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Abstract

This paper analyzes the commodity tax competition between two neighbouring countries in a two-dimensional Euclidean market space within the framework of non-cooperative Nash game. Three main conclusions are derived. First, some principal findings in the one-dimensional model by Kanbur and Keen(1993) and Ohsawa(1999) are still sustainable in a two-dimensional space. Second, as either the small country becomes more convex or the border becomes longer, the two countries are subject to fiercer competitive pressure. Finally, when the big country surrounds the small one (resp. the small country surrounds the big one), an increase in the border length raises (resp. reduces) the amount of cross-border shoppers. Thus, we show that the two-dimensional configurations play a significant role in tax competition game.

Keywords: tax competition; two dimensions; Nash game; cross-border shopping

JEL Classification: H71; H73; R51

1. Introduction

There is a growing literature concerning commodity tax competition between countries. One of the motivation for the literature may arise from the current grope of European Union (EU) after the topic how to attain the single European market. In particular, the commodity tax differences between countries induces cross-border shoppers, who pay the value-added tax (VAT) of foreign countries rather than the VAT of their domestic countries. Since the amount of such cross-border shopping affects directly the government revenues, it is worth whole examining the commodity tax competition closely. To explore analytically the effects of such competition, Kanbur and Keen(1993), Trandel(1994), Nielsen(1998) and Ohsawa(1999) have formulated commodity tax competition model between countries in the one-dimensional space within the framework of non-cooperative Nash game. They clarified the impact of the spatial characteristics on the tax rates, the tax revenues, and the amount of the cross-border shoppers from one country to another one. The common underlying mechanism of these models is that when physical border control is eliminated, each country faces the problem as to how it should choose its commodity tax rate to maximize its tax revenue. If the country raises its tax rate relative to those in competing countries, part of its domestic customers would evade the tax by shopping abroad. Otherwise, its tax revenue from the domestic customers would be small.

Kanbur and Keen(1993) formulated a two-country model, where although the two countries have identical size, the density of customers differs. Then they analyzed how the difference in that density affects tax rates and revenues in equilibrium. They also considered the effect of cooperative tax policies such as the minimum standard rate and the tax equalization. Since a seminal work by Kanbur and Keen(1993), Nielsen(1998) extended their analysis by treating the two respects: the transportation cost of the commodity; border inspection. On the other hand, Ohsawa(1999) considered a multi-country model, where the density of customers is fixed at unity over the whole market, but country sizes may differ. He explored how country size and country location produce an effect on tax rates and revenues in equilibrium. Although these three papers assume that the firms are continuously distributed, Trandel(1994) set up a two-county model where each country has just one firm and its location is fixed. He also assumed that the density of customers is constantly increasing over the linear market. He demonstrated how Nash equilibrium is affected by whether welfare-maximizing taxes or revenue-maximizing taxes is employed. He also showed how Nash equilibrium is affected by whether these two firms make use of marginal-cost or profit-maximizing pricing.

One of limitations of these previous analyses is the assumption of a linear market, though countries in the real world are located in the two-dimensional space. In fact, to the best of my knowledge, surprisingly analytical approaches on the tax competition in two-dimensional space have received remarkably little attention form researchers. This paper is an attempt to overcome this shortcoming. The aim of this paper is to set up a Nash tax competition game between two neighburing countries in the two-dimensional space, and to analyze how the two-dimensional market space affects Nash equilibrium. In particular, we focus on the impacts on the tax difference between these two countries in equilibrium, which may be regarded as the criterion to measure the intensity of the competition, and the amount of the cross-border shoppers, which may be considered as the criterion to measure the tax evasion due to shopping abroad. The problem of obtaining Nash equilibrium increases when one turns form the one-dimensional space to the two-dimensional space. The essential difference between the one-dimensional and the two-dimensional spaces is the shape of the countries such as the length and curvature of their boundaries. In order to examine the effects of such spatial characteristics, we employ the simplest model.

The remainder of this paper is organized as follows. In the next section, a Nash game in two-dimensional space is formulated, and Nash equilibrium is characterized. Section 3 discusses how the spatial configurations affect the tax differences and the amount of cross-border shoppers in detail. The final section contains our conclusions and suggests direction for future research. All proofs are collected in the Appendix for convenience.

2. Model

The setup in this paper is similar to that of Ohsawa(1999). Consider a study area Ω composed of big and small countries K_B, K_S, in a spatially homogeneous two-dimensional space. Let S_B and S_S be their areas such that $0 < S_S \leq S_B$. Assume that the border between these two countries has a finite length, and let L denote its length. Customers and firms are continuously distributed over Ω at a uniform density of one. All firms produce a homogeneous commodity at zero marginal production cost. These two countries levy their commodity tax rates, denoted by p_B and p_S respectively, on the firms located within the corresponding country. Each customer has a completely inelastic demand for one unit of the commodity from firms, and buys from the firm for which the tax rate plus the travel costs is smallest.

The travel cost equals to $\gamma(>0)$ per unit Euclidean distance. Assuming the firms to be non-cooperative, each would price at its tax-inclusive marginal cost, that is to say, all firms in the big (resp. small) country would charge the same and constant mill prices, p_B (resp. p_S). Thus, the current model differs from Kanbur and Keen(1993), and Ohsawa(1999) in the dimension of the study area.

It follows from our assumptions that the amounts of the customers who buy from a firm within big and small countries, denoted as $D_B(p_B,p_S)$ and $D_S(p_B,p_S)$, correspond to the area of the markets of the big and small countries. The *revenue* of a country is defined here as the amount of the taxes which it levies, referred to as $\Pi_B(p_B,p_S)$ and $\Pi_S(p_B,p_S)$, respectively. Hence, $\Pi_B(p_B,p_S)=p_BD_B(p_B,p_S)$ and $\Pi_S(p_B,p_S)=p_SD_S(p_B,p_S)$. In this paper, these two countries play a Nash game in tax rates to maximize their revenues. As usual, an asterisk $*$ denotes evaluation in equilibrium. Thus, $p^*{}_B$ and $p^*{}_S$ are in equilibrium if and only if $p_B^*>0, p_S^*>0$, $\Pi_B(p_B^*,p_S^*)\geq \Pi_B(p_B,p_S^*)$ for all $p_B>0$, and $\Pi_S(p_B^*,p_S^*)\geq \Pi_S(p_B^*,p_S)$ for all $p_S>0$. Let $C(p_B,p_S)$ denote the amount of tax-induced cross-border shoppers from the big country to the small one. Accordingly, we have

$$C(p_B,p_S)=D_S(p_B,p_S)-S_S=S_B-D_B(p_B,p_S). \tag{1}$$

Hence, if $p_B<p_S$, then $C(p_B,p_S)$ is negative. Thus, if $p_B>p_S$ (resp. $p_B<p_S$), $C(p_B,p_S)$ coincides with the area of the extended market, which is located just outside the periphery of the small (resp. big) country. Let $B(p_B,p_S)$ be the length of the boundary between two market areas.

Lemma 1 *If an equilibrium exists, then*

$$B(p_B^*,p_S^*)p_B^*=\gamma D_B(p_B^*,p_S^*), \qquad B(p_B^*,p_S^*)p_S^*=\gamma D_S(p_B^*,p_S^*). \tag{2}$$

Note that for general shape of two countries, more than one equilibria may exist.

Proposition 1 *If an equilibrium exists, then*

$$p_S^*\leq p_B^*, \qquad \Pi_S(p_B^*,p_S^*)\leq \Pi_B(p_B^*,p_S^*), \qquad \frac{\Pi_S(p_B^*,p_S^*)}{S_S}\geq\frac{\Pi_B(p_B^*,p_S^*)}{S_B}. \tag{3}$$

Each equality holds if and only if $S_B=S_S$.

Some of main conclusions to be drawn by Kanbur and Keen(1993), Ohsawa(1998) are that 1) the smaller country sets a lower equilibrium tax rate than the bigger one, 2) the revenue of small country is less than that of big one, and 3) per capita revenue is larger in the smaller country. Accordingly, it can be concluded that although we move from a one-dimensional market to a two-dimensional market, their conclusions are still true. The intuition for the first result is simple, as pointed out in Ohsawa(1999). The big country gets most revenue from its domestic market, so it would set its rate in this purpose. On the other hand, the small country considers its domestic market as insignificant, so it would set lower rate to encroach on the foreign market.

In view of (1), (2) can be rewritten as follows:

$$
\begin{aligned}
B(p_B^*, p_S^*)(p_B^* + p_S^*) &= \gamma S, && (4)\\
B(p_B^*, p_S^*)(p_B^* - p_S^*) &= \gamma\left(S_B - S_S - 2C(p_B^*, p_S^*)\right). && (5)
\end{aligned}
$$

The revenue difference $\Pi_B(p_B^*, p_S^*) - \Pi_S(p_B^*, p_S^*)$, the revenue ratio $\frac{\Pi_B(p_B^*, p_S^*)}{\Pi_S(p_B^*, p_S^*)}$ and the rate ratio $\frac{p_B^*}{p_S^*}$, which may be regarded as another criteria to evaluate the tax competition, can be analytically expressed either in $p_B^* - p_S^*$ or in $C(p_B^*, p_S^*)$ as follows:

$$
\begin{aligned}
\Pi_B(p_B^*, p_S^*) - \Pi_S(p_B^*, p_S^*) &= S(p_B^* - p_S^*), && (6)\\
\frac{\Pi_B(p_B^*, p_S^*)}{\Pi_S(p_B^*, p_S^*)} &= \left(\frac{p_B^*}{p_S^*}\right)^2 = \left(\frac{S_B - C(p_B^*, p_S^*)}{S_S + C(p_B^*, p_S^*)}\right)^2. && (7)
\end{aligned}
$$

The derivation of these formulas is given in Appendix. The formula (6) states that their revenue difference is directly proportional to their tax one. The formula (7) means that their revenue ratio is equated with the duplicate their tax one. Since $p_B^* \geq p_S^*$, the revenue ratio is equal to or greater than their rate ratio. It also follows from (7) that both the revenue and rate ratios are decreasing in $C(p_B^*, p_S^*)$.

3. Characterization

3.1. Spatial Configurations and Nash Equilibrium

So long as the general spatial configurations are maintained, no unambiguous conclusions can be derived as to how the two-dimensional characteristics affect on the tax competition. Therefore, our analysis is confined to the following three types of spatial configurations:

(Case 1) the whole market Ω, big and small countries K_B, K_S are rectangles;

(Case 2) the small country consists of m pairwise disjoint convex regions such that the big country surround these regions;

(Case 3) the big country consists of n pairwise disjoint convex regions such that these regions are surrounded by a small country.

Each case is illustrated in Figure 1, where each small country indicated as the shaded region has the same area. Note that the first case is essentially identical with the one-dimensional model by Ohsawa(1999). The second (resp. third) case may correspond to an imaginary situation where the EU with a common tax rate competes against Switzerland (resp. European non-member countries of the EU).

If $p_S < p_B$, then the small country encroaches on the big country within distance from the border of $(p_B - p_S)/\gamma$. The bold curves and hatched regions in Figure 1, do indeed give the boundaries between the two market areas and the extended market areas, respectively. In general, for any p_B and any p_S, it will be very complicated to get the exact expressions of the length of that boundary and the area of that extended market. Consider that p_B and p_S satisfy the following two conditions; (c1) for the second case, the market boundary consists of just m closed simple curves; (c2) for the third case, $(p_B - p_S)/\gamma$ is less than the minimum of the radius of curvature of the border. The first (resp. second) condition is equivalent to D (resp. R) $\geq (p_B - p_S)/\gamma$, where D is the minimum of a) half of the distances between any two inside regions; or b) the distances between an inside region and the boundary of the study area. R is the minimum of the radius of curvature of the border. Then, for an arbitrary integer I, $C(p_B, p_S)$ and $B(p_B, p_S)$ can be expressed analytically as follows: see Santaló(1976).

$$C(p_B, p_S) = L\frac{p_B - p_S}{\gamma} + \pi I \left(\frac{p_B - p_S}{\gamma}\right)^2, \tag{8}$$

$$B(p_B, p_S) = L + 2\pi I \frac{p_B - p_S}{\gamma}, \tag{9}$$

where

$$I = \begin{cases} 0, & \text{for the first case;} \\ m, & \text{for the second case;} \\ -n, & \text{for the third case.} \end{cases}$$

Thus, for the first case, these expressions have just one term, and that $C(p_B, p_S)$ is proportional to $p_B - p_S$, and $B(p_B, p_S)$ is constant. The important point to note is that for the second (resp. third) case, $C(p_B, p_S)$ increases more (resp. less) than proportionately with $p_B - p_S$, and that $B(p_B, p_S)$ is direct (resp. inverse) proportion to $p_B - p_S$. For given L, S_B and S_S, the greater m (resp. n) is, the more convex (resp. concave) the frontier of the small country is. Therefore we may regard I as a measure of the convexity (or concavity) of the frontier of the small country. So, we call I the *frontier index.*

It follows from the first inequality in (3) that if an equilibrium exists, then $p_S^* \leq p_B^*$. Hence, if p_B^* and p_S^* fulfill the conditions (c1) and (c2), then substituting $C(p_B^*, p_S^*)$ in (8) and $B(p_B^*, p_S^*)$ in (9) into (4) and (5), one arrives at the following system of simultaneous quadratic equations:

$$\left(L + 2\pi I \frac{p_B^* - p_S^*}{\gamma}\right) \frac{p_B^* + p_S^*}{\gamma} = S, \tag{10}$$

$$\left(3L + 4\pi I \frac{p_B^* - p_S^*}{\gamma}\right) \frac{p_B^* - p_S^*}{\gamma} = S_B - S_S. \tag{11}$$

Only the solutions to this system are candidates for equilibrium. The existence and uniqueness of equilibrium is addressed by the following.

Lemma 2 *For the first case, a unique Nash equilibrium always exists. For the second (resp. third) case, if D (resp. R) $> \frac{1}{8\pi I}\left(-3L + \sqrt{9L^2 + 16\pi I(S_B - S_S)}\right)$, then a unique Nash equilibrium exists.*

Note that each inequality in this lemma ensures that the conditions (c1) and (c2) hold. Since the explicit expressions for p_B^*, p_S^* and $C(p_B^*, p_S^*)$ with respect to L, S_S, S_B, γ and I are rather messy, we concentrate on the sensitivity analysis of the convexity of the small country I and the border length L on $p_B^* - p_S^*$ and $C(p_B^*, p_S^*)$ instead.

3.2. Effects of Convexity of Inside Country

We shall evaluate how the frontier index I has an influence on the tax difference $p_B^* - p_S^*$ and the amount of cross-border shoppers $C(p_B^*, p_S^*)$, *ceteris paribus.*

Proposition 2 *If a Nash equilibrium exists, both $p_B^* - p_S^*$ and $C(p_B^*, p_S^*)$ are decreasing in the frontier index I.*

This proposition indicates that increasing I reduces $p_B^* - p_S^*$, i.e., the intensity of competition increases. Thus, we see that increasing the frontier index I eliminates more geographical trade barrier between the two countries. This is just what we expect. It follows from (8) that for given p_B, p_S, γ and L, so $C(p_B, p_S)$ rises with I, i.e., the market of the small country expands with I. Hence, we can recognize that as I is increased, the small country would be able to more effectively encroach on the big one, that is to say, the small country has more incentive to employ an aggressive policy by cutting its tax rate. Thus, as I is increased, these two countries are subject to stronger competition pressure.

On the other hand, the intuition for the effect of the frontier index I on $C(p_B^*, p_S^*)$ is not simple. As we have just pointed out, increasing I reduces $p_B^* - p_S^*$. On the other hand, it follows from (8) that for given p_B and p_S, $C(p_B, p_S)$ is increasing in I. However, this proposition ensures that $C(p_B^*, p_S^*)$ is always decreasing in I.

3.3. Effects of Border Length

We shall evaluate how the border length L affects $p_B^* - p_S^*$ and $C(p_B^*, p_S^*)$, *ceteris paribus.* Note that a closed curve of a given length which encloses the greatest area is a circle. This is known as the isoperimetric problem: see Santaló(1976). Therefore, as L becomes shorter, each convex region of the inside county approaches a circle having the identical area.

Proposition 3 *If a Nash equilibrium exists, then (a) $p_B^* - p_S^*$ is decreasing in the border length L, (b) for the first (resp. second, third) case $C(p_B^*, p_S^*)$ is independent of (resp. increasing in, decreasing in) the border length L.*

Part (a) states that, for any case, lengthening L decreases the tax difference, and it leads to intense competition. Therefore, we see that lengthening the border length L eliminates more geographical trade barrier. Part (b) implies that the effect, however, on the amount of cross-border shoppers $C(p_B^*, p_S^*)$ depends on spatial configurations. For the first case, the amount of the cross-border shoppers is independent on the change in L; for the second case, it rises with L; for the third case, it falls with L. This does not seem to be in accord with intuition. We plot in Figures 2 and 3, $p_B^* - p_S^*$ and $C(p_B^*, p_S^*)$ versus the border length L, where $\gamma = 1$, $S_B = 3/4$ and $S_S = 1/4$. Note that when $L = \sqrt{\pi}$, then K_S is a circle with its radius $1/(2\sqrt{\pi})$. Three different functions are plotted, corresponding to the three values of I $(I = -1, 0, 1)$.

Let us offer intuition for the effect of L on $p_B^* - p_S^*$, which may be explained from the two viewpoints; the magnitude of the firms near the border; geometrical configuration of country shape. Let us discuss from the first point of view. Lengthening L also leads to the greater the firms in the vicinity of the border. As a result, there is more competition, so it would reduce $p_B^* - p_S^*$. Let us consider from the second point of view. As L becomes longer, the inside country becomes more oblong. This means that both the second and third cases approach the first case in the sense of the country shape. This together with Proposition 2 implies that for the second (resp. third) case, lengthening L would raise (resp. reduce) $p_B^* - p_S^*$. Based on these two viewpoints, for the second case, i.e, $I > 0$, the impact on $p_B^* - p_S^*$ is ambiguous. However, this proposition guarantees that lengthening L always reduces $p_B^* - p_S^*$.

The intuitive explanation of the impact of L on $C(p_B^*, p_S^*)$ for the first case, which will

Table 1: Effects of the border length L on tax rates and revenues

	$p_B^* - p_S^*$	$\frac{p_B^*}{p_S^*}$	$\Pi_B(p_B^*, p_S^*) - \Pi_S(p_B^*, p_S^*)$	$\frac{\Pi_B(p_B^*, p_S^*)}{\Pi_S(p_B^*, p_S^*)}$	$C(p_B^*, p_S^*)$
case 1 ($I = 0$)	$-$	0	$-$	0	0
case 2 ($I > 0$)	$-$	$-$	$-$	$-$	$+$
case 3 ($I < 0$)	$-$	$+$	$-$	$+$	$-$

be the benchmark for the other two cases, is as follows. From the first point of view, if L is short, the small country encroaches on the depth of the big one, as we have seen. Otherwise, the small one encroaches on the big one only in the vicinity of the border. Thus, the change in L, i.e., the length of such encroachment is completely offset by the change in the depth of the encroachment, so the amount of cross-border shoppers in total, which is $L(p_B^* - p_S^*)/\gamma$ from (8), become to always be the same. The effects on $C(p_B^*, p_S^*)$ for the second and third cases are interpreted, by way of the benchmark recorded, as follows. From the second point of view, Proposition 2 states that for the second (resp. third) case, lengthening L would raise (resp. reduce) $C(p_B^*, p_S^*)$. Combining this together with the benchmark yields that for the second (resp. third) case, $C(p_B^*, p_S^*)$ would rise (resp. fall) with L.

4. Conclusions

In this paper, we have assumed two dimensions in order to highlight how two-dimensional spatial configurations such as the length and curvature of the border affect tax competition. Although our theoretical analysis is based on the simplest situation, the following three findings seem to be helpful for designing the European Union cooperative tax policy. First, this paper confirms that some principal results in the one-dimensional model by Kanbur and Keen(1993) and Ohsawa(1999) remain valid even in the two-dimensional space. This is very general in the sense that it is independent of the shape of the two countries. The second finding of this paper is that the more convex the small country and the longer the border, the smaller the tax difference. Therefore, the two countries are subject to fiercer competitive pressure. The final finding is that the impact of the border length on $p_B^* - p_S^*$ (i.e., the extent of tax competition) is not necessarily consistent with that on $C(p_B^*, p_S^*)$ (i.e., the amount of tax evasion due to shopping abroad), as shown in Table 1. The signes, which are constructed by combining the results of Propositions 2 and 3 with (6) and (7), show the increase or decrease. The last two findings reflect the two-dimensional market, and they can not be explained by any one-dimensional model. This point deserves explicit emphasis. Note also that these two findings are general in the sense that they are independent of the position and shape of the inside convex country.

In this paper, the frontier index I has been restricted to integers. However, we should not overlook that Lemma 2 and Propositions 2 and 3 may still hold for every real number I. It is easy to check that the four spatial configurations in Figure 4 fulfill both (8) and (9). Therefore, these results can be directly applicable to such spatial configurations.

The model presented in this paper has wide application. For example, this geographical model seems to be applied to problems in product differentiation: see Anderson et al.(1992). Further extension to more general spatial configurations for multi-country will be the subject of a future paper. Although a slight increase in county shape complexity generates an intractable increase in mathematical complexity, the recent results to construct the Minkowski sum of polygons in Computational Geometry seem to be helpful: see de Berg et al.(1997).

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Appendix

Proof of Lemma 1

Now suppose we reduce p_S^* by sufficiently small amount $\epsilon > 0$. This extends the market of the small country by $\epsilon B(p_B^*, p_S^*)/\gamma$, and thus, increases its revenue from the newly extended market by $\epsilon p_S^* B(p_B^*, p_S^*)/\gamma$. On the other hand, this reduction simultaneously decrease its revenue from the former market by $\epsilon D_S(p_B^*, p_S^*)$. Evidently, in equilibrium that gain has to be equate with that loss, i.e., $\epsilon p_S^* B(p_B^*, p_S^*)/\gamma = \epsilon D_S(p_B^*, p_S^*)$. This establishes the first equation in (2). Similarly, the second equation in (2) can be established. □

Proof of Proposition 1

At equilibrium, one country does not undercut another one, because the undercut country can get positive revenue by setting its tax rate with its competitor. This indicates $B(p_B^*, p_S^*) > 0$ Suppose now that $p_S^* > p_B^*$. This together with the first inequality in (2) yields

$$p_S^* = \frac{\gamma D_S(p_B^*, p_S^*)}{B(p_B^*, p_S^*)} < \frac{\gamma S_S}{B(p_B^*, p_S^*)} \leq \frac{\gamma S_B}{B(p_B^*, p_S^*)} < \frac{\gamma D_B(p_B^*, p_S^*)}{B(p_B^*, p_S^*)} = p_B^*,$$

which contradicts $p_S^* > p_B^*$.

It follows from (2) that $B(p_B^*, p_S^*)(p_B^* - p_S^*) = \gamma(D_B(p_B^*, p_S^*) - D_S(p_B^*, p_S^*))$. This together with the first inequality in (3) yields that $D_B(p_B^*, p_S^*) \geq D_S(p_B^*, p_S^*)$. Combining this inequality again with the first inequality yields the second inequality in (3).

Since p_S^* is the best reply against p_B^*, $\Pi_S(p_B^*, p_S^*) = p_S^* D_S(p_B^*, p_S^*) > p_B^* S_S$. Also, it follows from the first inequality that $\Pi_B(p_B^*, p_S^*) = p_B^* D_B(p_B^*, p_S^*) < p_B^* S_B$. Combining these result yields $\frac{\Pi_S(p_B^*,p_S^*)}{S_S} > p_B^* > \frac{\Pi_B(p_B^*,p_S^*)}{S_B}$, so we have the last inequality in (3). □

Derivation of Formulas (6) and (7)

It follows from (2) that

$$\gamma\Pi_B(p_B^*,p_S^*) = B(p_B^*,p_S^*)(p_B^*)^2, \qquad \gamma\Pi_S(p_B^*,p_S^*) = B(p_B^*,p_S^*)(p_S^*)^2. \tag{12}$$

Subtracting the first equation in (12) by the second one in (12), while using (4), yields the formula (6). Dividing the first equation in (12) by the second one in (12) establishes the first equality in the formula (7). Dividing the first equation in (2) by the second one in (2), while taking account of (1), gives the second equality in the formula (7). □

Proof of Lemma 2

The proof proceeds in two stages. First, we shall prove that if an equilibrium exists, it is unique. If $I = 0$, substituting $I = 0$ into (11) yields $p_B^* - p_S^* = \frac{\gamma}{3L}(S_B - S_S)$. Otherwise, solving the quadratic equation for $p_B^* - p_S^*$ in (11) yields the following two roots:

$$p_B^* - p_S^* = \frac{\gamma}{8\pi I}\left(-3L \pm \sqrt{9L^2 + 16\pi I(S_B - S_S)}\right).$$

However, the minus sign preceding the square root must be disregarded, because, for $I > 0$, the root leads to $p_B^* < p_S^*$, which yields a contradiction. For $I < 0$, substituting the root into (9), while taking account of the isoperimetric inequality: see Santaló(1976), gives $B(p_B^*,p_S^*) = L - (3L + \sqrt{9L^2 + 16\pi I(S_B - S_S)})/4 \le (L - \sqrt{9 + 4I}L)/4$. Therefore, $B(p_B^*,p_S^*)$ is negative or cannot be defined. This gives a contradiction. Thus, for arbitrary integer I,

$$p_B^* - p_S^* = \frac{\gamma}{8\pi I}\left(-3L + \sqrt{9L^2 + 16\pi I(S_B - S_S)}\right). \tag{13}$$

Both this and the equation obtained by substituting this into (10) are linear in p_B^* and p_S^*, so p_B^* and p_S^* uniquely exists. On the other hand, it follows from (13) that D (resp. R) $> (p_B^* - p_S^*)/\gamma$ is identical with D (resp. R) $> \left(-3L + \sqrt{9L^2 + 16\pi I(S_B - S_S)}\right)/(8\pi I)$.

Next, in order to prove that p_S^* and p_B^* which fulfill (10) and (13) are indeed in equilibrium, it suffices from Lemma 1 to prove that 1) p_B^* and p_S^* are relative maxima of $\Pi_B(p_B,p_S^*)$ and $\Pi_S(p_B^*,p_S)$, respectively, 2) the small country cannot obtain higher revenue by violating the conditions either (c1) or (c2). In the neighbourhood of p_S^*, employing (1) and (8) yields

$$\Pi_S(p_B^*,p_S) = p_S D_S(p_B^*,p_S)) = p_S(S_S + C(p_B^*,p_S)) = p_S(S_S + \frac{L}{\gamma}(p_B^* - p_S) + \frac{\pi I}{\gamma^2}(p_B^* - p_S)^2).$$

Taking the second derivative of $\Pi_S(p_B^*, p_S)$ with respect to p_S and evaluating this derivative at p_S^* yields the expression

$$\frac{\mathrm{d}^2\Pi_S(p_B^*, p_S)}{\mathrm{d}p_S^2}|_{p_S=p_S^*} = -\frac{1}{4\gamma}(5L + \sqrt{9L^2 + 16\pi I(S_B - S_S)}) < 0.$$

This guarantees that p_S^* is relative maximum. Also, note that for a given p_B^*, if p_S violates the conditions either (c1) or (c2), then $C(p_B^*, p_S)$ is below $L\frac{p_B^*-p_S}{\gamma} + \pi I\left(\frac{p_B^*-p_S}{\gamma}\right)^2$, so $D_S(p_B^*, p_S)$ is below $S_S + L\frac{p_B^*-p_S}{\gamma} + \pi I\left(\frac{p_B^*-p_S}{\gamma}\right)^2$. This indicates that $\Pi_S(p_B^*, p_S)$ can not exceed the revenue at the tax rate which maximizes $p_S(S_S + L\frac{p_B^*-p_S}{\gamma} + \pi I\left(\frac{p_B^*-p_S}{\gamma}\right)^2)$, i.e., $\Pi_S(p_B^*, p_S^*)$, so the small country cannot raise its revenue by violating one of these conditions. □

Proof of Proposition 2

Taking the partial derivative of $p_B^* - p_S^*$ in (13) with respect to I gives

$$\frac{\partial(p_B^* - p_S^*)}{\partial I} = \frac{\gamma}{8\pi}\left(\frac{-9L^2 - 8\pi I(S_B - S_S) + 3L\sqrt{9L^2 + 16\pi I(S_B - S_S)}}{I^2\sqrt{9L^2 + 16\pi I(S_B - S_S)}}\right) < 0. \tag{14}$$

The insertion of (11) into (8) gives

$$C(p_B^*, p_S^*) = \frac{1}{4\gamma}\left(L(p_B^* - p_S^*) + \gamma(S_B - S_S)\right). \tag{15}$$

This together with (14) implies that $C(p_B^*, p_S^*)$ is decreasing in I, as required. □

Proof of Proposition 3

Taking the partial derivative of $p_B^* - p_S^*$ in (13) with respect to L gives

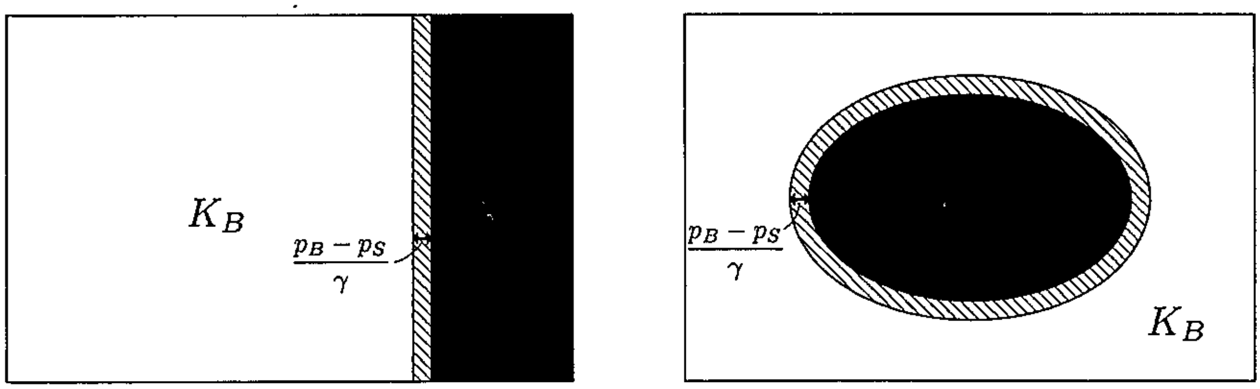
$$\frac{\partial(p_B^* - p_S^*)}{\partial L} = \frac{3\gamma}{8\pi I}\left(-1 + \frac{3L}{\sqrt{9L^2 + 16\pi I(S_B - S_S)}}\right) < 0, \tag{16}$$

completing the proof of part (a).

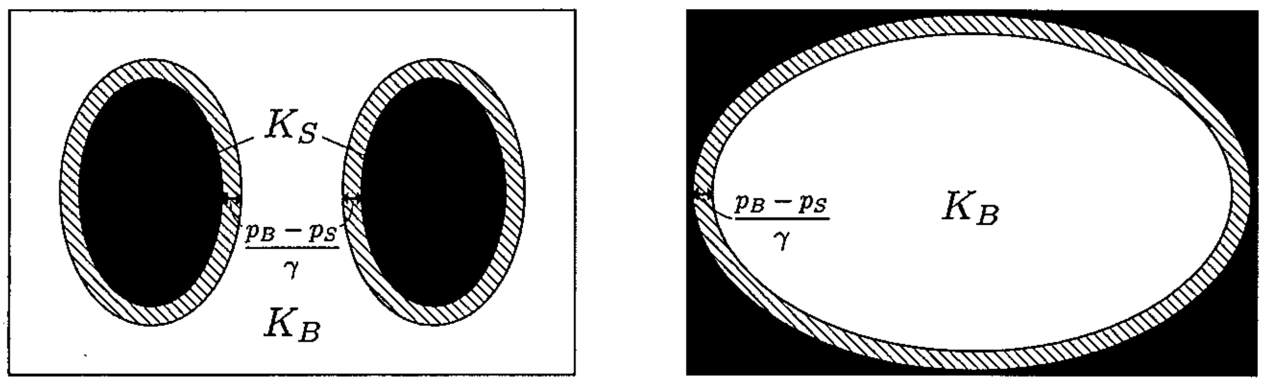
Taking the partial derivative of $C(p_B^*, p_S^*)$ in (15) with respect to L, while using (13) and (16), yields

$$\begin{aligned}\frac{\partial C(p_B^*, p_S^*)}{\partial L} &= \frac{1}{4\gamma}\left((p_B^* - p_S^*) + L\frac{\partial(p_B^* - p_S^*)}{\partial L}\right)\\ &= \frac{3}{16\pi I}\left(-L + \frac{3L^2 + (8/3)\pi I(S_B - S_S)}{\sqrt{9L^2 + 16\pi I(S_B - S_S)}}\right)\begin{cases} = 0, & \text{for } I = 0;\\ > 0, & \text{for } I > 0;\\ < 0, & \text{for } I < 0,\end{cases}\end{aligned}$$

which comeletes the proof of part (b). □



(1) First Case (i.e., $I = 0$) (2) Second Case for $m = 1$ (i.e., $I = 1$)



(3) Second Case for $m = 2$ (i.e., $I = 2$) (4) Third Case for $n = 1$ (i.e., $I = -1$)

Figure 1: Spatial Configurations and Market Areas

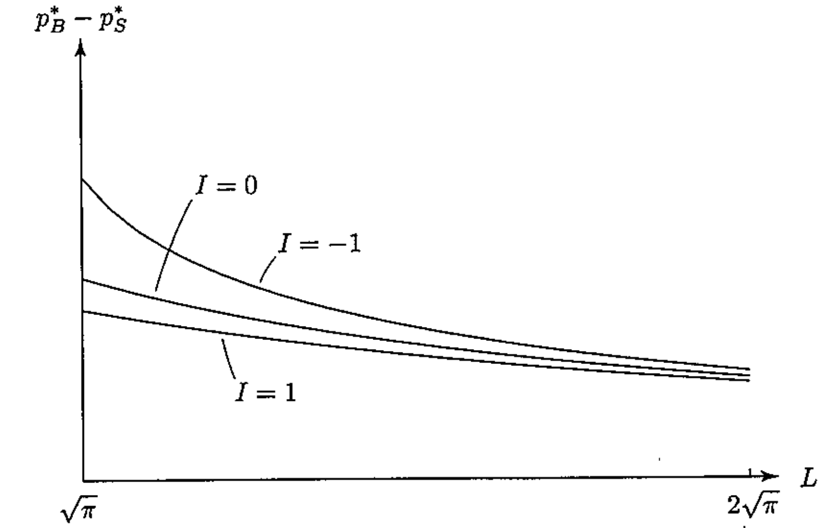


Figure 2: Tax Differences with respect to Border Length

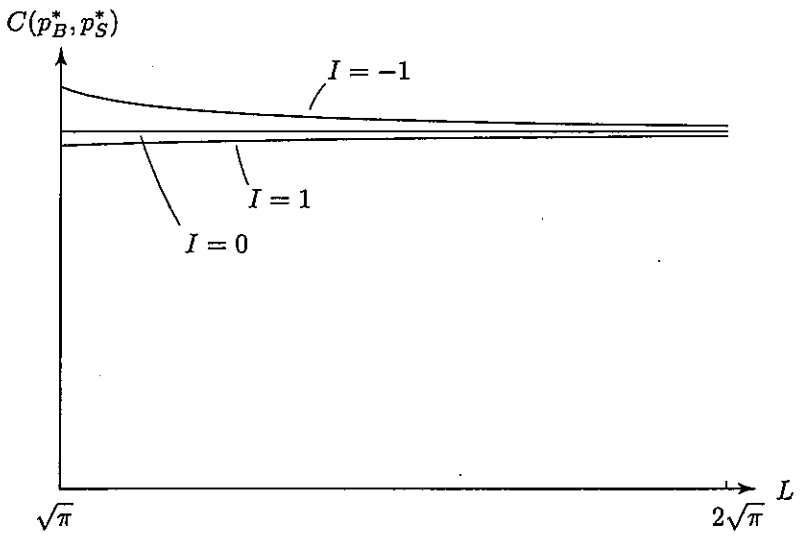
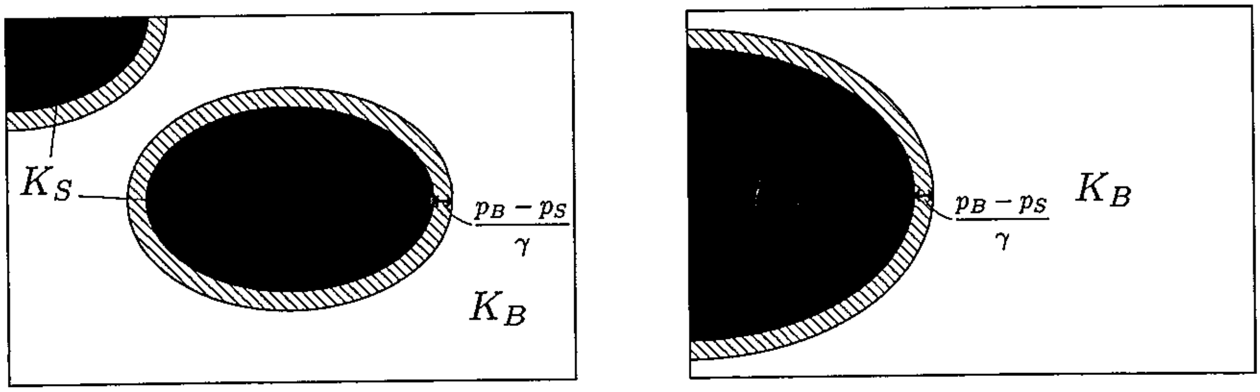
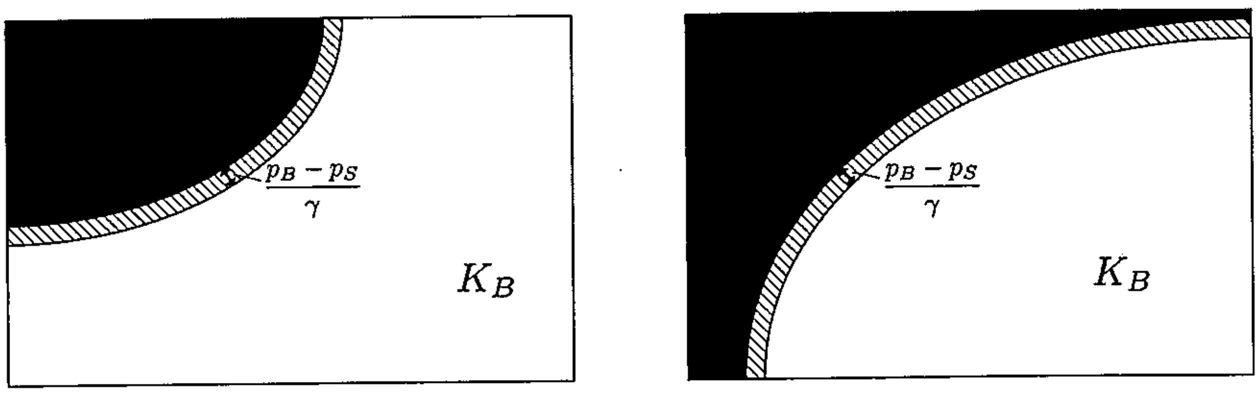


Figure 3: Amount of Cross-Border Shoppers with respect to Border Length



(1) $I = 1.25$ (2) $I = 0.5$



(3) $I = 0.25$ (4) $I = -0.25$

Figure 4: Another Spatial Configurations