(i) \geq (<)\, 0$ for all $i \geq 1$ and $t \geq 1$.*

Proof. Since $s = 0$, Eq. (5.1) becomes

$$\begin{aligned} V_1(i) &= (1-z)^i - (1-p)^i \\ &= \Big((1-z) - (1-p)\Big)\Big((1-z)^{i-1} + (1-z)^{i-2}(1-p) + \cdots + (1-z)(1-p)^{i-2}(1-p)^{i-1}\Big) \\ &= (p-z)\Big((1-z)^{i-1} + (1-z)^{i-2}(1-p) + \cdots + (1-z)(1-p)^{i-2}(1-p)^{i-1}\Big). \end{aligned} \tag{5.13}$$

From this and Eq. (5.12), the assertion for $t = 1$ becomes true. Accordingly, if $p \geq q/(q+r)$, i.e., $V_1(i) \geq 0$ for all i from Eq. (5.12), hence $V_t(i) \geq 0$ for all $i \geq 1$ and $t \geq 1$ from Lemma 5.4, and if $p < q/(q+r)$, i.e., $V_1(i) < 0$ for all i, hence $V_t(i) < 0$ for all $i \geq 1$ and $t \geq 1$ from Lemma 5.3. Thus the assertion holds. ∎

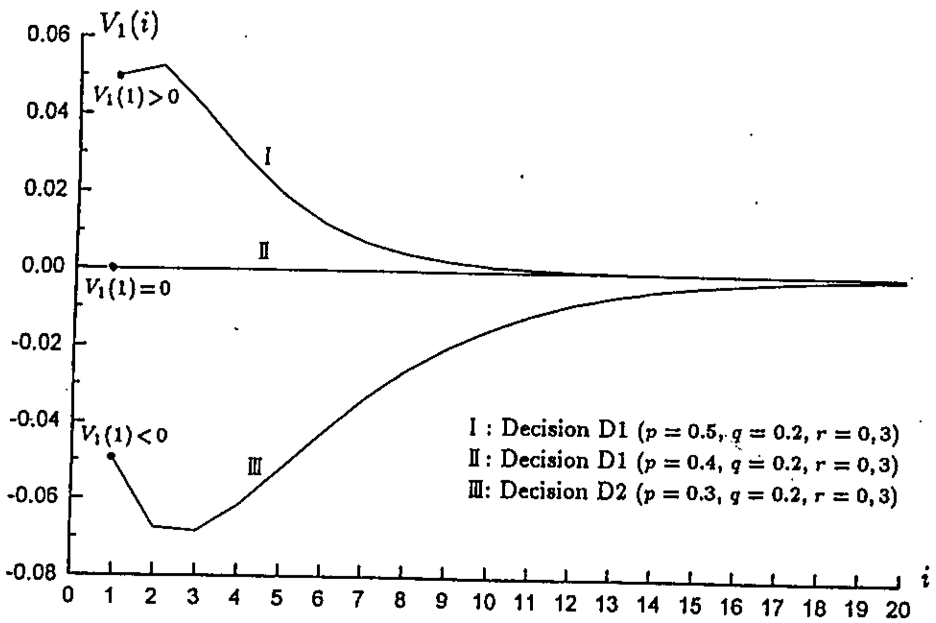

Figure 5.1: The graph of $V_1(i)$ with i $(s = 0)$

5.3 Case of $s > 0$ and $p \geq (1-s)z$

Lemma 5.7 *Let $s > 0$ and $p \geq (1-s)z$. Then, $V_t(i) > 0$ for all $i > 1$ and $t \geq 1$.*

PROOF. Assume $s > 0$.

Let $p = (1-s)z$. Then, from Eq. (5.2) we have $V_1(1) = 0$. Now, since $z > 0$ and $s > 0$, clearly $z > (1-s)z = p$, so $1 - p > 1 - z$. Then, from Eq. (5.7) we get for all $i > 1$,

$$V_1(i+1) - V_1(i) = p(1-p)^i - p(1-z)^i = p\Big((1-p)^i - (1-z)^i\Big) > 0,$$

hence it follows that $V_1(i)$ is strictly increasing in i, so $V_1(i) > 0$ for all $i > 1$.

Let $p > (1-s)z$. Then, from Eq. (5.2) we have $V_1(1) > 0$. If $p < z$, so $1 - p > 1 - z$. Then, from Eq. (5.7) we have, for all $i > 1$,

$$\begin{aligned} V_1(i+1) - V_1(i) &> (1-s)z(1-p)^i - (1-s)z(1-z)^i \\ &= (1-s)z\Big((1-p)^i - (1-z)^i\Big) > 0, \end{aligned}$$

hence it follows that $V_1(i)$ is strictly increasing in i, so $V_1(i) > 0$ for all $i > 1$. If $p \geq z$. Then, from (a) of Corollary 5.1 we obtain, for all $i \geq 1$, that if Eq. (5.8) is satisfied, then Eq. (5.9) must necessarily occur because the limit of $V_1(i)$ in i is positive due to the assumption $s > 0$ and Lemma 5.5. This is a contradiction; therefore, Eq. (5.8) does not occur at all; in other words, it must be that $V_1(i) > 0$ for all $i > 1$.

Thus, it follows that $V_1(i) > 0$ for all $i > 1$ if $p \geq (1-s)z$. Accordingly, from Lemma 5.4 we get $V_t(i) \geq V_1(i) > 0$ for all $i > 1$ and $t \geq 1$. ∎

5.4 Case of $s > 0$ and $p < (1-s)z$

By $i(t)$ let us define the i satisfying

$$V_t(i-1) < 0 \leq V_t(i), \quad i > 1, \quad t \geq 1, \tag{5.14}$$

if it exists.

Lemma 5.8 *Let $s > 0$ and $p < (1-s)z$. Then:*

(a) *If* $V_1(i) < 0$ *and* $V_1(i+1) = 0$ *for a certain* $i \geq 1$, *we have* $V_1(i+2) > 0$ *for that* i.

(b) *There exists a unique* $i(1)$, *hence* $V_1(i) < 0$ *for* $i < i(1)$, $V_1(i(1), 1) \geq 0$ *and* $V_1(i) > 0$ *for* $i > i(1)$.

(c) $V_t(1) < 0$ *for all* t.

(d) *If* $V_1(i') < 0$ *for a certain* $i' \geq 1$, *then* $V_t(i) = V_1(i) < 0$ *for all* $i \leq i'$ *and* $t \geq 1$.

(e) $i(t)$ *is unique for all* $t \geq 1$ *and independent of* t.

PROOF. Let $s > 0$ and $p < (1-s)z$. Then, from the fact of $z > (1-s)z$ due to $z > 0$ and $s > 0$, we get $z > p$, and from Eq. (5.2) we have $V_1(i) < 0$.

(a) If $V_1(i) < 0$ and $V_1(i+1) = 0$ for a certain i, then from $b(i) = V_1(i+1) - V_1(i)$ and $b(i+1) = V_1(i+2) - V_1(i+1)$ by definition (See the proof of Lemma 5.2) we immediately get, for that i,

$$b(i) > 0, \tag{5.15}$$

$$V_1(i+2) = b(i+1). \tag{5.16}$$

From Eqs. (5.15) and (5.7) we have

$$p(1-p)^i > (1-s)z(1-z)^i. \tag{5.17}$$

Noting Eqs. (5.17) and (5.7), Eq. (5.16) can be expressed as follows.

$$\begin{aligned} V_1(i+2) &= p(1-p)^{i+1} - (1-s)z(1-z)^{i+1} \\ &> (1-s)z(1-z)^i(1-p) - (1-s)z(1-z)^{i+1} \\ &= (1-s)z(1-z)^i(1-p-1+z) \\ &= (1-s)z(1-z)^i(z-p) > 0. \end{aligned}$$

(b) Since $V_1(1) < 0$, and the fact that $V_1(i) > 0$ for a sufficiently large i due to Lemma 5.5, clearly, $i(1)$ exists. Now, if Eq. (5.10) is satisfied, then, Eq. (5.11) does not occur at all. Noting this and (a) we immediately obtain that $i(1)$ is unique. Thus $V_1(i) < 0$ for $i < i(1)$, $V_1(i(1), 1) \geq 0$ and $V_1(i) > 0$ for $i > i(1)$.

(c)Since $V_1(1) < 0$. Hence, the assertion is true for $t = 1$. Suppose $V_{t-1}(1) < 0$. Then, from Eq. (3.13) we get $V_t(1) = V_1(1) < 0$. This completes the induction.

(d) From (b) we immediately obtain that the assertion is true if $t = 1$. Suppose the assertion is true for $t-1$; that is, $V_{t-1}(i) = V_1(i) < 0$ for $i \leq i'$, so $V_{t-1}(i-\ell) < 0$ for $0 \leq \ell \leq i-1$ and

$i \leq i'$. Accordingly, from Eq. (3.13) we immediately get $V_t(i) = V_1(i) < 0$. This completes the induction.

(e) From (c) and Lemma 5.5 we immediately get $i(t)$ exists, and from (b) we have a unique $i(1)$, hence $V_1(i) < 0$ for $i \leq i(1) - 1$, $V_1(i(1)) \geq 0$, and $V_1(i) > 0$ for $i > i(1)$. Now, letting $i' = i(1) - 1$, from (d) we have $V_t(i) = V_1(i) < 0$ for $i \leq i'$, and from Lemma 5.4 we have $V_t(i(1)) \geq V_1(i(1)) \geq 0$ and $V_t(i) \geq V_1(i) > 0$ for $i > i(1)$. Accordingly, by the definition of $i(t)$ it follows that $i(t) = i(1)$. This implies that $i(t)$ is unique and independent of t for all $t \geq 1$. ∎

Now, noting Eq. (5.2), we immediately have

$$p > (= (<))\ (1-s)z \iff V_1(1) > (= (<))\ 0. \tag{5.18}$$

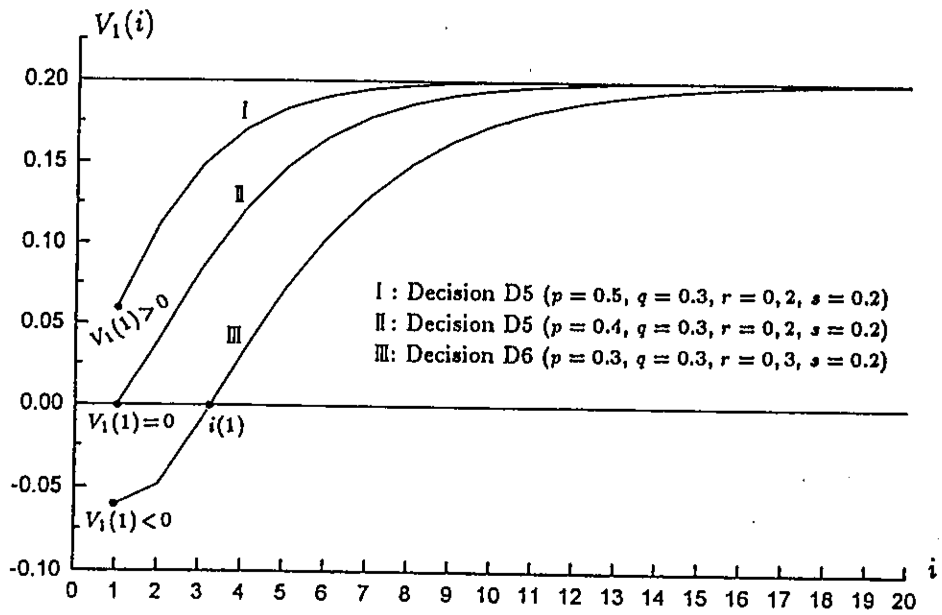

Figure 5.2: The graph of $V_1(i)$ with i $(s > 0)$

6 Concluding Remarks

6.1 Case of $s = 0$

In hostage plots perpetrated by a person who is determined to go through with it no matter what, and not surrender on any terms, he knows that, if arrested, he will be condemned to death or life imprisonment. This can be regarded as the case of $s = 0$. From Lemma 5.6 the optimal decision rule for any given i and t is independent of i and can be described as follows.

D1: *If $p \geq q/(q+r)$, wait up to time $t = 0$, deadline.*

D2: *If $p < q/(q+r)$, attempt a rescue immediatly when a hostage plot occurs.*

6.2 Case of $s > 0$ and $i = 1$

The most typical hostage plots are the case of $i = 1$. In this case, if $p \geq (1-s)z$, then $V_1(1) \geq 0$ from Eq. (5.2), hence it follows from Lemma 5.4 that $V_t(1) \geq 0$ for all $t \geq 1$, and if $p < (1-s)z$, then $V_1(1) < 0$ from Eq. (5.2), hence $V_t(1) < 0$ for all $t \geq 1$ from Lemma 5.3. Accordingly, if $i = 1$, the optimal decision rule for any given t can be stated as follows.

D3: *If $p \geq (1-s)z$, wait up to time 0, deadline.*

D4: *If $p < (1-s)z$, attempt a rescue immediately a hostage plot occurs.*

6.3 Case of $s > 0$

This is the case where a criminal might surrender, which in reality is the most possible case. In this case, if $p \geq (1-s)z$, then $V_t(i) > 0$ for all i and t from Lemma 5.7, and if $p < (1-s)z$, from (e) of Lemma 5.8 there exists a unique i^* such that $V_t(i) < (\geq) 0$ for $i < (\geq) i^*$, where $i^* = i(t) = i(1)$. Then, the optimal decision rule can be stated as follows.

D5: *If $p \geq (1-s)z$, wait up to time 0, deadline, irrespective of i, the number of remaining hostages.*

D6: *If $p < (1-s)z$, wait if $i < i^*$ and attempt a rescue if $i \geq i^*$.*

7 Future Studies

In our paper we propose a basic mathematical model for an optimal rescuing problem involving hostages. Taking different real hostage situations into account, we feel a need to modify the model from the following viewpoints:

1. In many real cases, criminals operate with confused motives. This causes the probabilities p, q, r and s to change randomly from one minute to the next. This consideration leads us to the model in which p, q, r and s are random variables with a known or unknown distribution function $F(p, q, r, s)$. When it is unknown, we can and must update its unknown parameters by using Bays' theorem.
2. In real hostage plots, some courses of action can be considered: whether or not to submit to the demands to be airlifted to another country, to provide a means of escape, to pay the ransom, to release comrades in prison, and so on. Taking such courses of action will influence the probabilities p, q, r and s to a greater or lesser degree. A problem arises as to when to act and what course of action to propose in order to maximize the probability of no hostage being killed.

3. In the present paper, all the hostages are implicitly assumed to be homogenous. As seen in many hostages crises, however, special considerations are given for females, the aged, the sick, children, and so on. Models in which such nonhomogenous classes of hostages are taken into consideration should also be proposed.
4. Cases where the deadline can not always be known.

Finally, in order for the model to be realistically effective, the probabilities p, q, r and s for each hostage crisis must be measured and known in advance. Although such a measurement would be a very difficult task, it should be tackled through united efforts of researchers in different fields, say, psychologists, sociologists, political scientists, engineers.