**No. 856**

# Unbiased Tests for Location and Scale Parameters
-Case of Cauchy Distribution-

by

Yoshiko Nogami

April 2000

Unbiased Tests for Location and Scale Parameters

----Case of Cauchy Distribution----

By Yoshiko Nogami

Abstract.

In this paper we deal with Cauchy distribution with the density

$$f(x|\theta,\xi)=\xi\pi^{-1}\{\xi^2+(x-\theta)^2\}^{-1},\quad \text{for } -\infty<x<\infty$$

where $-\infty<\theta<\infty$ and $\xi>0$.

We first consider $\xi=1$. Based on a random sample of size n from $f(x|\theta,1)$ we consider the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative $H_1:\theta\neq\theta_0$ for some constant $\theta_0$. We propose the test with the acceptance region derived from inverting the shortest confidence interval (C. I.) for $\theta_0$ and check if this test is unbiased.

We secondly consider $\theta=0$. This time we consider the problem of testing $H_0:\xi=\xi_0$ versus $H_1:\xi\neq\xi_0$ for some constant $\xi_0$. We again propose the test with acceptance region derived from inverting the C. I. for $\xi_0$ and check if this test is unbiased.

## §1. Introduction.

In this paper we deal with Cauchy distribution whose density is given as follows:

(1) $f(x|\theta,\xi) = \xi\pi^{-1}\{\xi^2 + (x-\theta)^2\}^{-1}$ for $-\infty<x<\infty$

provided that $-\infty<\theta<\infty$ and $\xi>0$.

Let $\doteq$ be the defining property. We first consider the density $f(x|\theta)\doteq f(x|\theta,1)$. Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density $f(x|\theta)$. We find in Section 2 the confidence interval(C. I.) for the location parameter $\theta$ with the shortest length using Lagrange's method. In Section 3 we consider the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant $\theta_0$. We propose the test with the acceptance region derived from inverting the shortest C. I. for $\theta_0$. Let $\alpha$ be a real number such that $0<\alpha<1$. When n=2m+1 with m a nonnegative integer, we show that our test is unbiased and of size $\alpha$. But, when n=2m, because we use conventional method to get the C. I. for $\theta$, we cannot show unbiasedness of our test. (However, for large m our test becomes almost unbiased as the test in case of n=2m+1 shows.)

In the second half We consider the density $f(x|\xi)\doteq f(x|0,\xi)$. Based on a random sample of size n from the density $f(x|\xi)$ we find in Section 4 the C. I. for the scale parameter $\xi$. In Section 5 we consider the problem of testing $H_0:\xi=\xi_0$ versus $H_1:\xi\neq\xi_0$ for some constant $\xi_0$. Again we propose the test with acceptance region derived from inverting the C. I. for $\xi_0$. When n=2m+1, we show that our test is unbiased and of size $\alpha$. But, in the same reason as that for $\theta$ our test is not unbiased when n=2m. (However, for large m our test becomes almost unbiased as the test in case of n=2m+1 shows.)

## §2. The Interval Estimation for $\theta$.

In this section we deal with the density

$$(2) \qquad f(x|\theta) \doteq f(x|\theta,1) = \pi^{-1}\{1+(x-\theta)^2\}^{-1}, \qquad \text{for } -\infty<x<\infty$$

where $-\infty<\theta<\infty$. We find the shortest C. I. for $\theta$ using Lagrange's method.

Let $n=2m+1$ with $m$ a nonnegative integer, until (15). Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We estimate $\theta$ by $Y \doteq X_{(m+1)}$. To get the shortest C. I. for $\theta$ we first find the density of Y. Let $F(x|\theta)$ be the cumulative distribution function(c. d. f. ) of X. Then, by (2) we get

$$(3) \qquad F(x) \doteq F(x|\theta) = \pi^{-1}\tan^{-1}(x-\theta) + 2^{-1}, \qquad \text{for } -\infty<x<\infty.$$

Hence, the density of Y is of form

$$(4) \qquad g_Y(y|\theta) = k(F(y))^m(1-F(y))^m f(y|\theta), \qquad \text{for } -\infty<y<\infty$$

where

$$(5) \qquad k = \Gamma(2m+2)/(\Gamma(m+1))^2.$$

Let $\alpha$ be a real number such that $0<\alpha<1$. Let $r_1$ and $r_2$ be real numbers such that $r_1<r_2$. To find the shortest C. I. for $\theta$ at confidence coefficient $1-\alpha$ we want to minimize $r_2-r_1$ under the condition that

$$(6) \qquad P_\theta[r_1 < Y-\theta < r_2] = 1-\alpha.$$

But, it follows by a variable transformation $W=F(Y)$ that

$$\begin{aligned} &\text{the left hand side of (6)} = P_\theta[r_1+\theta < Y < r_2+\theta] \\ (7) \qquad &= P_\theta[F(r_1+\theta) < W < F(r_2+\theta)] = 1-\alpha. \end{aligned}$$

Hence, we want to minimize $r_2-r_1$ under the condition (7). To do so we use Lagrange's method. Let $\lambda$ be a real number and define

$$(8)\qquad L\doteq L(r_1,r_2;\lambda)=r_2-r_1-\lambda\{\int_{F(r_1+\theta)}^{F(r_2+\theta)} h_W(w)\,dw-1+\alpha\}$$

where $h_W(w)$ is the density of W given by

$$(9)\qquad h_W(w)=kw^m(1-w)^m,\qquad \text{for } 0<w<1$$

where k is given by (5). The right hand side of (9) is the probability density function (p.d.f.) of Beta distribution Beta(m+1,m+1) with (m+1,m+1) degrees of freedom. Then, by Lagrange's method we have that

$$(10)\qquad \begin{cases} \partial L/\partial r_1 = -1+\lambda h_W(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=0 \\ \\ \partial L/\partial r_2 = 1-\lambda h_W(F(r_2+\theta))f(r_2+\theta\,|\,\theta)=0 \end{cases}$$

By (10) we get

$$(11)\qquad h_W(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=h_W(F(r_2+\theta))f(r_2+\theta\,|\,\theta)(=\lambda^{-1}),\qquad \forall\theta.$$

Taking

$$(12)\qquad F(r_1+\theta)=\beta(\alpha/2)\quad\text{and}\quad F(r_2+\theta)=1-\beta(\alpha/2)$$

where $\beta(\alpha/2)$ is given by

$$(13)\qquad \int_0^{\beta(\alpha/2)} h_W(w)\,dw=\alpha/2$$

we obtain by (3) that $r_1=-r_2\doteq-r$ where

$$(14)\qquad r=F^{-1}(1-\beta(\alpha/2))-\theta=\tan[(2^{-1}-\beta(\alpha/2))\pi].$$

We also have that $h_W(F(-r+\theta))=h_W(F(r+\theta))$ and $f(-r+\theta\,|\,\theta)=f(r+\theta\,|\,\theta)$ with r given by (14). Thus, (11) and (7) are satisfied for $r_1=-r_2=-r$ with r given by (14). Therefore, the shortest C. I. for $\theta$ at confidence coefficient $1-\alpha$ is given by

(15) $$(Y-r,\ Y+r)\doteq(Y-\tan[(2^{-1}-\beta(\alpha/2))\pi],\ Y+\tan[(2^{-1}-\beta(\alpha/2))\pi]).$$

Let $n=2m$. This time we estimate $\theta$ by $Y\doteq X_{(m)}$. In the similar way to the above we get the density of Y as follows:

(16) $$g_Y(y\,|\,\theta)=k_1(F(y))^{m-1}(1-F(y))^m f(y\,|\,\theta),\quad \text{for } -\infty<y<\infty$$

where

(17) $$k_1\doteq\Gamma(2m+1)/\{\Gamma(m)\Gamma(m+1)\}.$$

Putting $W\doteq F(Y)$ we minimize $r_2-r_1$ under the condition (7). However, since the density of W is now of form

(18) $$h_1(w)=k_1w^{m-1}(1-w)^m,\quad \text{for } 0<w<1$$

which is the p. d. f. of the Beta(m, m+1) distribution with $k_1$ defined by (17), it is difficult to get exact values for $F(r_i+\theta)$, i=1, 2 which satisfy

(19) $$h_1(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=h_1(F(r_2+\theta))f(r_2+\theta\,|\,\theta).$$

Hence, we use conventional values for $F(r_i+\theta)$, i=1, 2. Those are

(20) $$F(r_1+\theta)=\beta_{m,\,m+1}(\alpha/2) \text{ and } F(r_2+\theta)=1-\beta_{m+1,\,m}(\alpha/2)$$

where $\beta_{m,\,m+1}(\alpha/2)$ and $\beta_{m+1,\,m}(\alpha/2)$ are respectively determined by

(21) $$\int_0^{\beta_{m,\,m+1}(\alpha/2)} h_1(w)\,dw=\alpha/2=\int_0^{\beta_{m+1,\,m}(\alpha/2)} k_1w^m(1-w)^{m-1}\,dw.$$

Thus, by (3) $r_1$ and $r_2$ are respectively given by

$$(22)\qquad \begin{cases} r_1=F^{-1}(\beta_{m,m+1}(\alpha/2))-\theta = -\tan[(2^{-1}-\beta_{m,m+1}(\alpha/2))\pi], \\ \\ r_2=F^{-1}(1-\beta_{m+1,m}(\alpha/2))-\theta = \tan[(2^{-1}-\beta_{m+1,m}(\alpha/2))\pi]. \end{cases}$$

Therefore, the C. I. for $\theta$ at confidence coefficient $1-\alpha$ is

$$(23)\qquad (Y-r_2, Y-r_1) \doteq (Y-\tan[(2^{-1}-\beta_{m+1,m}(\alpha/2))\pi],\ Y+\tan[(2^{-1}-\beta_{m,m+1}(\alpha/2))\pi]).$$

In the next section we check if the tests with the acceptance regions derived from inverting the C. I.'s (15) for $n=2m+1$ and (23) for $n=2m$ are unbiased and of size $\alpha$.

## §3. Two-Sided Test for $\theta$.

In this section we consider the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant $\theta_0$. We propose the two-sided tests with the acceptance regions derived from inverting the (shortest) C. I.'s for $\theta_0$ obtained in Section 2. When $n=2m+1$, we show that our test is unbiased and of size $\alpha$. When $n=2m$, our test is not unbiased because of usage of conventional method for constructing the C. I. for $\theta$.

Let $n=2m+1$. As in Section 2 we define $Y \doteq X_{(m+1)}$. By inverting the shortest C. I. (15) for $\theta_0$ our test is to reject $H_0$ if $Y\in(-\infty, \theta_0-r]\cup[\theta_0+r, +\infty)$ and to accept $H_0$ if $Y\in(\theta_0-r, \theta_0+r)$ where r is given by (14). Now, we show that this test is unbiased and of size $\alpha$.

Let $y_1{}^0$ and $y_2{}^0$ be real numbers depending on $\theta_0$ such that $y_1{}^0<y_2{}^0$. Define $\psi(\theta)$ by

$$(24)\qquad \psi(\theta) \doteq P_\theta[Y<y_1{}^0 \ \text{ or } \ y_2{}^0<Y] = 1 - \int_{y_1{}^0}^{y_2{}^0} g_Y(y|\theta)\,dy$$

where $g_Y(y|\theta)$ is defined by (4).

To get unbiased size-$\alpha$ test with the acceptance region $(y_1{}^0, y_2{}^0)$ we choose $y_1{}^0$ and $y_2{}^0$ which satisfy

$$(25) \qquad \psi(\theta_0) = 1 - P_{\theta_0}[y_1{}^0 < Y < y_2{}^0] = \alpha$$

and minimize $\psi(\theta)$ at $\theta=\theta_0$; namely

$$(26) \qquad d\psi(\theta)/d\theta \Big|_{\theta=\theta_0} = g_Y(y_2{}^0|\theta_0) - g_Y(y_1{}^0|\theta_0) = 0.$$

We consider the test with the acceptance region $(\theta_0-r, \theta_0+r)$. Since from the construction the equality (11) with $r_1=-r$, $r_2=r$ and $\theta=\theta_0$ is satisfied, it follows from (4) and (9) that $g_Y(\theta_0-r|\theta_0)=g_Y(\theta_0+r|\theta_0)$; (26) is satisfied for $y_1{}^0$ and $y_2{}^0$ replaced by $\theta_0-r$ and $\theta_0+r$, respectively. (25) with $y_1{}^0$ and $y_2{}^0$ replaced by $\theta_0-r$ and $\theta_0+r$, respectively is the same as (6) except for $\theta$, $r_1$ and $r_2$ replaced by $\theta_0$, $-r$ and $r$, respectively. Therefore, our test with the acceptance region $(\theta_0-r, \theta_0+r)$ is unbiased and of size $\alpha$.

Let n=2m. As in Section 2 we define $Y \doteq X_{(m)}$. Again, by inverting the C. I. (23) for $\theta_0$ our test is to reject $H_0$ if $Y \in (-\infty, \theta_0+r_1] \cup [\theta_0+r_2, +\infty)$ and to accept $H_0$ if $Y \in (\theta_0+r_1, \theta_0+r_2)$ where $r_1$ and $r_2$ are given by (22). In this case our test depends on the conventional values for $F(r_i+\theta)$, $i=1,2$. Hence, we have that $g_Y(\theta_0+r_1|\theta_0) \neq g_Y(\theta_0+r_2|\theta_0)$. Furthermore, (25) with $y_1{}^0$ and $y_2{}^0$ replaced by $\theta_0+r_1$ and $\theta_0+r_2$, respectively is the same as (6) except for $\theta$ replaced by $\theta_0$. Thus, our test is of size $\alpha$, but is not unbiased. However, for large m our test becomes almost unbiased as the test in the case of n=2m+1 shows.

In the next two sections we deal with the scale parameter $\xi$. In Section 4 we obtain the C. I. for $\xi$ and in Section 5 we check if two-sided test with acceptance region derived from inverting the C. I. for $\xi_0$ is unbiased.

## §4. The Interval Estimation for $\xi$.

In this section we consider the density (1) with $\theta=0$;

$$(27) \qquad f(x|\xi) \doteq f(x|0,\xi) = \xi\pi^{-1}\{\xi^2+x^2\}^{-1}, \qquad \text{for } -\infty < x < \infty$$

provided that $\xi > 0$.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the population with density $f(x|\xi)$. Again, we first consider the case of $n=2m+1$ with m a nonnegative integer and secondly the case of $n=2m$. Putting $\xi^* \doteq \ln \xi$ we have

$$f(x|\xi)=\pi^{-1}e^{-\xi^*}\{1+e^{2(\ln|x|-\xi^*)}\}^{-1}, \quad \text{for } -\infty<x<\infty.$$

Thus, letting $Z \doteq \ln|X|$ and $Z_{(i)}$ be the i-th smallest observation of $Z_1, \ldots, Z_n$ we estimate $\xi^*$ by $Y \doteq Z_{(m+1)}$ when $n=2m+1$ and by $Y \doteq Z_{(m)}$ when $n=2m$, respectively. We find the C. I.'s for $\xi$ according to these estimates.

We beforehand derive the distribution of Z. Since $x=e^z$ for $x>0$; $x=-e^z$ for $x<0$; $z=-\infty$ for $x=0$, by a variable transformation $Z=\ln|X|$ the density of Z is obtained as follows:

$$q_Z(z) \doteq q_Z(z|\xi)=f(e^z|\xi)|de^z/dz| + f(-e^z|\xi)|d(-e^z)/dz|$$

$$(28) \qquad =2\pi^{-1}\frac{e^{z-\xi^*}}{1+e^{2(z-\xi^*)}}, \qquad -\infty<z<\infty$$

where $-\infty<\xi^*<\infty$. Since $q_Z(2\xi^*-z)=q_Z(z)$, $q_Z(z)$ is symmetric about $z=\xi^*$ and the unimodal function with the mode $\xi^*$.

Now, we let $n=2m+1$ until (37). We estimate $\xi^*$ by $Y \doteq Z_{(m+1)}$. Letting $Q_Z(z)$ be the c.d.f. of Z we obtain by (28) that

$$(29) \qquad Q_Z(z) \doteq Q_Z(z|\xi)= 2\pi^{-1}\tan^{-1}(e^{z-\xi^*}), \text{ for } -\infty<z<\infty.$$

The p.d.f. $g_Y(y|\xi)$ of Y is derived as follows:

$$(30) \qquad g_Y(y|\xi)=k(Q_Z(y))^m(1-Q_Z(y))^m q_Z(y), \quad \text{for } -\infty<y<\infty.$$

Let $\alpha$ be a real number such that $0<\alpha<1$. Let $r_1$ and $r_2$ be real numbers such that $0<r_1<r_2$. To find the C. I. for $\xi$ at confidence coefficient $1-\alpha$ we want to find $r_1$ and $r_2$ under the condition that

$$(31) \qquad P_\xi[r_1e^Y< \xi <r_2e^Y] = 1-\alpha.$$

But, it follows by a variable transformation $W=Q_Z(Y)$ that

$$\text{the left hand side of (31)}=P_\xi[-\ln r_2 < Y-\xi^* < -\ln r_1]$$

$$=P_\xi[Q_Z(\xi^*-\ln r_2) < W < Q_Z(\xi^*-\ln r_1)]=1-\alpha. \tag{32}$$

Hence, we want to find $r_1$ and $r_2$ which minimize $Q_Z(\xi^*-\ln r_1)-Q_Z(\xi^*-\ln r_2)$ under the condition (32). To do so we use Lagrange's method. Let $\lambda$ be a real number and define

$$L \doteq L(Q_Z(\xi^*-\ln r_1), Q_Z(\xi^*-\ln r_2);\lambda)$$

$$\doteq Q_Z(\xi^*-\ln r_1)-Q_Z(\xi^*-\ln r_2)-\lambda\left\{\int_{Q_Z(\xi^*-\ln r_2)}^{Q_Z(\xi^*-\ln r_1)} h_W(w)\,dw -1 +\alpha\right\} \tag{33}$$

where $h_W(w)$ is defined by (9). Then, by Lagrange's method we have that

$$\begin{cases} \partial L/\partial Q_Z(\xi^*-\ln r_1) = 1-\lambda\ h_W(Q_Z(\xi^*-\ln r_1)=0 \\ \\ \partial L/\partial Q_Z(\xi^*-\ln r_2) = -1+\lambda\ h_W(Q_Z(\xi^*-\ln r_2)=0 \end{cases} \tag{34}$$

By (34) we get

$$h_W(Q_Z(\xi^*-\ln r_1))=h_W(Q_Z(\xi^*-\ln r_2))\ (=\lambda^{-1}), \qquad \forall\xi. \tag{35}$$

Taking

$$Q_Z(\xi^*-\ln r_2)=\beta(\alpha/2) \quad \text{and} \quad Q_Z(\xi^*-\ln r_1)=1-\beta(\alpha/2)$$

where $\beta(\alpha/2)$ is given by (13), we obtain by (29) that

$$\begin{cases} r_1=[\tan\{2^{-1}\pi(1-\beta(\alpha/2))\}]^{-1} \\ \\ r_2=[\tan\{2^{-1}\pi\beta(\alpha/2)\}]^{-1} \end{cases} \tag{36}$$

and furthermore (35) and (32) are satisfied for $r_1$ and $r_2$ given by (36). Therefore, the C. I. for $\xi$ is given by

(37) $$(r_1e^Y, r_2e^Y)\doteq([\tan\{2^{-1}\pi(1-\beta(\alpha/2))\}]^{-1}e^Y,\ [\tan\{2^{-1}\pi\beta(\alpha/2)\}]^{-1}e^Y).$$

We now consider the case of n=2m. In this case we estimate $\xi^*$ by $Y\doteq Z_{(m)}$. Then, the p. d. f. of Y is given by

(38) $$g_Y(y|\xi)=k_1(Q_Z(y))^{m-1}(1-Q_Z(y))^m q_Z(y), \quad \text{for } -\infty<y<\infty$$

where $k_1$ is given by (17). To find the C. I. for $\xi$ at confidence coefficient $1-\alpha$ we want to find $r_1$ and $r_2$ with $0<r_1<r_2$ under the condition that

(39) $$P_\xi[r_1e^Y<\xi<r_2e^Y]=1-\alpha.$$

But, it follows by a variable transformation $W=Q_Z(Y)$ that

$$\text{the left hand side of (39)}=P_\xi[-\ln r_2<Y-\xi^*<-\ln r_1]$$

(40) $$=P_\xi[Q_Z(\xi^*-\ln r_2)< W < Q_Z(\xi^*-\ln r_1)]=1-\alpha.$$

Hence, we want to find $r_1$ and $r_2$ which minimize $Q_Z(\xi^*-\ln r_1)-Q_Z(\xi^*-\ln r_2)$ under the condition (40). Going through the similar process to (33) through (35), we get

(41) $$h_1(Q_Z(\xi^*-\ln r_1))=h_1(Q_Z(\xi^*-\ln r_2)) \ (=\lambda^{-1}), \quad \forall\xi$$

where $h_1(w)$ is the density of W given by (18). However, again it is difficult to get exact values of $Q_Z(\xi^*-\ln r_i)$, i=1, 2 which satisfy (41) (and furthermore $q_Z(\xi^*-\ln r_1)=q_Z(\xi^*-\ln r_2)$). Hence, we use conventional values for $Q_Z(\xi^*-\ln r_i)$, i=1, 2. Those are

(42) $$Q_Z(\xi^*-\ln r_2)=\beta_{m,\,m+1}(\alpha/2) \quad \text{and} \quad Q_Z(\xi^*-\ln r_1)=1-\beta_{m+1,\,m}(\alpha/2)$$

where $\beta_{m,m+1}(\alpha/2)$ and $\beta_{m+1,m}(\alpha/2)$ are respectively determined by (21). Thus, by (29) we obtain

$$(43)\qquad \begin{cases} r_1=[\tan\{2^{-1}\pi(1-\beta_{m+1,m}(\alpha/2))\}]^{-1}, \\ r_2=[\tan\{2^{-1}\pi\beta_{m,m+1}(\alpha/2)\}]^{-1}. \end{cases}$$

Therefore, the C. I. for $\xi$ is

$$(44)\qquad (r_1e^Y,\ r_2e^Y)$$

where $r_1$ and $r_2$ are given by (43).

## §5. Two-Sided Test for $\xi$.

In this section we consider the problem of testing the hypothesis $H_0:\xi=\xi_0$ versus the alternative hypothesis $H_1:\xi\neq\xi_0$ for some constant $\xi_0$. We propose the test with the acceptance region derived from inverting the C. I. for $\xi_0$. Let n be the size of the random sample $X_1, \ldots, X_n$. When n=2m+1 with m a nonnegative integer, we show that this test is unbiased and of size $\alpha$. When n=2m, our test is of size $\alpha$, but cannot be unbiased because we use the conventional device to determine the C. I. for $\xi$. However, it will be almost unbiased for large m.

Let n=2m+1. As in Section 4 we let $Z\doteq\ln|X|$ and $Z_{(i)}$ be the i-th smallest observation of $Z_1, \ldots, Z_n$. Let $\xi_0^*\doteq\ln\xi_0$ and define $Y\doteq Z_{(m+1)}$. By inverting the C. I. (37) for $\xi_0$ our test is to reject $H_0$ if $Y\in(-\infty,\xi_0^*-\ln r_2]\cup[\xi_0^*-\ln r_1,+\infty)$ and to accept $H_0$ if $Y\in(\xi_0^*-\ln r_2,\ \xi_0^*-\ln r_1)$ where $r_1$ and $r_2$ are given by (36). Now, we show that this test is unbiased and of size $\alpha$.

Let $y_1^0$ and $y_2^0$ be real numbers depending on $\xi_0$ such that $y_1^0<y_2^0$. Define $\psi(\xi)$ by

$$\psi(\xi)\doteq P_\xi[Y<y_1^0 \text{ or } y_2^0<Y]$$

$$(45)\qquad = 1-\int_{y_1^0}^{y_2^0} g_Y(y|\xi)\,dy$$

where $g_Y(y|\xi)$ is given by (30). To get unbiased size-$\alpha$ test with acceptance region $(y_1^0, y_2^0)$ we choose $y_1^0$ and $y_2^0$ which satisfy

$$(46) \qquad \psi(\xi_0) = 1 - P_{\xi_0}[y_1^0 < Y < y_2^0] = \alpha$$

and minimize $\psi(\xi)$ at $\xi=\xi_0$; namely

$$(47) \qquad d\psi(\xi)/d\xi \Big|_{\xi=\xi_0} = \xi_0^{-1} g_Y(y_2^0|\xi_0) - \xi_0^{-1} g_Y(y_1^0|\xi_0) = 0$$

Let $y_1^{\#} = \xi_0^* - \ln r_2$ and $y_2^{\#} = \xi_0^* - \ln r_1$. Then, since $q_Z(y_1^{\#}|\xi_0) = \pi^{-1}\sin\{\pi\beta(\alpha/2)\}$ $= \pi^{-1}\sin\{\pi(1-\beta(\alpha/2))\} = q_Z(y_2^{\#}|\xi_0)$, and since, from construction and (35), $h_W(Q_Z(y_1^{\#})) = h_W(Q_Z(y_2^{\#}))$, we obtain by (30) and (9) that $g_Y(y_1^{\#}|\xi_0) = g_Y(y_2^{\#}|\xi_0)$. Therefore, $(y_1^{\#}, y_2^{\#})$ satisfies (47). On the other hand, (46) with $y_1^0$ and $y_2^0$ replaced by $y_1^{\#}$ and $y_2^{\#}$, respectively is the same as (40) except for $\xi$ replaced by $\xi_0$. Therefore, our test with the acceptance region $(y_1^{\#}, y_2^{\#})$ is unbiased and of size $\alpha$.

Let $n=2m$. As in Section 4 we define $Y \doteq Z_{(m)}$. Again, by inverting the C. I. (44) for $\xi_0$ our test is to reject $H_0$ if $Y \in (-\infty, \xi_0^* - \ln r_2] \cup [\xi_0^* - \ln r_1, +\infty)$ and to accept $H_0$ if $Y \in (\xi_0^* - \ln r_2, \xi_0^* - \ln r_1)$ where $r_1$ and $r_2$ are determined by (43). In this case our test depends on the conventional values for $Q_Z(\xi^* - \ln r_i)$, $i=1,2$. So, we have $g_Y(\xi_0^* - \ln r_2|\xi_0) \neq g_Y(\xi_0^* - \ln r_1|\xi_0)$. Furthermore, (46) with $y_1^0$ and $y_2^0$ replaced by $\xi_0^* - \ln r_2$ and $\xi_0^* - \ln r_1$, respectively is the same as (40) except for $\xi$ replaced by $\xi_0$. Thus, our test is still of size-$\alpha$, but is not unbiased. However, for large m our test becomes almost unbiased as the test in case of $n=2m+1$ shows.