No. 857

Unbiased Test for a Location Parameter
-Case of Logistic Distribution-

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April 2000

Unbiased Test for a Location Parameter (2).

---Case of Logistic Distribution---

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Abstract.

In this paper we deal with the Logistic distribution with density

$$f(x|\theta)=\frac{e^{-(x-\theta)}}{\{1+e^{-(x-\theta)}\}^2}, \qquad \text{for } -\infty<x<\infty$$

where $-\infty<\theta<\infty$. Based on a random sample $X_1, \ldots, X_n$ of size n from the density $f(x|\theta)$ we consider the problem of the testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0. We propose the test with the acceptance region derived from inverting the shortest confidence interval for θ_0 and check if this test is unbiased.

§1. Introduction.

In this paper we deal with Logistic distribution whose density is given as follows:

$$(1) \qquad f(x|\theta)=\frac{e^{-(x-\theta)}}{\{1+e^{-(x-\theta)}\}^2}, \qquad \text{for } -\infty<x<\infty$$

provided that $-\infty<\theta<\infty$. Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density $f(x|\theta)$. We find in Section 2 the confidence interval (C. I.) for θ with the shortest length using Lagrange's method. In Section 3 we consider the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0. We propose the test with acceptance region derived from inverting the shortest C. I. for θ_0. Let α be a real number such that $0<\alpha<1$. When n=2m+1 with m a nonnegative integer, we show that our test is unbiased and of size α. But, when n=2m, because we use conventional device to get the C. I. for θ, we cannot show unbiasedness of our test. However, for large m our test becomes almost unbiased as the test in case of n=2m+1 shows.

Let $\doteq$ be the defining property.

§2. The Interval Estimation for θ.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the population with the density (1). We find the shortest C. I. for θ using Lagrange's method.

Let n=2m+1 with m a nonnegative integer, until (14). Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We estimate θ by $Y\doteq X_{(m+1)}$. To get the shortest C. I. for θ we first find the density of Y. Let $F(x|\theta)$ be the cumulative distribution function (c.d.f.) of X. Then, by (1) we get

$$(2) \qquad F(x)\doteq F(x|\theta)=\{1+e^{-(x-\theta)}\}^{-1}, \qquad \text{for } -\infty<x<\infty.$$

Hence, the density of Y is of form

(3) $g_Y(y|\theta)=k(F(y))^m(1-F(y))^m f(y|\theta)$, for $-\infty<y<\infty$.

where

(4) $k=\Gamma(2m+2)/\{\Gamma(m+1)\}^2$.

Let α be a real number such that $0<\alpha<1$. Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the shortest C. I. for θ at confidence coefficient $1-\alpha$ we want to minimize r_2-r_1 under the condition that

(5) $P_\theta[r_1<Y-\theta<r_2]=1-\alpha$.

But, it follows by a variable transformation $W=F(Y)$ that

(6) the left hand side of (5) $= P_\theta[r_1+\theta<Y<r_2+\theta]$

$=P_\theta[F(r_1+\theta)<W<F(r_2+\theta)]=1-\alpha$.

Hence, we want to minimize r_2-r_1 under the condition (6). To do so we use Lagrange's method. Let λ be a real number and define

(7) $L\doteq L(r_1,r_2;\lambda)\doteq r_2-r_1-\lambda\{\int_{F(r_1+\theta)}^{F(r_2+\theta)} h_W(w)\,dw-1+\alpha\}$

where $h_W(w)$ is the density of W given by

(8) $h_W(w)=kw^m(1-w)^m$, for $0<w<1$

where k is given by (4). The right hand side of (8) is the probability density function(p. d. f.) of Beta distribution Beta(m+1,m+1) with (m+1,m+1)degrees of freedom. Then, by Lagrange's method we have that

$$(9)\qquad \begin{cases} \partial L/\partial r_1 = -1 + \lambda h_W(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=0 \\ \partial L/\partial r_2 = 1 - \lambda h_W(F(r_2+\theta))f(r_2+\theta\,|\,\theta)=0 \end{cases}$$

By (9) we get that

$$(10)\qquad h_W(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=h_W(F(r_2+\theta))f(r_2+\theta\,|\,\theta)\ (=\lambda^{-1}),\qquad \forall\theta.$$

Taking

$$(11)\qquad F(r_1+\theta)=\beta(\alpha/2)\quad\text{and}\quad F(r_2+\theta)=1-\beta(\alpha/2)$$

where $\beta(\alpha/2)$ is given by

$$(12)\qquad \int_0^{\beta(\alpha/2)} h_W(w)\,dw=\alpha/2,$$

we obtain by (2) that $r_1=-r_2\doteq -r$ where

$$(13)\qquad r\doteq F^{-1}(1-\beta(\alpha/2))-\theta = \ln[\{1-\beta(\alpha/2)\}/\beta(\alpha/2)].$$

We also have $h_W(F(-r+\theta))=h_W(F(r+\theta))$ and $f(-r+\theta\,|\,\theta)=f(r+\theta\,|\,\theta)$ with r given by (13). Thus, (10) and (6) are satisfied for $r_1=-r_2=-r$ with r given by (13). Therefore, the shortest C. I. for θ at confidence coefficient $1-\alpha$ is given by

$$(14)\qquad (Y-r,Y+r)\doteq(Y-\ln[\{1-\beta(\alpha/2)\}/\beta(\alpha/2)],Y+\ln[\{1-\beta(\alpha/2)\}/\beta(\alpha/2)]).$$

Let n=2m. This time we estimate θ by $Y\doteq X_{(m)}$. In the similar way to the above we get the density of Y

$$(15)\qquad g_Y(y\,|\,\theta)=k_1(F(y))^{m-1}(1-F(y))^m f(y\,|\,\theta),\quad \text{for } -\infty<y<\infty$$

where

(16) $\quad k_1=\Gamma(2m+1)/\{\Gamma(m)\Gamma(m+1)\}$.

Putting $W=F(Y)$ we minimize r_2-r_1 under the condition (6). However, since the density of W is now of form

(17) $\quad h_1(w)=k_1 w^{m-1}(1-w)^m, \quad \text{for } 0<w<1$

which is the p.d.f. of the Beta(m,m+1) distribution with k_1 defined by (16), it is difficult to get exact values for $F(r_i+\theta)$, i=1,2 which satisfy

(18) $\quad h_1(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=h_1(F(r_2+\theta))f(r_2+\theta\,|\,\theta)$.

Hence, we use conventional values for $F(r_i+\theta)$, i=1,2. Those are

(19) $\quad F(r_1+\theta)=\beta_{m,m+1}(\alpha/2)$ and $F(r_2+\theta)=1-\beta_{m+1,m}(\alpha/2)$

where $\beta_{m,m+1}(\alpha/2)$ and $\beta_{m+1,m}(\alpha/2)$ are respectively determined by

(20) $$\int_0^{\beta_{m,m+1}(\alpha/2)} h_1(w)\,dw = \alpha/2 = \int_0^{\beta_{m+1,m}(\alpha/2)} k_1 w^m(1-w)^{m-1}\,dw.$$

Thus, r_1 and r_2 are respectively given by

(21) $$\begin{cases} r_1=F^{-1}(\beta_{m,m+1}(\alpha/2))-\theta = -\ln[\{1-\beta_{m,m+1}(\alpha/2)\}/\beta_{m,m+1}(\alpha/2)] \\ r_2=F^{-1}(\beta_{m+1,m}(\alpha/2))-\theta = \ln[\{1-\beta_{m+1,m}(\alpha/2)\}/\beta_{m+1,m}(\alpha/2)] \end{cases}$$

Threfore, the C. I. for θ at confidence coefficient $1-\alpha$ is

(22) $\quad (Y-r_2, Y-r_1)$,

where r_1 and r_2 are determined by (21).

In the next section we check if the tests with the acceptance regions derived from inverting the C. I.'s (14) for $n=2m+1$ and (22) for $n=2m$, respectively are unbiased and of size α.

§3. Two-Sided Test for θ.

In this section we consider the problem of testing the null hypothesis $H_0: \theta=\theta_0$ versus the alternative hypothesis $H_1: \theta\neq\theta_0$ for some constant θ_0. We propose the two-sided test with the acceptance region derived from inverting the shortest C. I. for θ_0. When $n=2m+1$ we show that our test is unbiased and of size α. When $n=2m$ our test is not unbiased because of usage of conventional method for constructing the C. I. for θ.

Let $n=2m+1$. As in Section 2 we define $Y \doteq X_{(m+1)}$. By inverting the shortest C. I. (14) for θ_0 our test is to reject $Y\in(-\infty, \theta_0-r]\cup[\theta_0+r, +\infty)$ and to accept H_0 if $Y\in(\theta_0-r, \theta_0+r)$ where r is given by (13). Now, we show that this test is unbiased and of size α.

Let $y_1{}^0$ and $y_2{}^0$ be real numbers depending on θ_0 such that $y_1{}^0<y_2{}^0$. Define $\psi(\theta)$ by

$$\psi(\theta)=P_\theta[Y<y_1{}^0 \quad \text{or} \quad y_2{}^0<Y] = 1-\int_{y_1{}^0}^{y_2{}^0} g_Y(y|\theta)\, dy \tag{23}$$

where $g_Y(y|\theta)$ is defined by (3). To get unbiased size-α test with the acceptance region $(y_1{}^0, y_2{}^0)$ we choose $y_1{}^0$ and $y_2{}^0$ which satisfy

$$\psi(\theta_0)=1-P_{\theta_0}[y_1{}^0<Y<y_2{}^0]=\alpha \tag{24}$$

and minimize $\psi(\theta)$ at $\theta=\theta_0$; namely

$$d\psi(\theta)/d\theta\Big|_{\theta=\theta_0} = g_Y(y_2{}^0|\theta_0)-g_Y(y_1{}^0|\theta_0)=0. \tag{25}$$

We consider the test with the acceptance region (θ_0-r, θ_0+r). Since from the construction the equality (10) with $r_1=-r$, $r_2=r$ and $\theta=\theta_0$ is satisfied, we obtain by (3) and (8) that $g_Y(\theta_0-r|\theta_0)=g_Y(\theta_0+r|\theta_0)$; (25) is satisfied for y_1^0 and y_2^0 replaced by θ_0-r and θ_0+r, respectively. (24) with y_1^0 and y_2^0 replaced by θ_0-r and θ_0+r, respectively is the same as (5) except for θ, r_1 and r_2 replaced by θ_0, $-r$ and r, respectively. Therefore, our test with the acceptance region (θ_0-r, θ_0+r) is unbiased and of size α.

Let $n=2m$. As in Section 2 we define $Y \doteq X_{(m)}$. Again, by inverting the C. I. (22) for θ_0 our test is to reject H_0 if $Y \in (-\infty, \theta_0+r_1] \cup [\theta_0+r_2, +\infty)$ and to accept H_0 if $Y \in (\theta_0+r_1, \theta_0+r_2)$ where r_1 and r_2 are given by (21). In this case our test depends on the conventional values for $F(r_i+\theta)$, $i=1,2$. Hence, we have that $g_Y(\theta_0+r_1|\theta_0) \neq g_Y(\theta_0+r_2|\theta_0)$. Furthermore, (24) with y_1^0 and y_2^0 replaced by θ_0+r_1 and θ_0+r_2, respectively is the same as (5) except for θ replaced by θ_0. Therefore, our test is still of size α, but not unbiased. However, for large m our test becomes almost unbiased as the test in case of $n=2m+1$ shows.