No. 84 (80-22)

The N-part Partition of Risks

by
Nozomu Matsubara

July 1980

The N-part Partition of Risks

1. Introductory Example

Let us suppose rather artificial situation that an investor is offered two options, I and II, of investment plans like:

Plan I; asset = estate

investment = 300 (in some monetary units)

net gain = A = 90

net loss = B = 60

probability of gain = p = 0.65

Plan II; assets = estate, bond, stocks

investments = 100 (each)

net gains = α = 30 (each)

net losses = β = 20 (each)

probabilities of gains = p = 0.65 (each)

(3 assets are assumed to be stochastically independent)

His financial status quo is, say, $x_o = 1000$. The Plan II is the "diversification" of the Plan I, usually intended to be the aversion of risks. The example is introduced here for us to discuss generally on this paper whether the "partition of risky decisions" (or "risks" in short) brings about the improvement in terms of expected utility.

Thus we calculate the expected utilities, e_I and e_{II}, associated with the Plan I and Plan II respectively. His possible financial states would be the set

$$\mathcal{L}(I) = \{m, m+3\sigma\} \tag{1}$$

for the Plan I with a current understanding that

$$\sigma = \alpha + \beta = 50, \quad m = x_0 - B = 940, \tag{2}$$

and the lattice set

$$\mathcal{L}(II) = \{m, m+\sigma, m+2\sigma, m+3\sigma\} \tag{3}$$

for the Plan II, which consists of 3-equi-partitional points of the interval (940, 1090), corresponding to $\mathcal{L}(I)$. States of two plans are distributed over $\mathcal{L}(I)$ and $\mathcal{L}(II)$ as

$$\text{Bi}(1, p): \quad \mathfrak{g}_I = (1-p, p) \tag{4}$$

$$\text{Bi}(3, p): \quad \mathfrak{g}_{II} = ((1-p)^3, 3p(1-p)^2, 3p^2(1-p), p^3) \tag{5}$$

$$e_{II} - e_I = p(1-p)\sigma[(1+p)\ \{u'(\gamma_2) - u'(\gamma_3)\} + (2-p)\ \{u'(\gamma_1) - u'(\gamma_2)\}] > 0, \qquad (11)$$

and hence

$$e_{II} > e_I, \qquad (12)$$

provided $u(\cdot)$ satisfies the "diminishing marginal utility" hypothesis, which implies that

$$u'(\gamma_1) > u'(\gamma_2) > u'(\gamma_3) \qquad (13)$$

Thus it seems that, given the parametric specification of investment plans, the assumption on the marginal utility, and not the concrete form of the utility function, is all that we need. In other words, given this assumption to be valid, parameters of investment plans are only sufficient information for such a comparison, and the comparison is thus utility-free in this context, which serves us the nice applicability

2. N-equi-partition (General Case)

This section is for the more general N-equi-partitional case with $\mathcal{G}_{II}$ not necessarily being binomial distribution. In order that wider class of problems be covered, the term "investment plan" is extended to a more theoretical term "a lottery."

Let us now consider following two lotteries:

Lottery I has two states

a net gain = $A > 0$,

a net loss = $-B < 0$;

with

the prob. of gain = p,

the prob. of loss = $1 - p$.

Lottery II consists of N identical, but not necessarily stochastically independent, sub-lotteries each of which has two states

a net gain = $A/N \equiv \alpha$,

a net loss = $-B/N \equiv -\beta$

with

the prob. of i gains = q_i, $i = 0, 1, \ldots, N$.

$(q_i \geqq 0, \quad \Sigma q_i = 1)$

For both lotteries, the gambler's financial status quo is represented by x_o. The state space of the Lottery I is then

$$\mathcal{L}(I) = \{ x_o - B, x_o + A\}. \qquad (14)$$

The outcome of Lottery II is represented by N-vector

$$\underset{\sim}{d} = (d_1, d_2, \ldots, d_N) \qquad (15)$$

with $d_i = 1$ or 0 accordingly as the i-th sub-lottery ends up with a gain or a loss ($i = 0, 1, \ldots, N$). As the i-th sub-lottery gives the gambler $(\alpha+\beta)d_i - \beta$, the Lottery II gives him in sum, when the outcome is $\underset{\sim}{d}$,

$$s(\underset{\sim}{d}) = x_o + \sum_{i=1}^{N} \{(\alpha + \beta)d_i - \beta\}$$

$$= x_o + (\alpha + \beta) \sum_{i=1}^{N} d^i - \beta$$

$$= (x_o - \beta) + \sigma \cdot n(\underset{\sim}{d}), \tag{16}$$

where σ is the "range" of the sub-lottery

$$\sigma = \alpha + \beta \tag{17}$$

and $n(\underset{\sim}{d})$ is the number of gains ("success count") among the outcome $\underset{\sim}{d}$ of N sub-lotteries of the Lottery II

$$n(\underset{\sim}{d}) = \sum_{i=1}^{N} d_i \tag{18}$$

Noting that

$$n(\underset{\sim}{d}) = 0, 1, \ldots, N \tag{19}$$

we have as the state space of the Lottery II

$$\mathcal{L}(\text{II}) = \{m + i\sigma;\ i = 0,\ 1,\ \ldots,\ N\} \tag{20}$$

where m, M are the extremal states

$$m = x_o - B, \quad M = x_o + A \ (= m + N\sigma). \tag{21}$$

This and (14) correspond to (1), (2) and (3).

Now to compare

$$\mathcal{q}_{\text{I}} = (1-p,\ 0,\ \ldots,\ 0,\ p) \tag{22}$$

on $\mathcal{L}(\text{I})$ (with the understood notation) and

$$\mathcal{q}_{\text{II}} = (q_o,\ q_1,\ \ldots,\ q_{N-1},\ q_N) \tag{23}$$

on $\mathcal{L}(\text{II})$, we calculate their expected utilities

$$\begin{aligned} e_{\text{I}} &= pu(m + N\sigma) + (1 - p)u(m), \\ &= \sum_{n=o}^{N} p(\Delta u)_n + u(m) \end{aligned} \tag{24}$$

and

$$\begin{aligned} e_{\text{II}} &= \sum_{n=o}^{N} q_n u(m + n\sigma) \\ &= \sum_{n=o}^{N} q_n\left(\sum_{i=o}^{n} (\Delta u)_i + u(m)\right), \end{aligned} \tag{25}$$

where $(\Delta u)_i$ are marginal utilities defined by

$$(\Delta u)_i = u(m + i\sigma) - u(m + (i-1)\sigma), \; i = 1, \ldots, N$$
$$(\Delta u)_o \equiv 0 \qquad (26)$$

Therefore, e_I and e_{II} are compared by

$$e_{II} - e_I = \sum_{n=o}^{N} \{(\sum_{i=n}^{N} q_i - p) \; (\Delta u)_n\}$$
$$= \sum_{n=1}^{N} [\sum_{i=n}^{N} (i-n+1)q_i - (N-n+1)p \;] \; (\Delta^2 u)_n \qquad (27)$$

where

$$(\Delta^2 u)_i = (\Delta u)_i - (\Delta u)_{i-1}, \quad i = 1, \ldots, N.$$

Throughout this paper, we confine ourselves to the family u of utility functions with the diminishing marginal utility :

$$(\Delta^2 u)_i \leqq 0, \quad i = 2, \ldots, N, \qquad (28)$$

for any $\mathcal{L}$(II) set of the form (20). Incidentally note that the differentiability is not assumed here and that $(\Delta^2 u)_1 \geqq 0$, independently of the above assumption. Thus from the expression (27) we arrive at the main theorem of the paper:

[Theorem 1] In order that $e_{II} \geqq e_I$ for <u>any</u> utility function with the diminishing marginal utility (28), it is necessary and sufficient that the following condition on $\mathcal{q}_I$ and $\mathcal{q}_{II}$ hold;

$$\sum_{i=1}^{N} iq_i \geqq Np \tag{29-1}$$

$$\left.\begin{array}{lcl} \sum_{i=2}^{N} (i-1)q_i & \leqq & (N-1)p \\ \cdots \quad \cdots & & \cdots \\ q_{N-1} + 2q_N & \leqq & 2p \\ q_N & \leqq & p \end{array}\right\} \tag{29-2}$$

The theorem provides the criterion stated essentially utility-free for an equi-partitioned lottery to be better than the original one. It is re-stated equivalently in a more probability-theoretic form:

[Theorem 2] Let n^* be the random variable to denote the "success count" introduced in (17) with the probability distribution

$$P(n^* = i) = q_i, \quad i = 0, 1, \ldots, N. \tag{30}$$

Then in order that $e_{II} \geqq e_I$ for <u>any</u> utility function with the diminishing marginal utility (28), it is necessary and sufficient that the following condition on n^* hold;

$$E(n^*) \geqq Np \tag{31-1}$$

$$E(n^* - j)^+ \leqq (N-j)p, \quad j = 1, \ldots, N, \tag{31-2}$$

where $x^+ = x$ $(x \geqq 0)$, $= 0$ $(x < 0)$ and $E(\cdot)$ denote the expectation in terms of the probability distribution $\mathcal{G}_{II}$.

The left side of (31-2) is in the well-known form of "the shortage function" of a random variable, X, defined by

$$G(\lambda) = E\,(\,(X - \lambda)^+) = \int_\lambda^\infty (x - \lambda)\, dF_X(x), \tag{32}$$

which is interesting in itself, but we do not discuss the implication here.

The right hand side Np of (31-1) is the expected success count during N-repetition of sub-lotteries as if under the probability distribution $\mathcal{G}_I$ on $\{0, 1\}$. Thus the (30-1) is the assertion to the effect that we expect larger number of success under $\mathcal{G}_{II}$ than under the N-repeated $\mathcal{G}_I$.

(29-2) and (31-2) correspond with (28) and are called the "convexity parts" or the "non-linearity parts" of the theorems, while (29-1) and (31-1) are called their "linearity parts" since they should hold even when $u(\cdot)$ is linear.

The convexity part of the Theorem seems simple but still rather hard to check for given probability distributions. More tractable conditions are to be sought and the following corollary would provide one of them:

[Corollary 1] In order that $e_{II} \geqq e_I$ for <u>any</u> utility function with the diminishing marginal utility (28), it is sufficient that the following condition on $\mathcal{q}_I$ and $\mathcal{q}_{II}$ hold:

$$\sum_{i=1}^{N} iq_i \geqq Np, \tag{33-1}$$

$$q_o + q_1 \geqq 1 - p \tag{33-2}$$

[Proof] (33-2) implies

$$q_2 + q_3 + \dots + q_N \leqq p \tag{34}$$

and hence

$$\sum_{i=n}^{N} q_i \leqq p, \quad n = 2, 3, \dots, N. \tag{35}$$

Adding up inequalities in (35) with $n=N, N-1, \dots, N-h$ yields the $(N-1) - h$ -th (from the top) inequality of (29-3).

3. N-equi-partition (Binomial Case etc.)

To check the intorudctory example, where sub-lotteries of of Lottery II (Plan II) are identical and independent, we try the result of the preceeding section 2 for the binomial case. Let $\mathcal{q}_I$, $\mathcal{q}_{II}$ be $Bi(1, \theta_1)$ and $Bi(N, \theta_2)$. Since

$$q_i = \binom{N}{i} \theta_2^{\,i} (1 - \theta_2)^{N-i}, \quad i = 0, 1, \ldots, N \tag{36}$$

the sufficient condition in the corollary 1 is written down as

$$N\theta_2 \geqq N\theta_1 \tag{37-1}$$

$$(1 - \theta_2)^N + N\theta_2(1 - \theta_2)^{N-1} \geqq 1 - \theta_1 \tag{37-2}$$

Therefore $e_{II} \geqq e_I$ when (θ_1, θ_2) belongs to the 2-dim. domain represented by inequalities (37-1) and (37-2). At least one possibility satisfies them. That is the case of $\theta_1 = \theta_2$, where it is easy to check the $>$ sign on (37-2). Thus the introductory example of the binomial case generally holds true with other parameter values also.

The most general case is that the outcomes $\underset{\sim}{d} = (d_1, d_2, \ldots, d_N)$ of N sub-lotteries of the Lottery II are neither identically nor indpendently distributed. The probability distribution $\pi(d_1, d_2, \ldots, d_N)$ then yeilds the $q_o, q_1, \ldots, q_N$ through (30) like

$$q_i = \sum_{D(i)} \pi(d_1, d_2, \ldots, d_N), \tag{38}$$

where the summation is over

$$D(i) = \{\underset{\sim}{d};\ \sum_{r=1}^{N} d_r = i\} \tag{39}$$

with respect to $\underset{\sim}{d} = (d_1, \ldots, d_N)$, $(i = 0, 1, \ldots, N)$. The random variable d_i^* to denote d_i has the marginal distribution

$$\pi_i(d_i) = \Sigma^{(i)}\pi(d_1, d_2, \ldots, d_N) \tag{40}$$

where the summation $\Sigma^{(i)}$ extends over all $\underset{\sim}{d}$'s with d_i fixed, and for short

$$\pi_i(1) = \pi_i, \quad \pi_i(0) = 1 - \pi_i \ (i = 1, 2, \ldots, N). \tag{41}$$

To apply the sufficient condition in the Corollary 1, we only have to have for (33-1)

$$\begin{aligned}\sum_{i=1}^{N} iq_i &= E(n^*) \\ &= E(\sum_{i=1}^{N} d_i^*) \\ &= \sum_{i=1}^{N} E(d_i^*) \\ &= \sum_{i=1}^{N} \pi_i\end{aligned} \tag{42}$$

and for (33-2)

$$\begin{aligned}q_o &= \pi(0, 0, \ldots, 0), \\ q_1 &= \pi(1, 0, \ldots, 0) + \pi(0, 1, 0, \ldots, 0) \\ &\quad + \ldots + \pi(0, \ldots, 0, 1)\end{aligned} \tag{43}$$

In case of independence, one has

$$\pi(d_1, d_2, \ldots, d_N) = \pi_1(d_1) \ldots \pi_N(d_N)$$
$$= \prod_{i=1}^{N} \pi_i^{d_i} (1 - \pi_i)^{1-d_i} \qquad (44)$$

and

$$q_o = (1 - \pi_1)(1 - \pi_2) \ldots (1 - \pi_N),$$
$$q_1 = \sum_{j=1}^{N} \{\pi_j \prod_{i \neq j}^{N} (1-\pi_i)\}, \qquad (45)$$

corresponding to (43)

4. Generalizations

There would be a wide possibility of generalizations to be discussed to this model:

i) Non-equi-partitional case that N sub-lotteries are all distinct with different gains, losses and probabilities,

ii) Comparison of any two lotteries I and II such that one is the partition of the other,

iii) Normal approximations to criteria by Theorems 1, 2,

iv) Continuous lotteries, i.e., with continuous states on the real line,

v) Possible applications to the real problem of risk assessments through the numerical illustrations, etc.

These problems suggested above would be all answered affirmatively, but not dealt with in this paper.

Errata to "The N-Part Partition of Risks" by N. Matsubara

Page 12, ℓ. 7, 8:

Add "to hold at least for $\theta_1 = \theta_2 < 1/(N-1)^2$" to "the > sign on (37-2)."

Page 12, ℓ.8:

"Thus the introductory example ..." should read "Thus it is suggested that the introductory example ..."

Page 12, ℓ.9:

Add "In fact, (29-2) hold for $N = 3$ and 4. It would need some task to check (29-2) for general N, however."