No. 83 (80-21)

PRICE-QUANTITY DYNAMICS IN A MONOPOLISTICALLY COMPETITIVE ECONOMY WITH SMALL INVENTORY COSTS

by

Yoshitsugu Kanemoto*

June, 1980

*I would like to thank Keizo Nagatani and Jan Magnus for useful discussions and Dave Ryan for editorial assistance.

ABSTRACT

A model of short-run macro dynamics is developed on the basis of rational price-quantity adjustment by monopolistic firms facing adjustment costs. Local behaviour of the monopolistic economy is characterized and it is shown that the economy is locally stable if the curvature of the inventory cost function is small. Short-run and long-run effects of demand and cost disturbances are also analyzed.

INTRODUCTION

This paper is an attempt to reconstruct the theory of short-run macro dynamics on the basis of rational disequilibrium behaviour of monopolistic firms. The macroeconomic framework used here is an extension of Barro and Grossman (1971), and hence of the income-expenditure approach of the textbook Keynesian model: labour supply is infinitely elastic at a fixed money wage rate, money is the only asset, and investment in fixed capital is ignored or taken as given. The extension is made in two directions. First, many different kinds of goods are produced and a firm producing a certain kind of good has monopoly power. Second, the consumer's decision problem is made explicitly intertemporal.

There are k firms producing k different goods. At each instant of time, each firm sets the price of its product and the employment level (and hence the production level). A representative consumer takes both price and employment levels as given and reacts to them by deciding demand for goods. Firms incur adjustment costs when they change price and employment. Thus, they do not adjust these variables instantaneously, and consequently current production does not necessarily meet demand. However, firms carry inventories of products and always satisfy demand through changes in these inventories.

A firm must decide how to adjust price and employment at each instant of time. Following the approach of Negishi (1961), (1972), I assume that each firm estimates a perceived demand curve from observations of consumer demand. Firms then calculate the optimal adjustment paths of price and employment levels, and change existing levels according to the optimal plan. At the next moment, however, these changes in price and employment levels induce a change in the consumer's demand for goods. Firms must therefore adjust their perceived demand curves, recalculate the optimal plans and hence determine new rates of change of price and employment levels. This same process now continues based on the new price and employment levels.

Since both firms and the representative consumer face intertemporal decision problems, an expectation formation mechanism must be specified. In this paper, I assume one of the simplest mechanism: static expectations. In the adjustment process, therefore, firms and the consumer find that their expectations are always wrong; expectations are only realized when they reach the steady state.

The specification of the model owes much to recent development in the general equilibrium theory of a monopolistic economy, represented by Negishi (1961), (1972), Arrow and Hahn (1971), Nikaido (1974), (1975 a,b), Silvestre (1977), and Benassy (1976), (1978). The major novelty lies in the fact that price and employment are adjusted by the firm on the basis of rational economic calculations. In order to handle this new element and to obtain more transparent results, I introduce the symmetry assumption: The consumer's utility function is symmetric in all goods and all firms have the same production function.

In terms of the treatment of the firm's behaviour, Maccini (1976) uses a similar approach to the one adopted here. There are two major differences, however. First, although his model includes inventory costs, it does not have adjustment costs of price and employment. Second, his specification of aggregate demand is simplistic and is not based on microeconomic foundations.

Barro (1972), Iwai (1974), and Sheshinski and Weiss (1977), (1979), have developed models of price adjustment of a monopolistic firm under uncertainty. They assume that the firm incurs a fixed cost whenever it changes the price, regardless of the magnitude of that change. Since characterizing the dynamic behaviour of an economy based on such a model requires much more complicated analysis, I use a deterministic model and assume that the adjustment cost is a smooth convex function of the rate of change of price.

The analysis in this paper is limited to local properties, the complexity of the model making global analysis difficult. This difficulty arises in two respects. First, stability results are hard to establish. All candidates of the Lyapunov function that I have tried did not work. Second, comparative dynamics are extremely difficult to carry out except in the neighbourhood of the steady state.

Most of the results are obtained under the assumption that the curvature of the inventory cost function is small. In contrast, Kanemoto (1978) dealt with the case where all goods are perishable so that inventories cannot be carried.

The organization of the paper is as follows. In section 1, the decision problem of the representative consumer is solved. The solution is characterized by the consumption function which determines the consumer's total expenditures on consumption goods and the demand function which determines demand for each good. In section 2, the firm's problem is solved. Using the Ricatti algebraic equation, the first order approximation to the optimal path is obtained. In section 3, by specifying how profits are distributed, the temporary equilibrium of the monopolistic economy is obtained. The aggregate demand function determines the total expenditures on consumption goods and the objective demand function determines demand for each good. In section 4, the adjustment by firms of price and employment levels is incorporated in the monopolistic economy and the local behaviour of the economy is characterized. It is shown that the monopolistic economy is stable if the number of firms is large and if the curvature of the inventory cost function is small.

Sections 5 and 6 carry out comparative dynamics of an individual firm and of the monopolistic economy respectively. In section 5, the long-run and short-run effects of changes in aggregate demand, prices of other goods and the wage rate are considered. One of the most important results is that if the marginal cost is constant, demand disturbances have no long-run effect on price, but are completely absorbed by a change in employment, whereas cost disturbances have significant effects on both price and employment. Since the short-run price adjustment is influenced by the levels of employment and inventories as well as that of price, the short-run effect on price of demand disturbances is not zero. However, if the curvature of the inventory cost function is small, the short-run effect is also small. Section 6 examines the effects on the economy as a whole of a change in demand and a change in the wage rate. Due to interactions between different firms, the long-run effects on the economy are different from those on an individual firm. The effects of demand disturbances are amplified by the "price multiplier effect," since demand disturbances change the prices of other goods, which in turn induces further shifts in the demand curves. Aside from the "price multiplier effect," the effects of demand disturbances on the economy are the same as those on an individual firm's optimal plan. The effect on price of a change in wage rate has the same price multiplier, but the effects on employment and inventories have a multiplier which is less than unity. This is caused by the fact that the indirect effect through a change in other prices counteracts the direct effect in these cases. The short-run effects of demand and cost disturbances are exactly the same as those on an individual firm, since in the short-run prices of other firms are fixed at the original levels.

Finally, section 7 contains a discussion of numerical examples.

1. The Consumer

The representative consumer has the instantaneous utility function, $U(C,m)$, defined over the vector of consumption goods, $C = (c_1,\cdots,c_k)^T$, and real money balance, m. For simplicity, the utility function is assumed to be separable:

$$U(C,m) \equiv u[\phi(C),m] \quad , \qquad \ldots . (1.1)$$

with $\phi(C)$ being symmetric and $u(\phi,m)$ being homothetic.

At each instant of time t, the consumer receives wage income $\Omega(t)$ and profit income $\Pi(t)$; holds nominal money balance $M(t)$; and faces the (nominal) price vector of consumption goods, $P(t) = (p_1(t),\cdots,p_k(t))^T$.

The consumer has static expactations with respect to these variables: the optimal consumption plan is calculated under the (erroneous) assumption that current values of these variables persist indefinitely.

The problem for the consumer is to maximize the discounted sum of future utilities,

$$\int_t^\infty U[C(s),\ M(s)/\bar{p}(t)]e^{-r(s-t)}\,ds \quad , \qquad \text{....(1.2)}$$

subject to the cash-flow constraint,

$$\dot{M}(s) \;=\; \Omega(t) + \Pi(t) - P(t)^T C(s) \quad , \qquad \text{....(1.3)}$$

where the initial nominal balance $M(t)$ is given. The variable $\bar{p}(t)$ may be any appropriate price index, but for simplicity it is taken to be the average price:

$$\bar{p}(t) \;\equiv\; \frac{1}{k} \sum_{i=1}^{k} p_i(t) \quad . \qquad \text{....(1.4)}$$

The control variable is $C(s)$ and the state variable is $M(s)$.

The problem can be solved in two stages. First, the optimal consumption vector is obtained for an arbitrarily given level of total expenditure on consumption goods, $Y = P(t)^T C$. Second, the optimal path of the expenditure, $Y(s)$, $t \leq s < \infty$, is chosen. The result of the first stage optimization may be summarized by the indirect utility function:

$$v(Y,P,m) \;\equiv\; \max_C \left\{U(C,m):\ Y = P^T C\right\} \quad . \qquad \text{....(1.5)}$$

Roy's Identity yields the uncompensated demand function,

$$C(Y,P,m) \equiv -\nabla_P v/v_Y \quad , \qquad \ldots .(1.6)$$

where $\nabla_P v \equiv (\partial v/\partial p_1,\cdots,\partial v/\partial p_k)^T$ and $v_Y \equiv \partial v/\partial Y$. Due to the assumption of separability of the utility function (1.1), the demand function is invariant in m and therefore can be written as

$$C(Y, P) \quad . \qquad \ldots .(1.7)$$

By the symmetry of the utility function, the demand function for the i^{th} good can be expressed by

$$c_i = c(Y,P^i,p_i) \quad , \qquad \ldots .(1.8)$$

where the same function c() can be used for all goods, and P^i is the k-1 vector of prices of all goods other than the i^{th} one:

$$P^i = (p_1,\cdots,p_{i-1},p_{i+1},\cdots,p_k)^T.$$

The symmetry of the utility function also implies that c() is symmetric in P^i.

For convenience, define η as the reciprocal of the own price elasticity of demand:

$$\eta(Y,P^i,p_i) \equiv -c(Y,P^i,p_i)/[p_i\partial c(Y,P^i,p_i)/\partial p_i] \quad . \qquad \ldots .(1.9)$$

Now, differentiating the identify

$$Y \equiv \sum_{i=1}^{k} p_i c(Y,P^i,p_i) \quad , \qquad \ldots .(1.10)$$

with respect to Y and p_j, $j \neq i$, and substituting η where appropriate the following useful results are obtained.

Lemma 1. *If all prices are equal, then*

$$c_Y \equiv \frac{\partial c(Y,P^i,p_i)}{\partial Y} = \frac{1}{kp} > 0 \ , \qquad \text{....(1.11a)}$$

$$\frac{\partial c(Y,P^i,p_i)}{\partial p_j} = \frac{1}{k-1}\,\frac{1-\eta}{\eta} > 0 \ , \quad j \neq i \ , \qquad \text{....(1.11b)}$$

$$c_p \equiv \frac{\partial c(Y,P^i,p_i)}{\partial p_i} = -\frac{1}{\eta}\,\frac{c}{p} < 0 \ . \qquad \text{....(1.11c)}$$

The second stage optimization is to maximize

$$\int_t^\infty v[Y(s),P(t),M(s)/\bar{p}(t)]\, e^{-r(s-t)}\, ds \qquad \text{....(1.12)}$$

subject to

$$\dot{M}(s) = \Omega(t) + \Pi(t) - Y(s) \ . \qquad \text{....(1.13)}$$

The control variable is now $Y(s)$. The optimal path of $Y(s)$ and, in particular, the optimal level of $Y(t)$ depend on $\Omega(t)+\Pi(t)$, $M(t)$, and $P(t)$. Thus, this optimization yields the consumption function in nominal terms

$$Y(t) = Y[\Omega(t) + \Pi(t),\ M(t),\ P(t)] \ . \qquad \text{....(1.14)}$$

The partial derivatives of (1.14) can be evaluated using the method developed by Oniki (1969), (1973).

Lemma 2. *In the neighbourhood of the optimal steady state, the consumption function,* $Y(\Omega + \Pi, M, P)$, *satisfies*

$$\gamma \equiv Y_{\Omega} = Y_{\Pi} < 1 , \qquad \text{....(1.15a)}$$

$$\mu = Y_{M} > 0 . \qquad \text{....(1.15b)}$$

In, in addition, all prices are equal, then

$$\frac{\partial Y}{\partial p_i} = 0 \qquad \textit{for any } i . \qquad \text{....(1.15c)}$$

Condition (1.15a) shows that the marginal propensity to consume, γ, is less than one, while (1.15b) reveals that our model contains a positive real balance (or Pigou) effect, μ. These results can be explained heuristically as follows. Since the homotheticity assumption implies that both money and the consumer goods are normal, a rise in income increases the steady state money balance. For this to occur, the increase in consumption should be less than that of income so that money balance can accumulate; that is, the marginal propensity to consume should be less than one. However, an increase in the initial holdings of money balances does not affect the steady state money balance, and in the process of moving toward the steady state, consumption must increase in order to reduce the money balance.

Condition (1.15c) is the consequence of the homotheticity assumption. When a price rises, the "price" of money and the "average" price of consumption goods rise by the same proportion. By homotheticity, the resulting decrease in consumption is proportionately equal to the decrease in the money balance. Hence, consumption decreases by the same proportion as the rise in price, and the steady state total expenditure remains the same. If the consumer is initially at the steady state, it is therefore natural that expenditure will not change.

The marginal propensity to consume may be negative but this seems very unlikely. I therefore assume

$$\gamma > 0 \ . \qquad \ldots .(1.15d)$$

Finally, combining (1.8) and (1.14) yields the demand function for the i^{th} good at time t:

$$c_i(t) = c\left[Y(\Omega(t) + \Pi(t), M(t), P(t)),\ P^i(t), p_i(t)\right] , \qquad i = 1,\cdots,k \ \ldots .(1.16)$$

2. The Firm

Consider the behaviour of a firm which believes -- rightly or wrongly -- that it faces the demand curve c(p) at time t. Being a price setter, the firm must calculate the optimal future path of both the price of its product, p(s), $t \leqq \infty$, and the production level. I again assume static expectations: the firm believes that the demand curve remains the same forever.

The production function is denoted f(n), where n is the employment of an input called labour, and the production function exhibits decreasing returns to labour: $f''(n) < 0$. Concentrating on short-run analysis, it is assumed that quantities of other inputs cannot be changed. The money wage rate is constant at w, and there is excess supply of labour at this wage rate so that the firm never faces a supply constraint.

It is costly to adjust price and employment. For simplicity, adjustment costs are assumed to take the form of disappeared products. If the adjustment speeds of price and employment are $\dot{p}$ and $\dot{n}$, the amount of the firm's product used up for adjustment is

$$g^p(\dot{p}) + g^n(\dot{n}) \quad , \qquad \ldots .(2.1)$$

where the adjustment cost function is assumed to be smooth and convex:

$$g^p(0) = g^n(0) = g^{p\prime}(0) = g^{n\prime}(0) = 0 \quad , \qquad \ldots .(2.2a)$$

$$g^{p\prime\prime}(\dot{p}) > 0 \ , \quad g^{n\prime\prime}(\dot{n}) > 0 \quad . \qquad \ldots .(2.2b)$$

Price and employment do not adjust instantaneously because of these adjustment costs so that the quantity demanded does not in general equal the quantity produced (net of adjustment and inventory costs). However, the firm carries inventories of products and this difference is met by a change in inventories. The inventory costs are also assumed to take the form of disappeared products. The inventory cost function is

$$g^z(z,c(p)) \geqq 0 \ , \qquad \ldots .(2.3a)$$

where z is the amount of inventories. In addition to pure carrying costs like warehousing costs, inventory costs also include such things as the cost of transporting the products to the market and the cost of not being able to meet demand when demand suddenly rises. Inventory costs therefore depend on the volume of sales as well as the amount of inventories. It is assumed that the cost function is convex in z:

$$g^z_{zz}(z,c) > 0 \ , \qquad \ldots .(2.3b)$$

and that, given c, it reaches the minimum at a positive $z = \bar{z}(c)$:

$$g^z_z(\bar{z}(c),c) = 0 \ , \quad \text{for} \quad \bar{z}(c) > 0 \ . \qquad \ldots .(2.3c)$$

Furthermore, the function is nondecreasing in c:

$$g^z_c(z,c) \geqq 0 \ , \qquad \ldots .(2.3d)$$

and the cross partial is positive:

$$g^z_{zc}(z,c) > 0 \ , \qquad \ldots .(2.3e)$$

so that the minimum cost level of inventories, $\bar{z}(c)$, increases as the volume of sales increases:

$$\bar{z}'(c) = -g^z_{zc} / g^z_{zz} > 0 \ . \qquad \text{....(2.3f)}$$

The holding of inventories increases by the quantity produced minus the quantity sold minus the quantity used as adjustment and inventory costs:

$$\dot{z} = f(n) - c(p) - g^p(\dot{p}) - g^n(\dot{n}) - g^z(z,c(p)). \qquad \text{....(2.4)}$$

The firm maximizes the discounted sum of the future profit stream,

$$\int_t^\infty \left[p(s)c((p(s)) - \bar{w}n(s)\right] e^{-r(s-t)}\, ds \qquad \text{....(2.5)}$$

subject to the constraint (2.4), given the initial values, p(t), n(t) and z(t), of p, n and z. The restriction that z be nonnegative is ignored here since this paper deals only with local analysis and if the firm is initially close to the optimal steady state, it never runs out of inventories along the optimal path. Although the same notation is used, the firm and the household may have different discount rates.

Control variables are

$$\dot{p}(s) = \nu_p(s) \ , \qquad \text{....(2.6)}$$

$$\dot{n}(s) = \nu_n(s) \ , \qquad \text{....(2.7)}$$

and state variables are p(s), n(s) and z(s). The present value Hamiltonian is

$$H = pc(p) - \bar{w}n + q_z \Big\{f(n) - c(p) - g^p(\nu_p) - g^n(\nu_n)$$

$$- g_z(z,c(p))\Big\} + q_p\nu_p + q_n\nu_n \ , \qquad \text{....(2.8)}$$

where q_p, q_n, and q_z are costate variables associated with p, n, and z, respectively. The necessary conditions for the optimum are

$$-\dot{q}_p = \partial H/\partial p = c(p) + \Big(p - q_z[1 + g_c^z(z,c(p))]\Big)\ c_p - rq_p \quad \ldots(2.9a)$$

$$-\dot{q}_n = \partial H/\partial n = -\bar{w} + q_z f'(n) - rq_n \quad \ldots(2.9b)$$

$$-\dot{q}_z = \partial H/\partial z = -q_z g_z^z(z,c(p)) - rq_z\ , \quad \ldots(2.9c)$$

$$\partial H/\partial \nu_p = q_p - q_z g^{p\prime}(\nu_p) = 0 \quad \ldots(2.9d)$$

$$\partial H/\partial \nu_n = q_n - q_z g^{n\prime}(\nu_n) = 0\ , \quad \ldots(2.9e)$$

where

$$c_p(p) = dc(p)/dp\ .$$

The transversality condition requires the optimal path to converge to the optimal steady state at which $\dot{p} = \dot{n} = \dot{z} = \dot{q}_p = \dot{q}_n = \dot{q}_z = 0$.

The optimal steady state satisfies

$$p^*\Big(1 - \eta(p^*)\Big)f'(n^*) = \bar{w}\Big(1 + g_c^z(z^*,c(p^*))\Big)\ , \quad \ldots(2.10a)$$

$$f(n^*) = c(p^*) + g^z(z^*,c(p^*))\ , \quad \ldots(2.10b)$$

$$g_z^z(z^*,c(p^*)) + r = 0\ , \quad \ldots(2.10c)$$

$$q_p^* = q_n^* = 0\ , \quad \ldots(2.10d)$$

$$q_z^* = \frac{\bar{w}}{f'(n^*)} = p^*[1 - \eta(p^*)]\ , \quad \ldots(2.10e)$$

where $\eta(p)$ is the reciprocal of the demand elasticity as defined in (1.9):

$$\eta(p) \quad = \quad -c(p) / [pc_p(p)] \quad .$$

Condition (2.10a) shows that static monopoly theory holds at the steady state: marginal revenue equals the marginal cost, where marginal cost here includes inventory costs. According to (2.10b), steady state production equals the demand for the product plus the quantity of the product used for inventory management. Condition (2.10c) implies that the steady state inventory holding is less than the minimum-inventory-cost level $\bar{z}$. Defining $z^*(c)$ by

$$g_z^z(z^*(c),c) + r \quad = \quad 0 \quad ,$$

Condition (2.10c) can be replaced by

$$z^* \quad = \quad z^*(c(p^*)) \quad , \qquad \ldots .(2.10c')$$

where

$$\gamma_z \quad \equiv \quad z^{*\prime}(c) \quad = \quad -g_{zc}^z / g_{zz}^z \quad > \quad 0 \quad . \qquad \ldots .(2.11)$$

In order to simplify exposition, inventory costs are assumed to be small compared with production costs so that g_c^z is negligibly small at the steady state:

$$g_c^z(z^*,c(p^*)) \quad = \quad 0 \quad . \qquad \ldots .(2.12a)$$

It is also assumed that

$$g_{cc}^z - (g_{cz}^z)^2 / g_{zz}^z \quad = \quad 0 \qquad \ldots .(2.12b)$$

at the steady state. These assumptions can easily be relaxed and provided the magnitudes of both g_c^z and the expression on the left hand side of (2.12b) remain small, the results do not change.

Let

$$x = (p,n,z)^T , \qquad \text{....(2.13a)}$$

$$q = (q_p,q_n,q_z)^T . \qquad \text{....(2.13b)}$$

Equations (2.4), (2.6), (2.7), and (2.9) define the system of differential equations for x and q. By the transversality condition at $s=\infty$, the optimal path is the stable solution of the differential equations if the discount rate is sufficiently small.

Define

$$x^* \equiv (p^*,n^*,z^*)^T , \qquad \text{....(2.14a)}$$

$$q^* \equiv (q_p^*,q_n^*,q_z^*)^T = (0,0,q_z^*)^T , \qquad \text{....(2.14b)}$$

$$A = \begin{pmatrix} 0 , & 0 , & 0 \\ 0 , & 0 , & 0 \\ -c_p , & f; , & 0 \end{pmatrix} , \qquad \text{....(2.14c)}$$

$$B = \begin{pmatrix} \alpha_p , & 0 , & 0 \\ 0 , & \alpha_n , & 0 \\ 0 , & 0 , & 0 \end{pmatrix} , \qquad \text{....(2.14d)}$$

$$D = \begin{pmatrix} d_1, & 0, & d_3 \\ 0, & d_2, & 0 \\ d_3, & 0, & \alpha_z \end{pmatrix} , \qquad \ldots .(2.14e)$$

$$\alpha_p \equiv 1 / [q_z g^{p}{}''(0)] > 0 , \qquad \ldots .(2.14f)$$

$$\alpha_n \equiv 1 / [q_z g^{n}{}''(0)] > 0 , \qquad \ldots .(2.14g)$$

$$\alpha_z \equiv q_z g^z_{zz}(z,c) > 0 , \qquad \ldots .(2.14h)$$

$$d_1 \equiv \frac{c}{p}\frac{1}{\eta}\left(1 - \eta - p\eta_p\right) + q_z\left(\frac{c}{\eta p}\right)^2 g^z_{cc} \qquad \ldots .(2.14i)$$

$$d_2 \equiv -p\ (1 - \eta)\ f'' > 0 , \qquad \ldots .(2.14j)$$

$$d_3 \equiv \alpha_z \gamma_z \frac{1}{\eta}\frac{c}{p} > 0 , \qquad \ldots .(2.14k)$$

$$\eta_p \equiv d\eta(p) / dp , \qquad \ldots .(2.14\ell)$$

$$\gamma_z \equiv z^{*\prime}(c) , \qquad \ldots .(2.14m)$$

with A, B, and D evaluated at the steady state (x^*, q^*). It is assumed that

$$d_1 > 0 , \qquad \ldots .(2.15a)$$

which is true if the marginal revenue curve is downward sloping and

$$g^z_{cc} > 0 \ .$$

In order to use results in quadratic control theory, it is also assumed that D is positive definite. This assumption is satisfied if

$$|D| = d_2\left(\alpha_z d_1 - (d_3)^2\right) > 0 \ . \qquad \ldots .(2.15b)$$

A standard technique in quadratic control theory can now be applied to obtain the optimal solution.[1]

<u>Lemma 3</u>. *In the neighbourhood of the optimal steady state x^*, the optimal path can be approximated by*

$$\dot{x} = (A + BE)\,(x - x^*) \ , \qquad \ldots .(2.16)$$

where $E \equiv \{e_{ij}\}$ *is a* 3×3 *symmetric, negative definite matrix satisfying*

$$(A^T - rI)E + EA + EBE = D \ . \qquad \ldots .(2.17)$$

The matrix Equation (2.17) is called the algebraic Riccati equation. If the matrix E were diagonal, then price and employment would be adjusted simply in the direction of the steady state levels:

$$\dot{p} = \alpha_p e_{11}(p - p^*)$$

$$\dot{n} = \alpha_n e_{22}(n - n^*) \ .$$

It can be shown, however, that under our assumptions, E cannot be diagonal.

Lemma 4. *Matrix* E *satisfying* (2.17) *is diagonal only if* $\alpha_z = 0$.

Thus as long as the curvature of the inventory cost function, g^z_{zz}, is positive, matrix E is not diagonal. However, if $\alpha_z = 0$, it can be demonstrated that E is necessarily diagonal.

Lemma 5. *If* $\alpha_z = 0$, *then the negative semi-definite solution of* (2.17) *is diagonal.*

In general, only restrictions of symmetry and negative definiteness can be placed on E and these are not strong enough to yield clear-cut results in the following analysis. However, definite results can be obtained if α_z is small, since in this case the off-diagonal elements of E can be shown to be negative and small relative to the diagonal elements. This result and the rest of the results in this section are summarized in the following proposition.

Proposition 1. *Near the optimal steady state* x^*, *the optimal path of* x *can be approximated by*

$$\dot{x} = G(x - x^*) \quad , \qquad \text{.....(2.18)}$$

where

$$G = A + BE$$

$$= \begin{pmatrix} \alpha_p e_{11}, & \alpha_p e_{12}, & \alpha_p e_{13} \\ \alpha_n e_{12}, & \alpha_n e_{22}, & \alpha_n e_{23} \\ -c_p(p^*), & f'(n^*), & 0 \end{pmatrix} \qquad \text{.....(2.19)}$$

and

$$e_{11} < 0 \,, \quad e_{22} < 0 \,, \quad (e_{12})^2 < e_{11}e_{22} \,. \qquad \ldots.(2.20)$$

If α_z *is small, then*

$$e_{ij} < 0 \qquad \textit{for any } i \textit{ and } j \,, \qquad \ldots.(2.21)$$

and e_{12}, e_{13}, *and* e_{23} *are small compared with* e_{11} *and* e_{22} *in absolute value.*

3. The Objective Demand Function

There are k firms in the economy, with the i^{th} firm producing the i^{th} good. Since different goods are imperfect substitutes, a producer has monopoly power in the product market. Although different firms produce different goods, I assume that they are otherwise identical: they all have the same technology and objective functional, and pay the same fixed wage rate.

At time t, producers set prices P(t) and employment levels N(t). The wage income of the representative household is then determined by

$$\Omega(t) = \sum_{i=1}^{k} \bar{w}n_i(t) \quad . \qquad \ldots .(3.1)$$

If profit income is specified, Equation (1.16) yields demand for the i^{th} good at time t (given the consumer's money balance at that time).

Profit income depends on divident policies of firms. I assume a myopic policy wherein a firm pays an amount equal to the current instantaneous profit (excluding the value of inventory accumulation). Thus at time t, firm i pays the divident:

$$\pi_i(t) = p_i(t)c\left[Y(t), P^i(t), p_i(t)\right] - \bar{w}n_i(t) \quad . \qquad \ldots .(3.2)$$

The profit income of the representative household is the sum of divident payments from all firms:

$$\Pi(t) = \sum_{i=1}^{k} \pi_i(t)$$

$$= Y(t) - \Omega(t) \quad , \qquad \ldots .(3.3)$$

where the second equality is obtained from (1.10) and (3.1).

Total income is the sum of wage income and profit income, which, from (3.3) equals the total expenditure on consumption goods:

$$\Omega(t) + \Pi(t) = Y(t) \quad . \qquad \ldots .(3.4)$$

Thus the savings of the household are zero and its money balance remains constant over time: $M(t) = \bar{M}$, since

$$\dot{M}(t) = \Omega(t) + \Pi(t) - P(t)^T C(t) = 0 \quad . \qquad \ldots .(3.5)$$

The consumption function (1.14) now becomes

$$Y(t) = Y[Y(t), \bar{M}, P(t)] \quad . \qquad \ldots .(3.6)$$

This equation may be solved to obtain Y(t) as a function of $\bar{M}$ and P(t):

$$Y(t) = \tilde{Y}[\bar{M}, P(t)] \quad , \qquad \ldots .(3.7)$$

The function $\tilde{Y}()$ corresponds to the aggregate demand function in Keynesian economics, the only difference being that $\tilde{Y}(\ )$ is expressed in nominal terms, while the aggregate demand function is usually formulated in real terms.

From Lemma 2, in the neighbourhood of the optimal steady state of the consumer's optimal plan, the aggregate demand function $\tilde{Y}()$ satisfies

$$\frac{\partial \tilde{Y}}{\partial \bar{M}} = \frac{\mu}{1-\gamma} > 0 \quad , \qquad \ldots .(3.8a)$$

and if all prices are equal,

$$\frac{\partial \tilde{Y}}{\partial p_i} = 0 \ , \qquad \text{for any} \quad i \ . \qquad \ldots.(3.8b)$$

Thus, if all prices are equal, the money balance is the sole determinant of total expenditure on consumption goods, that is, of the nominal consumption.

Finally, substitution of (3.7) into (1.16) yields the objective demand function.

<u>Proposition 2</u>. *The demand for the i^{th} good at time t is given by the objective demand function:*

$$c_i(t) = c\left[\tilde{Y}(\bar{M},P(t)),\ P^i(t),\ p_i(t)\right] \ , \qquad i = 1,\cdots,k. \ . \ .(3.9)$$

where, if all prices are equal, c() *satisfies* (1.11) *in Lemma 1, and if, in addition, the consumer is at the steady state of its optimal plan, then* $\tilde{Y}$() *satisfies* (3.8).

4. Dynamic Behaviour of the Monopolistic Economy

The objective demand function (3.9) determines demand for the i^{th} good. In order to calculate the optimal plan, firm i must estimate the shape of the demand curve it is facing. I make the Nash-type assumption that the firm takes prices of all other goods as given and at each instant of time estimates the demand function as a function of the price of its product only. For simplicity, the following two assumptions on the perceived demand function are made.

Assumptions

(a) The firm knows the quantity demanded correctly at each instant of time. Thus, if the perceived demand function at time t is $\tilde{c}_i(p_i,t)$, then this function satisfies

$$c_i\left(p_i(t),t\right) = c\left[\tilde{Y}\left(\bar{M},P(t)\right),\ P^i(t),\ p_i(t)\right] . \qquad \ldots .(4.1)$$

(b) The firm believes that the price elasticity of demand is constant, and the perceived elasticity is correct at the steady state of the economy:

$$\tilde{\eta}(p_i,t) \equiv -\tilde{c}_i / [p_i\, \partial\tilde{c}_i/\partial\tilde{p}_i]$$

$$= \eta[Y^{**},P^{i**},p_i^{**}] \equiv \eta \ , \qquad \ldots .(4.2)$$

where double asterisks denote the steady state values.

Assumption b is made to simplify calculations. The assumption can be interpreted in two ways. First, the firm may simply have an erroneous perception. Second, the consumer's preferences may be such that the own elasticity is constant. If the consumer has a CES utility function and if the number of commodities is large then the own elasticity is approximately constant. At any rate, the same results are obtained as long as a change in the elasticity caused by changes in Y and P^i is relatively small.

The dynamic behaviour of the monopolistic economy can be described as follows. At time t, the vectors of prices, employment levels and inventories are set at

$$X(t) = \left(x_1(t)^T,\cdots,x_i(t)^T,\cdots,x_k(t)^T\right)^T , \qquad \ldots.(4.3)$$

where

$$x_i(t) = \left(p_i(t),\ n_i(t),\ z_i(t)\right)^T , \qquad \ldots.(4.4)$$

and the money balance held by the household is constant at $\bar{M}$. Given these variables, the household determines demand for all goods. Firms then perceive the demand curve and calculate the optimal paths of price, employment and inventory holding as in Section 2, thus determining the direction of change of these variables. In the neighbourhood of the steady state, $x_i(t)$ moves according to

$$\dot{x}_i(t) = G_i(t)[x_i(t) - x_i^*(t)] , \qquad i = 1,\cdots,k , \ldots.(4.5)$$

where $G_i(t)$ is the matrix G in (2.19) and $x_i^*(t)$ denotes the steady state values of x_i in the firm's optimal plan. Note that both $G_i(t)$ and $x_i^*(t)$

are obtained given the perceived demand curve at time t, $\tilde{c}(p_i,t)$. At the next moment, X(t) is adjusted to slightly different levels, and, given the new prices, employment levels and inventories, the same process is repeated.

Since the perceived demand curve depends in general on X(t), both $G_i(t)$ and $x_i^*(t)$ in (4.5) are functions of X(t):

$$x_i^*(t) = x_i^* \left(X(t)\right) , \qquad \ldots .(4.6)$$

$$G_i(t) = G_i \left(X(t)\right) . \qquad \ldots .(4.7)$$

Equation (4.5) can now be written as

$$\dot{x}_i(t) = G_i\left(X(t)\right)\left[x_i(t) - x_i^*\left(X(t)\right)\right] , \quad i = 1,\cdots,k \quad \ldots .(4.8)$$

The steady state of the system (4.8), $X^{**} = (x_1^{**T},\cdots x_k^{**T})^T$, satisfies

$$x_i^{**} = x_i^*(X^{**}) . \qquad \ldots .(4.9)$$

Only the steady state at which all firms have the same vector x is considered:

$$x_i^{**} = x_j^{**} = x^{**} , \qquad \text{for any } i, j . \qquad \ldots .(4.10)$$

Under the symmetry assumption, minor regularity conditions are sufficient to ensure the existence of such a steady state. However, it seems that stronger assumptions are required to establish uniqueness.

Linearizing (4.8) around the steady state yields[2]

$$\dot{x}_i(t) = G[I - \nabla_{x_i} x_i^*][x_i(t) - x^{**}] - \sum_{j \neq i} [G\nabla_{x_j} x_i^*][x_j(t) - x^{**}] , \quad ..(4.11)$$

where $\nabla_{x_i} x_i^*$ and $\nabla_{x_j} x_i^*$ are 3×3 Jacobian matrices of partial derivatives of $x_i^*(X)$ with respect to x_i and x_j, evaluated at the steady state, X^{**}, and

$$G \equiv G_i(X^{**}) \quad . \quad ....(4.12)$$

From (2.10), (4.1), and (4.2), $x_i^*(X)$ is the solution to

$$p_i^*(1-\eta)f'(n_i^*) = \bar{w}\left(1 + g_c^z[z_i^*, c(Y(\bar{M},P), P^i, p_i^*)]\right) \quad ....(4.13a)$$

$$f(n_i^*) = c(\tilde{Y}(\bar{M},P), P^i, p_i^*) + g^z[z_i^*, c(\tilde{Y}(\bar{M},P), P^i, p_i^*)] \quad ....(4.13b)$$

$$z_i^* = z^*[c(\tilde{Y}(\bar{M},P), P^i, p_i^*)] \quad . \quad ....(4.13c)$$

Noting (2.11 a,b) and (3.8b), implicit differentiation of (4.13) yields

$$\nabla_{x_i} x_i^*(X^{**}) = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix} \quad ....(4.14a)$$

$$\nabla_{x_j} x_i^*(X^{**}) = \begin{pmatrix} \partial p^*/\partial p_j, & 0, & 0 \\ \partial n^*/\partial p_j, & 0, & 0 \\ \partial z^*/\partial p_j, & 0, & 0 \end{pmatrix}$$

$$\equiv L , \quad ....(4.14b)$$

where

$$\frac{\partial p^*}{\partial p_j} = \frac{1}{(k-1)\Delta} \frac{1-\eta}{\eta} (-cf'')(1-r\gamma_z) \quad , \qquad \ldots .(4.15a)$$

$$\frac{\partial n^*}{\partial p_j} = \frac{1}{(k-1)\Delta} \frac{1-\eta}{\eta} \frac{cf'}{p} (1-r\gamma_z) \qquad \ldots .(4.15b)$$

$$\frac{\partial z^*}{\partial p_j} = \frac{1}{(k-1)\Delta} \gamma_z \frac{1-\eta}{\eta} \frac{c(f')^2}{p} \quad , \qquad \ldots .(4.15c)$$

$$\Delta = (f')^2 - \frac{cf''}{\eta} (1-r\gamma_z) \quad , \qquad \ldots .(4.15d)$$

and γ_z is defined by (2.11) and is assumed to be less than $1/r$:

$$1 - r\gamma_z > 0 \ . \qquad \ldots .(4.15e)$$

Thus the system (4.11) can be written

$$\dot{X}(t) = Q[X(t) - X^{**}] \quad , \qquad \ldots .(4.16)$$

where

$$Q = \begin{pmatrix} G, & -GL, \cdots, & -GL \\ -GL, & G, \cdots, & -GL \\ \vdots & \ddots & \vdots \\ -GL, & \cdots\cdots, & G \end{pmatrix} . \qquad \ldots .(4.17)$$

The off-diagonal elements of Q represent interactions between different firms: the adjustment of price and employment of a firm is influenced by changes in prices of other goods since these changes cause a shift in the perceived demand curve.

The following Lemma is useful in examining the local stability of the monopolistic economy.

Lemma 6. *The eigenvalues of* Q *are*

(i) *the eigenvalues of* $G(I+L)$ *with multiplicity* $k-1$,

and

(ii) *the eigenvalues of* $G[I-(k-1)L]$ *with multiplicity* 1.

From (4.15), L approaches a zero matrix as k tends to infinity so that matrix $G(I+L)$ approaches matrix G. Since the firm's optimal plan is stable, G is a stable matrix, and by the continuity of eigenvalues with respect to the elements of the matrix, $G(I+L)$ is a stable matrix for a sufficiently large k.

Thus for a large enough k the economy is locally stable if $G[I-(K-1)L]$ is a stable matrix. For later use, define

$$S = I - (k-1)L$$

$$= \begin{pmatrix} s_{11}, & 0, & 0 \\ s_{21}, & 1, & 0 \\ s_{31}, & 0, & 1 \end{pmatrix} , \qquad \ldots .(4.18)$$

where

$$s_{11} = 1-(k-1)\,\partial p^*/\partial p_k = 1 - \frac{1}{\Delta}\frac{1-\eta}{\eta}(-cf'')(1-r\gamma_z) \qquad \ldots .(4.19a)$$

$$s_{21} = -(k-1)\,\partial n^*/\partial p_j = -\frac{1}{\Delta}\frac{1-\eta}{\eta}\frac{c}{p}f'(1-r\gamma_z) \qquad \ldots .(4.19b)$$

$$s_{31} = -\frac{1}{\Delta}\gamma_z\frac{1-\eta}{\eta}\frac{c(f')^2}{p} . \qquad \ldots .(4.19c)$$

It is easy to see that

$$0 < s_{11} < 1 , \qquad \ldots .(4.20a)$$

$$s_{21} < 0 \ , \qquad \text{....(4.20b)}$$

$$s_{31} < 0 \ . \qquad \text{....(4.20c)}$$

Further it can be shown that if the curvature of the inventory cost function, g^z_{zz}, is small, matrix GS is also stable. Thus

Proposition 3. *The monopolistic economy is locally stable if the number of firms, k, is large and if the curvature of the inventory cost function, g^z_{zz}, is small.*

Matrix GS has an interesting interpretation. If all prices are equal at the initial time, then obviously they remain equal throughout the adjustment process. In such a case the system (4.16) collapses to

$$\begin{aligned}\dot{x}(t) &= G[I-(k-1)L][x(t)-x^{**}] \\ &= GS[x(t) - x^{**}] \ , \qquad \text{....(4.21)}\end{aligned}$$

where $x(t) = x_i(t)$ for any i. The stability of GS therefore corresponds to the stability of the special case where all prices are equal.

5. Comparative Dynamics of the Optimal Plan of an Individual Firm

As a preliminary step toward examining the effects of demand and cost disturbances on the monopolistic economy, their effects on an individual firm's optimal plan are considered.

An individual firm may face two kinds of demand disturbances: one is a change in aggregate demand, Y, and the other is a change in the prices that other firms charge, P^i. Cost disturbances can be represented by a change in the money wage rate, $\bar{w}$. First, the long-run effects of these three types of disturbances are examined.

(a) The Long Run

Considering Y as an independent variable, Equation (4.13) can be solved to express the steady state levels of (p_i, n_i, z_i) as functions of Y, P^i, and $\bar{w}$:

$$p_i^* = p^*(Y, P^i, \bar{w}) \ , \qquad \ldots .(5.1a)$$

$$n_i^* = n^*(Y, P^i, \bar{w}) \ , \qquad \ldots .(5.1b)$$

$$z_i^* = z^*(Y, P^i, \bar{w}) \ . \qquad \ldots .(5.1c)$$

Implicit differentiation of (4.13) yields

$$\frac{\partial p^*}{\partial Y} = \frac{1}{k\Delta}(-f'')(1 - r\gamma_z) > 0 \ , \qquad \ldots .(5.2a)$$

$$\frac{\partial n^*}{\partial Y} = \frac{1}{k\Delta}\frac{f'}{p}(1 - r\gamma_z) > 0 \ , \qquad \ldots .(5.2b)$$

$$\frac{\partial z^*}{\partial Y} = \frac{1}{k\Delta}\gamma_z\frac{(f')^2}{p} > 0 \ , \qquad \ldots .(5.2c)$$

$$\frac{\partial p^*}{\partial p_j} = \frac{1}{(k-1)\Delta}\frac{1-\eta}{\eta}(-cf'')(1-r\gamma_z) = \frac{k(1-\eta)}{\eta(1-k)} c \frac{\partial p^*}{\partial Y} > 0 \ , \quad j \neq i, \quad \ldots (5.3a)$$

$$\frac{\partial n^*}{\partial p_j} = \frac{1}{(k-1)\Delta}\frac{1-\eta}{\eta}\frac{cf'}{p}(1-r\gamma_z) = \frac{k(1-\eta)}{\eta(1-k)} c \frac{\partial p^*}{\partial Y} > 0 \ , \quad j \neq i \ , \quad \ldots (5.3b)$$

$$\frac{\partial z^*}{\partial p_j} = \frac{1}{(k-1)\Delta}\frac{1-\eta}{\eta}\frac{\gamma_z c(f')^2}{p} = \frac{k(1-\eta)}{\eta(1-k)} c \frac{\partial p^*}{\partial Y} \ , \ j \neq i \ , \quad \ldots (5.3c)$$

$$\frac{\partial p^*}{\partial \bar{w}} = \frac{1}{\Delta}\frac{f'}{1-\eta} > 0 \ , \quad \ldots (5.4a)$$

$$\frac{\partial n^*}{\partial \bar{w}} = -\frac{1}{\Delta}\frac{1-r\gamma_z}{\eta(1-\eta)}\frac{c}{p} < 0 \ , \quad \ldots (5.4b)$$

$$\frac{\partial z^*}{\partial \bar{w}} = -\frac{1}{\Delta}\frac{\gamma_z f'}{\eta(1-\eta)}\frac{c}{p} < 0 \ , \quad \ldots (5.4c)$$

where Δ and γ_z are defined by (4.15d) and (2.11) respectively, and Lemma 1 was used by assuming that all prices were initially equal.

Equations (5.2) shows that when aggregate demand rises, price, employment and inventory holdings all rise. According to (5.3), the effects of a change in prices of other goods are proportional to the effects of a change in aggregate demand. This result is due to the assumption of constant price elasticity of demand. Obviously, if the elasticity does not change, all the demand disturbances should work in the same way. Equations (5.4) show that when the wage rate rises, the price level rises but employment and inventory holdings fall.

Two polar cases, one with $f'' = 0$ and the other with $\eta = 0$, are of interest. It is argued in the industrial organization literature that both marginal and average costs are constant in the long run. In our model this corresponds to the case of $f'' = 0$. Although there is no capital in our model, similar results are obtained in a model with capital. The case of $\eta = 0$ represents a perfectly competitive economy in which the demand curve faced by an individual firm is perfectly elastic. In terms of consumer preferences, this case is obtained if all goods are perfect substitutes.

For the case where $f'' = 0$, substitution in (4.15d) reveals Δ to be equal to $(f')^2$, and therefore positive. Hence, from (5.2a) and (5.3a), demand disturbances have no effect on the long-run price level. In this case demand disturbances are completely absorbed by a change in production. Again, this result is an immediate consequence of the constant elasticity assumption. In the long run, price and employment should be such that marginal revenue equals marginal cost. Since marginal cost is constant when $f'' = 0$, marginal revenue should also be constant. In the constant elasticity case, therefore, the price level cannot be affected by demand disturbances.

Next, consider the second polar case where $\eta = 0$. In this case, Δ is infinite unless f'' is zero, since in the limit as η approaches zero, $\eta\Delta = -cf''(1-r\gamma_z)$. From (5.2), therefore, a change in aggregate demand has no effect on the firm's long-run optimum. When the prices of perfect substitutes for its product are fixed, a particular firm cannot change the price level even if aggregate demand rises. Otherwise, this firm would lose all of its customers to other firms. Since the price level remains constant, the optimal employment and inventory levels also remain constant.

If other firms change prices, however, there is a significant change in the particular firm's long-run optimum. It can be seen from (5.3a) that if all other firms raise prices by equal amounts, then this firm raises the price by the same amount:

$$\sum_{j \neq i} \partial p^* / \partial p_j = 1 \quad . \qquad \qquad \ldots . (5.5)$$

From (5.4a), the price level is unaffected by a change in wage rate if $\eta = 0$. The employment level falls, however, to make the marginal revenue product equal to the higher wage rate. If, in addition to $\eta = 0$, f'' is small, then the effect on employment is very large, since

$$\frac{\partial n^*}{\partial \bar{w}} = \frac{1}{pf''} \quad .$$

The inventory holding also falls because of reduced production, and the fall is large if f'' is small.

(b) The Short Run

Next, consider the short-run, or impact, effects of demand and cost disturbances. Suppose that the firm is initially at the optimal steady state and examine what happens to price, employment and inventories at the moment when small disturbances occur.

By Proposition 1, the optimal path of firm i satisfies

$$\begin{pmatrix} \dot{p}_i \\ \dot{n}_i \\ \dot{z}_i \end{pmatrix} = G \begin{pmatrix} p_i - p_i^* \\ n_i - n_i^* \\ z_i - z_i^* \end{pmatrix} \qquad \text{....(5.6)}$$

where matrix G is defined by (2.19). A change in any parameter, say "a", changes G and (p_i^*, n_i^*, z_i^*). But, since initially $p_i = p_i^*$, $n_i = n_i^*$, and $z_i = z_i^*$, the change in G can be ignored. Thus

$$\begin{pmatrix} \partial\dot{p}_i/\partial a \\ \partial\dot{n}_i/\partial a \\ \partial\dot{z}_i/\partial a \end{pmatrix} = -G \begin{pmatrix} \partial p^*/\partial a \\ \partial n^*/\partial a \\ \partial z^*/\partial a \end{pmatrix} \qquad \text{....(5.7)}$$

Taking in turn Y, p_j, and $\bar{w}$ as the parameter "a" yields

$$\frac{\partial\dot{p}_i}{\partial Y} = -\alpha_p \left(e_{11}\frac{\partial p^*}{\partial Y} + e_{12}\frac{\partial n^*}{\partial Y} + e_{13}\frac{\partial z^*}{\partial Y} \right) > 0 \ , \qquad \text{...(5.8a)}$$

$$\frac{\partial\dot{n}_i}{\partial Y} = -\alpha_n \left(e_{12}\frac{\partial p^*}{\partial Y} + e_{22}\frac{\partial n^*}{\partial Y} + e_{23}\frac{\partial z^*}{\partial Y} \right) > 0 \ , \qquad \text{...(5.8b)}$$

$$\frac{\partial\dot{z}_i}{\partial Y} = c_p\frac{\partial p^*}{\partial Y} - f'\frac{\partial n^*}{\partial Y} < 0 \ , \qquad \text{....(5.8c)}$$

$$\frac{\partial\dot{p}_i}{\partial p_j} = \frac{k(1-\eta)c}{\eta(1-k)}\frac{\partial\dot{p}_i}{\partial Y} > 0 \ , \qquad j \neq i \qquad \text{....(5.9a)}$$

$$\frac{\partial\dot{n}_i}{\partial p_j} = \frac{k(1-\eta)c}{\eta(1-k)}\frac{\partial\dot{n}_i}{\partial Y} > 0 \ , \qquad j \neq i \qquad \text{....(5.9b)}$$

$$\frac{\partial \dot{z}_i}{\partial p_j} = \frac{k(1-\eta)c}{\eta(1-k)} \frac{\partial \dot{z}_i}{\partial Y} < 0 \ , \qquad j \neq i \qquad ....(5.9c)$$

$$\frac{\partial \dot{p}_i}{\partial \bar{w}} = -\alpha_p \left(e_{11} \frac{\partial p^*}{\partial \bar{w}} + e_{12} \frac{\partial n^*}{\partial \bar{w}} + e_{13} \frac{\partial z^*}{\partial \bar{w}} \right) , \qquad ....(5.10a)$$

$$\frac{\partial \dot{n}_i}{\partial \bar{w}} = -\alpha_n \left(e_{12} \frac{\partial p^*}{\partial \bar{w}} + e_{22} \frac{\partial n^*}{\partial \bar{w}} + e_{23} \frac{\partial z^*}{\partial \bar{w}} \right) , \qquad ....(5.10b)$$

$$\frac{\partial \dot{z}_i}{\partial \bar{w}} = c_p \frac{\partial p^*}{\partial \bar{w}} - f' \frac{\partial n^*}{\partial \bar{w}} = 0 \ , \qquad ....(5.10c)$$

where the inequalities in (5.8 a,b) and (5.9 a,b) hold if α_z is small so that e_{12}, e_{13}, and e_{23} are negative.

From (5.8 a,b) and (5.9 a,b), increases in aggregate demand and in other prices both raise price and employment in the short-run. Therefore, the short-run effects of demand disturbances are in the same direction as the long-run effects. From (5.8c) and (5.9c), however, the inventory holding moves in the opposite direction. When demand rises, the long-run inventory holding rises, but in the short-run the expansion of production lags behind the increase in demand, thus causing a short-run reduction in inventories.

Equation (5.10c) shows that a change in the wage rate does not cause any change in the inventory holding in the short-run although the inventory holding does decrease in the long-run as shown by (5.4c). This is due to the fact that in the short-run, a change in demand caused by a change in price is offset by a change in production.

The effects of cost disturbances on price and employment are ambiguous. Substitution of (5.4 a-c) into (5.10 a) yields

$$\frac{\partial \dot{p}_i}{\partial \bar{w}} = -\alpha_p \frac{1}{\Delta(1-\eta)} \left[e_{11} f' - e_{12} \frac{c}{\eta p} (1 - r\gamma_z) - e_{13} \frac{\gamma_z c f'}{\eta p} \right] \quad . . (5.10a')$$

The first term in the square bracket of (5.10a') is negative, but the second and the third terms are positive at least when α_z is small. Since e_{12} and e_{13} tend to zero as α_z approaches zero, (5.10a') is positive if α_z is small and η is <u>not</u> small. In such a case, the short-run change has the same direction as the long-run change. If, however, η is small and α_z is not small, the price may fall in the short run even though it rises in the long run. This anomaly is caused by the fact that short-run adjustment of price is influenced by employment and inventory levels as well as the price level. Since a rise in wage rate decreases long-run employment and inventory levels, the firm finds that current levels are too high. This provides an incentive for the firm to lower price and thereby increase demand.

Substituting (5.3 a-c) into (5.10 b) yields

$$\frac{\partial \dot{n}_i}{\partial \bar{w}} = -\alpha_n \frac{1}{\Delta(1-\eta)} \left[e_{12} f' - e_{22} \frac{c}{\eta p} (1 - r\gamma_z) - e_{23} \frac{\gamma_z c f'}{\eta p} \right] . \quad . (5.10b')$$

The short-run effect on employment is thus less ambiguous, since both the second and third terms in the square bracket are positive. If η becomes smaller, both the second and the third terms become larger, which reinforces the tendency for employment to fall. The employment level is therefore more likely to move in the direction of the long-run level.

Now, consider the polar case in which f" is small. In this case, demand disturbances have a small effect on the long-run price level, but the effects on employment and inventories are not small. In order for the rate of change of price to be small, e_{12} and e_{13} must also be small. This case is obtained if α_z is small. If α_z is not small, the rate of change is not small and price may overshoot the long-run level. It should be noted, however, that even if the rate of change of price is not small, the absolute change in price may be small, since the long-run change is small.

In the other polar case where η is small, the short-run effects of a change in aggregate demand on price and employment, (5.8 a,b), are small, since long-run effects, $\partial p^*/\partial Y$, $\partial n^*/\partial Y$ and $\partial z^*/\partial Y$, are all small. However, the effect on inventory holding, (5.8c), is not small because $c_p = c/(\eta p)$ tends to infinity as the price elasticity approaches infinity; a change in price becomes infinitesimal, but the infinitesimal change in price causes a finite change in demand.

The short-run effects on price and employment of a change in other prices, (5.9 a,b), do not become small even if η becomes small, and the effect on inventories, (5.9c), approaches infinity as η tends to zero.

Although the long-run effect of a change in wage rate on the price level, (5.4a), is small if η is small, the short-run effect does not become small even in this case since indirect effects through employment and inventories remain finite. As seen before, however, the short-run effect on price may become negative as η approaches zero.

The effect on employment does not become small either, although the short-run effect is in the same direction as the long-run effect.

6. Comparative Dynamics of the Monopolistic Economy

The entire economy reacts to demand and cost disturbances differently from an individual firm because of interactions between firms. In our model these interactions work through the dependence of the demand for an individual firm's product on the prices that other firms charge.

As in the preceding section, both demand and cost disturbances are considered, but the formulation of demand disturbances must be modified. In the case of an individual firm, the aggregate demand, Y, and prices of other goods, P^i, were taken as independent variables, and perturbations of these variables were considered. In the entire economy, however, these variables are endogenous and cannot be perturbed independently. Consider, therefore, a change in the preferences of the representative consumer that results in an increase in propensity to consume. Specifically, demand disturbances take the form of perturbations of the consumption function (3.6) by a parameter y:

$$Y(t) \;=\; Y[Y(t),M(t),P(t)] + y \;. \qquad \ldots .(6.1)$$

The aggregate demand function (3.7) now becomes

$$Y(t) \;=\; \tilde{Y}[\tilde{M},P(t),y] \;, \qquad \ldots .(6.2)$$

where from (1.15a),

$$\partial\tilde{Y}/\partial y \;=\; 1/(1-\gamma) \;. \qquad \ldots .(6.3)$$

Equation (6.3) shows the standard multiplier effect: a unit increase in autonomous expenditures increases the aggregate demand by an amount equal to the multiplier, $1/(1-\gamma)$, which is the reciprocal of the marginal propensity to save.

(a) The Long Run

In long-run equilibrium, all firms have the same levels of price, employment and inventories, (p^{**},n^{**},z^{**}). The long-run equilibrium therefore satisfies

$$p^{**} = p^{*}\left(\tilde{Y}(\bar{M},p^{**}1_k,y),\ p^{**}1_{k-1},\bar{w}\right) , \qquad \ldots .(6.4a)$$

$$n^{**} = n^{*}\left(\tilde{Y}(\bar{M},p^{**}1_k,y),\ p^{**}1_{k-1},\bar{w}\right) , \qquad \ldots .(6.4b)$$

$$z^{**} = z^{*}\left(\tilde{Y}(\bar{M},p^{**}1_k,y),\ p^{**}1_{k-1},\bar{w}\right) , \qquad \ldots .(6.4c)$$

where functions, $p^*()$, $n^*()$ and $z^*()$, are defined in (5.1), and 1_k is a column vector of k unit elements: $1_k = (1,\cdots,1)$,

Total differentiation of (6.4) yields

$$S \begin{pmatrix} dp^{**} \\ dn^{**} \\ dz^{**} \end{pmatrix} = \frac{1}{1-\gamma} \begin{pmatrix} \partial p^*/\partial Y \\ \partial n^*/\partial Y \\ \partial z^*/\partial Y \end{pmatrix} dy + \begin{pmatrix} \partial p^*/\partial \bar{w} \\ \partial n^*/\partial \bar{w} \\ \partial z^*/\partial \bar{w} \end{pmatrix} d\bar{w} \qquad \ldots . .(6.5)$$

where S is defined in (4.18). The determinant of S,

$$|S| = s_{11} = \frac{(f')^2 - cf''(1-r\gamma_z)}{(f')^2 - \frac{1}{\eta}\, cf''(1-r\gamma_z)} \qquad \ldots .(6.6)$$

plays an important role later. From (4.20a), the determinant satisfies

$$0 < |S| < 1 , \qquad \ldots .(6.7)$$

where $|S|$ approaches one as f" approaches zero, and approaches zero as η approaches zero when $f'' < 0$.

From (6.5), the long-run effects of demand and cost disturbances satisfy

$$\frac{\partial p^{**}}{\partial y} = \frac{1}{|S|}\frac{1}{1-\gamma}\frac{\partial p^*}{\partial Y} \geqq \frac{1}{1-\gamma}\frac{\partial p^*}{\partial Y} > 0 \qquad \ldots .(6.8a)$$

$$\frac{\partial n^{**}}{\partial y} = \frac{1}{|S|}\frac{1}{1-\gamma}\frac{\partial n^*}{\partial Y} \geqq \frac{1}{1-\gamma}\frac{\partial n^*}{\partial Y} > 0 \qquad \ldots .(6.8b$$

$$\frac{\partial z^{**}}{\partial y} = \frac{1}{|S|}\frac{1}{1-\gamma}\frac{\partial z^*}{\partial Y} \geqq \frac{1}{1-\gamma}\frac{\partial z^*}{\partial Y} > 0 \qquad \ldots .(6.8c)$$

$$\frac{\partial p^{**}}{\partial \bar{w}} = \frac{1}{|S|}\frac{\partial p^*}{\partial \bar{w}} \geqq \frac{\partial p^*}{\partial \bar{w}} > 0 \qquad \ldots .(6.9a)$$

$$\frac{\partial n^{**}}{\partial \bar{w}} = \frac{1}{|S|}\left(s_{11}\frac{\partial n^*}{\partial \bar{w}} - s_{21}\frac{\partial p^*}{\partial \bar{w}}\right)$$

$$= \frac{\eta}{|S|}\frac{\partial n^*}{\partial \bar{w}} < 0 \qquad \ldots .(6.9b)$$

$$\frac{\partial z^{**}}{\partial \bar{w}} = \frac{1}{|S|}\left(s_{11}\frac{\partial z^*}{\partial \bar{w}} - s_{31}\frac{\partial p^*}{\partial \bar{w}}\right)$$

$$= \frac{\eta}{|S|}\frac{\partial z^*}{\partial \bar{w}} < 0 \ . \qquad \ldots .(6.9c)$$

Thus, the directions of long-run effects on the economy are the same as those on the optimal plan of an individual firm, but the magnitudes are different.

First, consider the effects of demand disturbances. As can be seen from (6.8), the effects are amplified by what may be called the "price multiplier effect" represented by $1/|S|$, in addition to the usual multiplier effect. An increase in the aggregate demand tends to raise prices. The induced rise in prices that other firms charge raises the demand curve that an individual firm is facing, thus magnifying the initial effect.

As f'' approaches zero, $|S|$ approaches one and hence the price multiplier also approaches one. The reason for this is that if f'' is small, the optimal price level of an individual firm is mainly determined by the cost condition and consequently the effects of a change in other prices are small. As in the individual firm case, therefore, demand disturbances have small long-run effects on the price level if f'' is small.

As η approaches zero, the price multiplier tends to infinity. Therefore, even though the effects on an individual firm vanish when η becomes zero, the effects on the economy as a whole do not vanish. This simply reflects the fact that prices of other goods are fixed in the case of an individual firm, whereas all prices change simultaneously when the economy as a whole is considered.

Now, let us turn to the effects of cost disturbances, (6.9). The effect on price, (6.9a), has the same price multiplier as in the case of demand disturbances, but the multiplier for employment and inventories, $\eta/|S|$, is smaller. It can be easily seen that this latter multiplier is less than one. This results from the fact that the effect of a change in prices of other goods counteracts the direct effect: the primary effect of a rise in wage rate is a reduction in both employment and inventories, but the indirect effect through a rise in other prices tends to raise the levels of employment and inventories.

It can be seen that the effects on price, employment and inventories all remain finite even if η tends to zero, whereas, in the case of an individual firm, the effect on price vanishes although the effects on employment and inventories remain finite.

Substitution of (5.4b) into (6.9b) yields

$$\frac{\partial n^{**}}{\partial \bar{w}} = - \frac{c}{\left[(f')^2 - cf''(1-r\gamma_z)\right]p(1-\eta)} \quad .$$

Therefore, even if both η and f'' approach zero, the effect on employment remains finite in contrast to the individual firm case where the effect tends to infinity. The same result is obtained for inventories.

Finally, the effect of a change in money supply is proportional to that of a change in y for the obvious reason:

$$\nabla_M x^{**} = \mu \nabla_y x^{**} \quad . \qquad \qquad . \; . \; . \; . \; .(6.10)$$

(b) The Short Run

Suppose the economy is perturbed from the long-run equilibrium by demand or cost disturbances. Assume, for simplicity, that the disturbances affect all firms equally. Hence, all firms adjust prices, employment and inventories in exactly the same manner, so that these variables remain equal throughout the adjustment process. In such a case, the adjustment process (4.16) can be reduced to

$$\dot{x} = GS(x - x^{**}) \quad , \qquad \text{....(6.11)}$$

by setting $x_i = x$ for any i. From (6.5), the impact effect of a change in a parameter, "a", satisfies

$$\nabla_a \dot{x} = -GS(\nabla_a x^{**}) = -G(\nabla_a x^*) \quad . \qquad \text{....(6.12)}$$

Thus the short-run effect on the entire economy is the same as the effect on an individual firm. This is quite natural since at the instant when a disturbance occurs, prices that other firms charge are at their old equilibrium levels. As time passes the prices change, causing a change in the behaviour of an individual firm.

7. Numerical Calculations

Numerical examples of the linearized system have been calculated. Figures 1-4 describe the effects of demand and cost disturbances on the dynamic behaviour of an individual firm and the monopolistic economy. The values of parameters at the initial steady state are

$$\alpha_p = 0.1, \quad \alpha_n = 0.1, \quad \alpha_z = 0.0001$$

$$\eta = 0.1, \quad f'' = -1, \quad r = 0.005$$

$$\bar{w} = 13.5, \quad c = 20, \quad p = 1, \quad \gamma_z = 2.$$

Changes of ten percent in demand and in the wage rate are considered. The unit time period is taken to be a month with the discount rate, r, set at 0.5 percent per month. The inventory-sales ratio, γ_z, at the margin is 2: an increase in sales by a unit per month causes an increase in the optimal inventory level by two units. The curvature of the price adjustment cost function, $g^{p''}$, is equal to that of the employment adjustment cost function, $g^{n''}$. The curvature of the inventory cost function is small as is the curvature of the production function which is set at -1. The price elasticity of demand is 10. The number of workers employed by a firm is normalized to be one at the initial steady state. Thus, wage income is 67.5% of sales.

Figure 1 depicts the effect on the optimal plan of the firm of a 10 percent fall in demand. The origin is adjusted so that the initial steady state is zero and the new steady state levels of price,

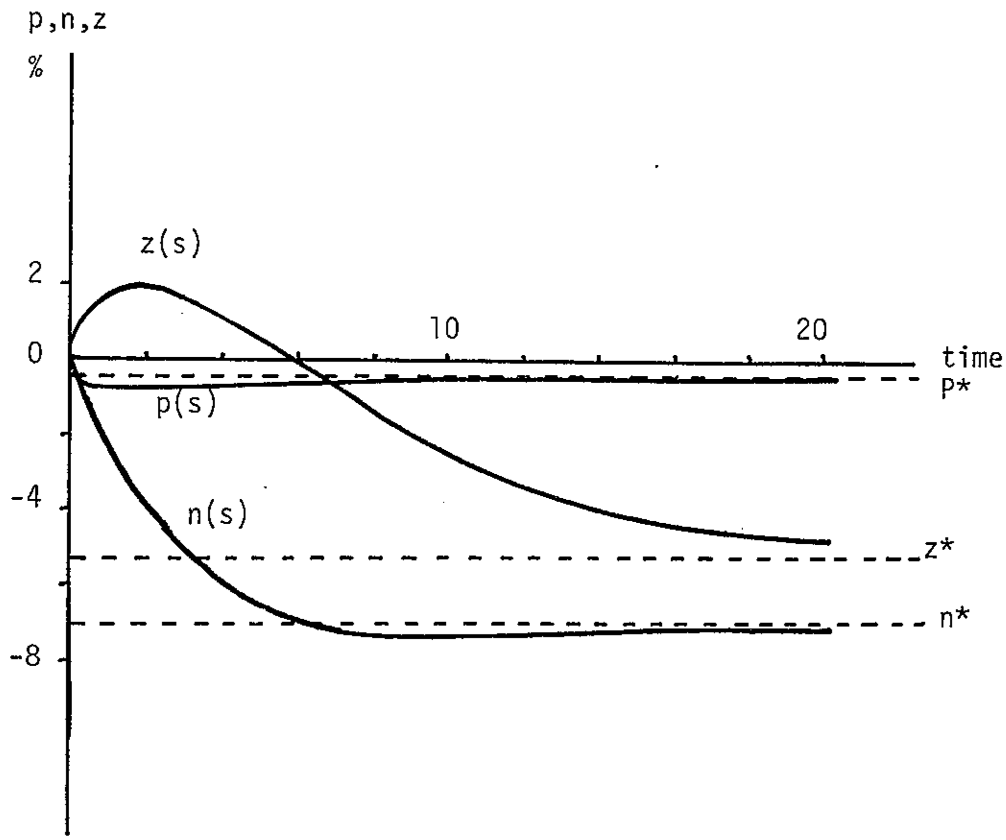

Figure 1 The effect of a fall in demand on the optimal plan of an individual firm

employment and inventories are given by lines p*, n* and z* respectively. Reflecting the small curvature of the production function, the steady state price level falls by only 0.47 percent. The steady state employment and inventory levels fall by 7.02 percent and 5.31 percent respectively.

Both price and employment overshoot the new steady state levels in the short run. However, the price level adjusts much faster than the employment level: price falls below the new steady state level within a month but employment takes about six months to reach its new steady state level. Although the price adjustment is small in absolute magnitude, it is fast in the short run. This result is caused by the assumption that the price adjustment function is smooth at the origin. This assumption implies that the marginal cost of changing the price is very small at the origin. If the adjustment cost function has a kink or if price adjustment involves a fixed cost, the price may not change at all.

Although the inventory level rises in the short run due to the adjustments of employment and price lagging behind the decrease in demand, it eventually falls to the lower steady state level. The convergence of inventory holding to this new level is slow because the curvature of the inventory cost function is small. It takes about six months for the inventory level to reach the old steady state level again and about twenty months to become close to the new steady state level.

The effect of a 10% rise in the wage rate is illustrated in Figure 2. The steady state price level rises by 5.32%; the employment level falls by 70.2%; and the inventory level falls by 53.2%. The fall

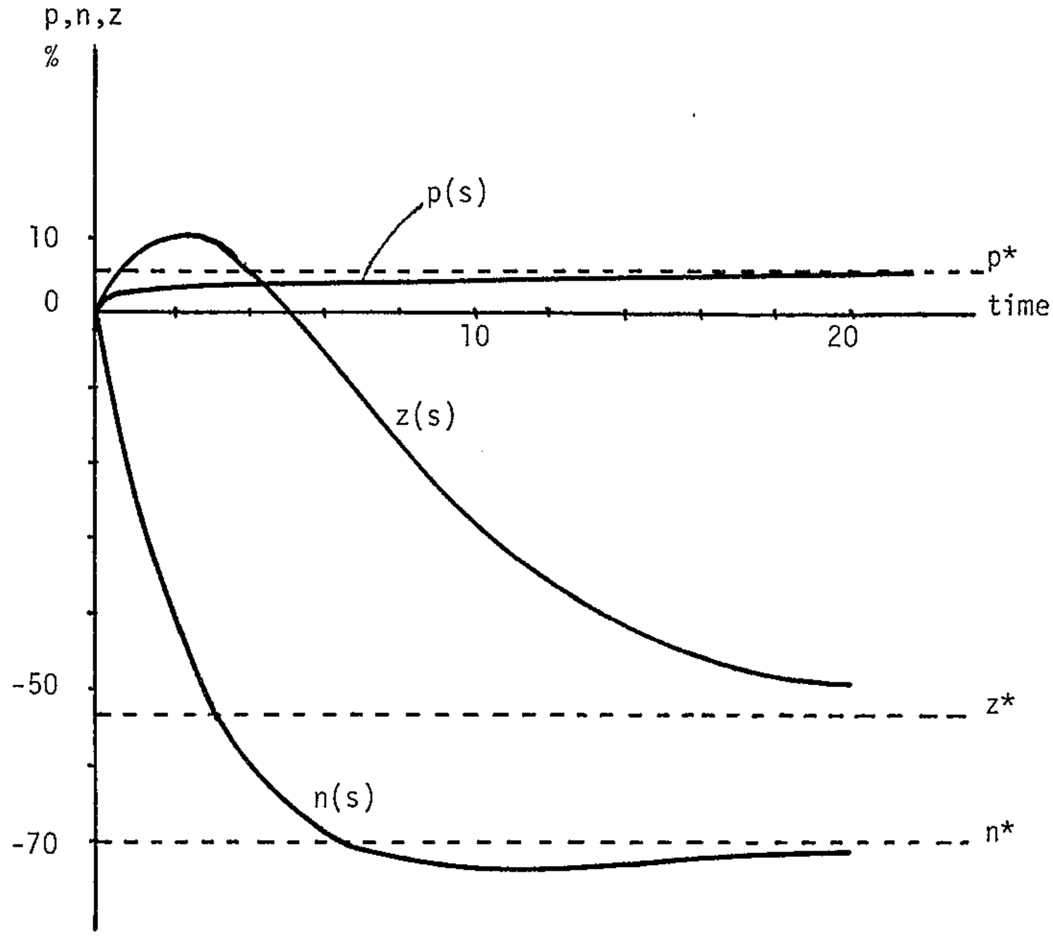

Figure 2 The effect of a rise in wage rate on the optimal plan of an individual firm

in employment and inventories is large because the curvature of the production function is small and prices of other goods are fixed. In the short run, the employment level overshoots the steady state level, while the inventory level increases in the short run before falling to the lower steady state level. The price rises monotonically until it reaches the steady state.

The effects on the monopolistic economy are depicted in Figures 3 and 4. Figure 3 describes the effect of a 10% fall in aggregate demand. The usual multiplier is ignored to make the result comparable with the case of an individual firm. the price multiplier is 1.73 so that the change in the steady state is 1.73 times larger than in the individual firm case. It can be seen from Figures 1 and 3 that the overshooting of price and employment is more pronounced in the monopolistic economy than in the optimal plan of an individual firm.

As shown in Figure 4, a 10% rise in the wage rate raises the steady state price level by 9.2% and lowers the employment and inventory levels by 12.1% and 9.2% respectively. In the short run the employment level overshoots the steady state level. In contrast to the individual firm case, the inventory level does not rise in the short run.

Many other examples have been calculated and results in previous sections have been confirmed. For example, as α_z becomes smaller, the off-diagonal elements of the matrix E become smaller. Other Notable findings are:

(a) The solution may have cycles if f" or η is small. Oscillation occurs more often in the case of the economy as a whole than in the case of an individual firm;

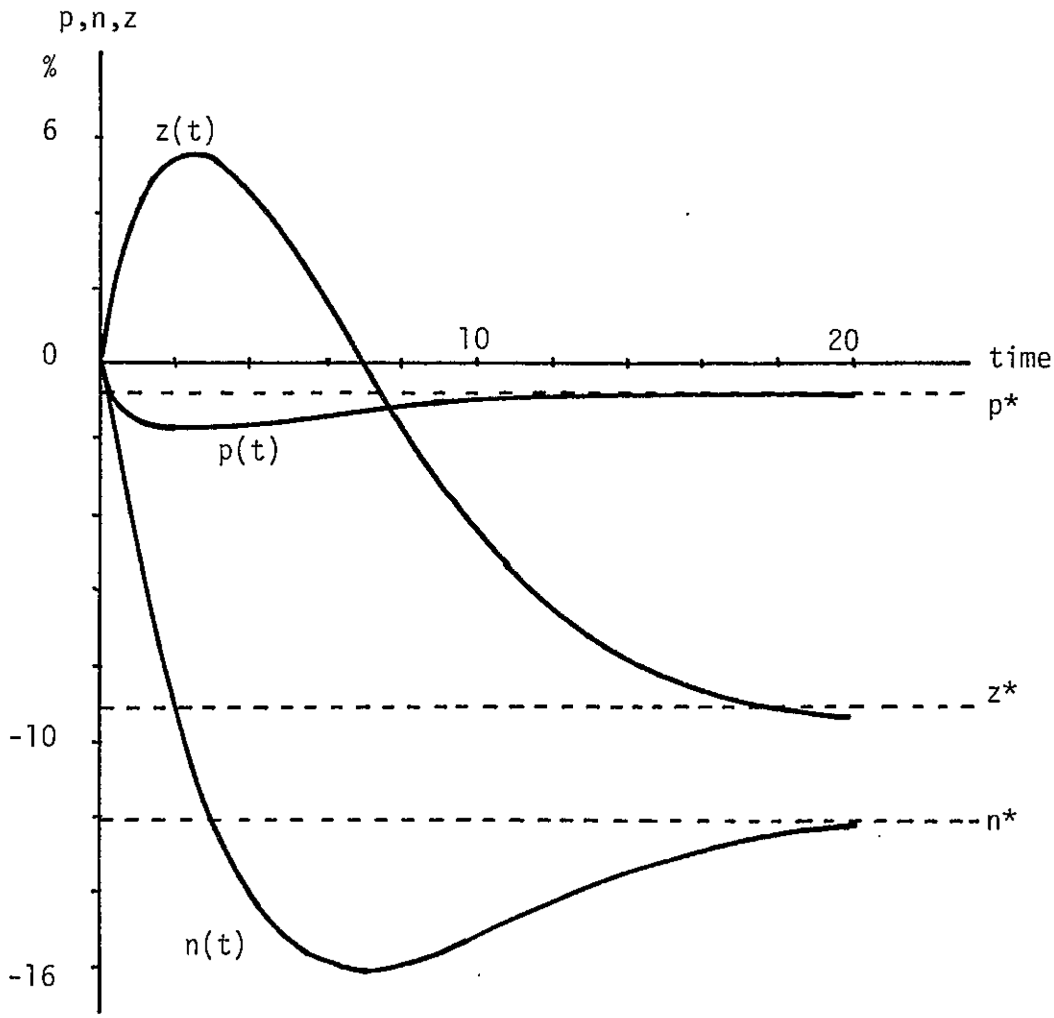

Figure 3 The effect of a fall in demand on the monopolistic economy

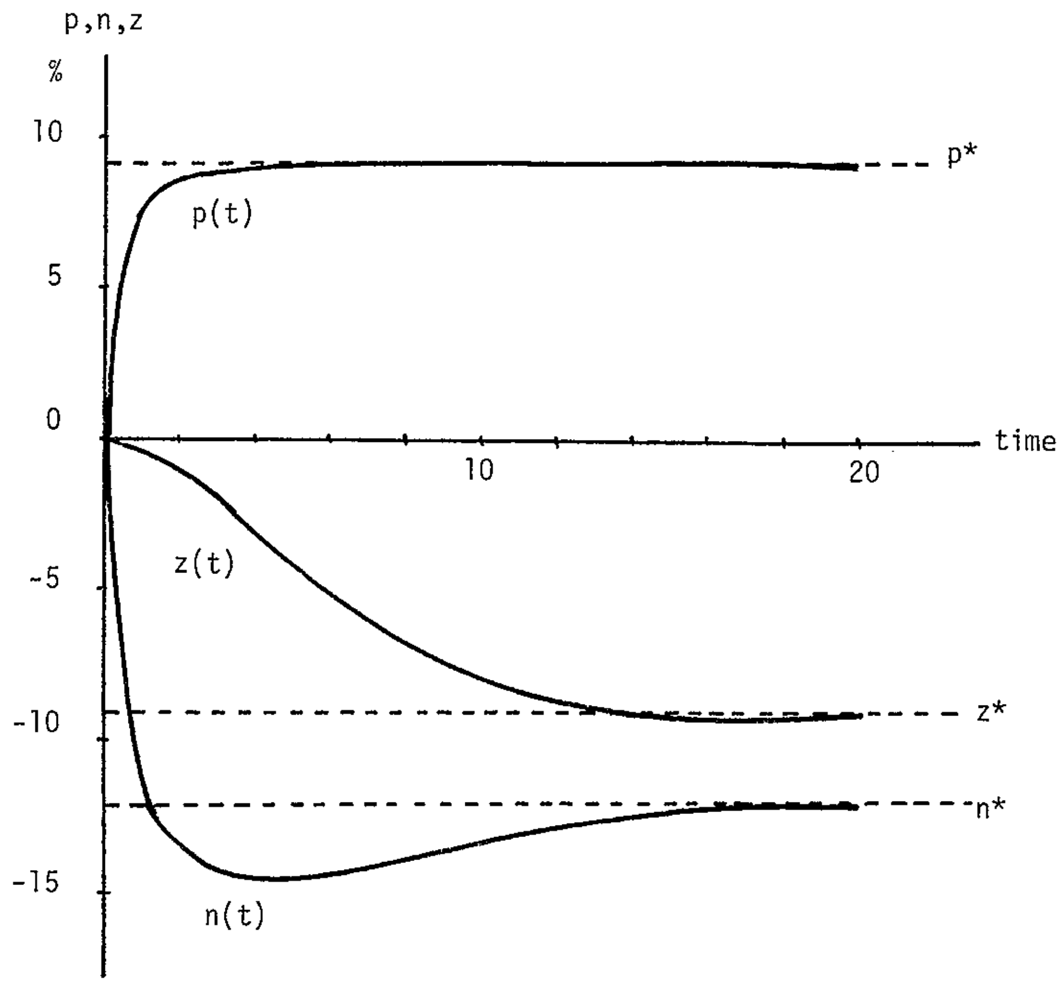

Figure 4 The effect of a rise in wage rate on the monopolistic economy

(b) All cases that have been calculated are stable if the matrix D is positive definite;

(c) If α_z is not small, there are cases where some off-diagonal elements of E are positive.

FOOTNOTES

[1]The approach adopted here is similar to the one developed by Epstein (1980).

[2]This linear approximation involves a somewhat subtle mathematical problem. The linear approximation of the firm's optimal plan can be justified, since the residual term tends to zero faster than $x - x^*$. The first order approximation of the path of the economy requires that the residual term tends to zero faster than $X - X^{**}$. It can be shown that if $x_i^*(X)$ is continuously differentiable, the former implies the latter.

[3]See Magnus (1980) for the proof of a more general form of this result.

REFERENCES

Arrow, K.J., and F.H. Hahn, 1971, General Competitive Analysis, Holden Day, San Francisco.

Barro, R.J., 1972, "A Theory of Monopolistic Price Adjustment," Review of Economic Studies 39, 17-26.

Barro, R.J. and H.I. Grossman, 1971, "A General Disequilibrium Model of Income and Employment," American Economic Review 61, 82-93.

Benassy, J.P., 1976, "The Disequilibrium Approach to Monopolistic Price Setting and General Monopolistic Equilibrium," Review of Economic Studies 43, 69-81.

Benassy, J.P., 1978, "A Neokeynesian Model of Price and Quantity Determination in Disequilibrium," in Equilibrium and Disequilibrium in Economic Theory (G. Schwödiauer, Ed.), D. Reidel Publishing Company, Dordrecht, Holland.

Epstein, L.G., 1980, "Comparative Dynamics in the Adjustment-Cost Model of the Firm," Working Paper No. 8004, Institute for Policy Analysis, University of Toronto.

Iwai, K., 1974, "The Firm in Uncertain Markets and Its Price, Wage and Employment Adjustments," Review of Economic Studies 41, 257-276.

Kanemoto, Y., 1978. "A Theory of Disequilibrium Dynamics," Discussion Paper No. 78-17, Department of Economics, University of British Columbia.

Long, N.V., 1979, "Two Theorems on Generalized Diminishing Returns and Their Applications to Economic Analysis," Economic Record, 58-63.

Maccini, L.J., 1976, "An Aggregate Dynamic Model of Short-Run Price and Output Behaviour," Quarterly Journal of Economics 90, 177-196.

Magnus, J., 1980,

Negishi, T., 1961, "Monopolistic Competition and General Equilibrium," Review of Economic Studies 28, 196-201.

Negishi, T., 1972, General Equilibrium Theory and International Trade, North-Holland, Amsterdam.

Nikaido, H., 1974, "What is an Objective Demand Function?" Zeitschrift fur Nationalokonomie 34, 291-307.

Nikaido, H., 1975a, Monopolistic Competition and Effective Demand, (Princeton, Princeton University Press).

Nikaido, H., 1975b, "Economic Adjustments under Noncompetitive Pricing," in Adaptive Economic Models (R.H. Day and T. Groves, Eds.), Academic Press, New York.

Oniki, H., 1969, "Comparative Dynamics in the Theory of Optimal Growth," Keizaigaku (Tohoku Economic Journal) 30, 48-57.

Oniki, H., 1973, "Comparative Dynamics (Sensitivity Analysis) in Optimal Control Theory," Journal of Economic Theory 6, 265-283.

Sheshinski, E., and Y. Weiss, 1977, "Inflation and Costs of Price Adjustment," Review of Economic Studies 44, 287-304.

Sheshinski, E., and Y. Weiss, 1979, "Demand for Fixed Factors, Inflation and Adjustment Costs," Review of Economic Studies 46, 31-46.

Silvestre, J., "A Model of General Equilibrium with Monopolistic Behaviour," Journal of Economic Theory 16, 425-442.

APPENDIX

Proofs of Lemmas and Propositions are provided in this appendix. Those of Lemma 1 and Proposition 2 are omitted as they are obvious from the main text.

Lemma 2. In the neighbourhood of the optimal steady state, the consumption function, $Y(\Omega + \Pi, M, P)$, satisfies

$$\gamma \equiv Y_\Omega = Y_\Pi < 1 \ ,$$

$$\mu \equiv Y_M > 0 \ .$$

If, in addition, all prices are equal, then

$$\frac{\partial Y}{\partial p_i} = 0 \qquad \text{for any } i \ .$$

Proof.

The first order condition for the problem of maximizing (1.12) subject to (1.13) is

$$\frac{d}{ds} v_Y = -v_M + r v_Y \ . \qquad \ldots . (A.1)$$

Noting that

$$\frac{d}{ds} v_Y = v_{YY} \dot{Y} + v_{YM} \dot{M}$$

and substituting for $\dot{M}$ from (1.13), Equation (A.1) can be written as

$$\dot{Y} = -\frac{1}{v_{YY}}\left[v_M - rv_Y + v_{YM}\,(\Omega + \Pi - Y)\right] .$$

The optimal path converges to the steady state (Y*,M*) given by

$$Y^* = \Omega + \Pi \qquad \text{....(A.2)}$$

$$rv_Y\left(Y^*, P, \frac{M^*}{\bar{p}}\right) - v_M\left(Y^*, P, \frac{M^*}{\bar{p}}\right) = 0 . \qquad \text{....(A.3)}$$

The steady state money balance, M*, can then be written

$$M^* = M^*(\Omega + \Pi, P) \quad ,$$

where

$$M^*_{\Omega} = M^*_{\Pi} = \frac{v_Y v_{MY} - v_M v_{YY}}{v_M v_{YM} - v_Y v_{MM}} > 0 .$$

The inequality is a consequence of normality of Y and M, the latter being implied by our homotheticity assumption.

First, examine the effect of a change in Ω (or Π). In the neighbourhood of the steady state, the variational differential equations with respect to Ω are

$$\dot{M}_{\Omega} = 1 - Y_{\Omega} \quad ,$$

$$\dot{Y}_{\Omega} = -\frac{1}{v_{YY}}\left[(v_{MM} - rv_{YM})\, M_{\Omega} - rv_{YY}Y_{\Omega} + v_{YM}\right] \quad ,$$

and the following boundary conditions are satisfied:

$$M_\Omega(t) = 0$$

$$M_\Omega(\infty) = \partial M^*/\partial\Omega > 0$$

$$Y_\Omega(\infty) = 1 \ .$$

The locus of $\dot{M}_\Omega = 0$ is the horizontal line, $Y_\Omega = 1$, and the locus of $\dot{Y}_\Omega$ is upward sloping due to the normality of money balance. The phase diagram is given in Figure 5. Since (M_Ω, Y_Ω) must converge to $(\partial M^*/\partial\Omega, 1)$, the path of (M_Ω, Y_Ω) must be one of the stable branches. But since $M_\Omega = 0$ at time t, $Y_\Omega(t)$ is the value of Y_Ω at the intersection point between the vertical axis and a stable branch. Hence,

$$Y_\Omega(t) = Y_\Omega(\Omega + \Pi, M, P) < 1 \ .$$

Second, consider a change in M. The variational differential equations with respect to M(t) is

$$\dot{M}_M = - Y_M$$

$$\dot{Y}_M = - \frac{1}{v_{YY}} \left[(v_{MM} - rv_{YM}) M_M - rv_{YY} Y_M \right] ,$$

where

$$M_M(t) = 1$$

$$M_M(\infty) = 0$$

$$Y_M(\infty) = 0 \ .$$

As in Figure 6, the locus of $\dot{M}_M$ is the M_M axis and the locus of $\dot{Y}_M$ is upward sloping. Again, the path of (M_Ω, Y_Ω) must be one of the stable

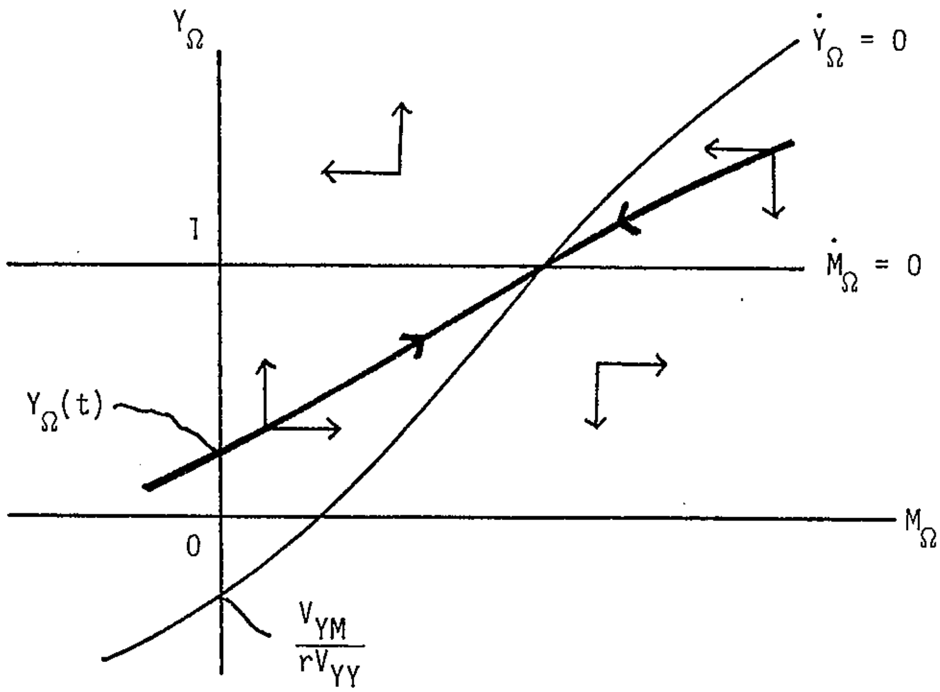

Figure 5 Comparative dynamics with respect to $\Omega(t)$

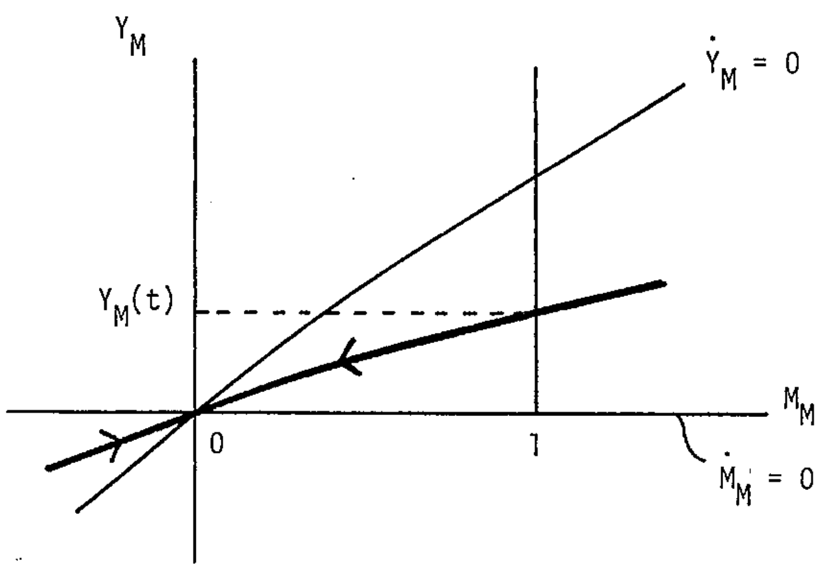

Figure 6 Comparative dynamics with respect to M(t)

branches and since $M_M(t) = 1$, $Y_M(t)$ must be positive. Hence,

$$Y_M(\Omega + \Pi, M, P) > 0 \ .$$

Finally, the variational differential equations with respect to p_i are

$$\dot{M}_p = - Y_p$$

$$\dot{Y}_p = - \frac{1}{v_{YY}} \left[(v_{MM} - rv_{YM}) M_p - rv_{YY} Y_p + v_{Mp_i} - rv_{Yp_i} \right] \ ,$$

where

$$M_p(t) = 0$$

$$M_p(\infty) = \partial M^* / \partial p_i$$

$$Y_p(\infty) = 0 \ .$$

From Figure 7, it can be seen that if $\partial M^*/\partial p_i = 0$, then $Y_p(t) = 0$. Since

$$\frac{\partial M^*}{\partial p_i} = \frac{rv_{Yp_i} - v_{MP_i}}{v_{MM} - rv_{YM}} \ ,$$

it suffices to show that

$$v_M v_{Yp_i} - v_Y v_{Mp_i} = 0 \ . \qquad \ldots . (A.4)$$

The utility function (1.1) can be rewritten as

$$u[y(C), m] \ ,$$

with a linearly homogeneous function $y(C)$. The indirect utility function satisfies

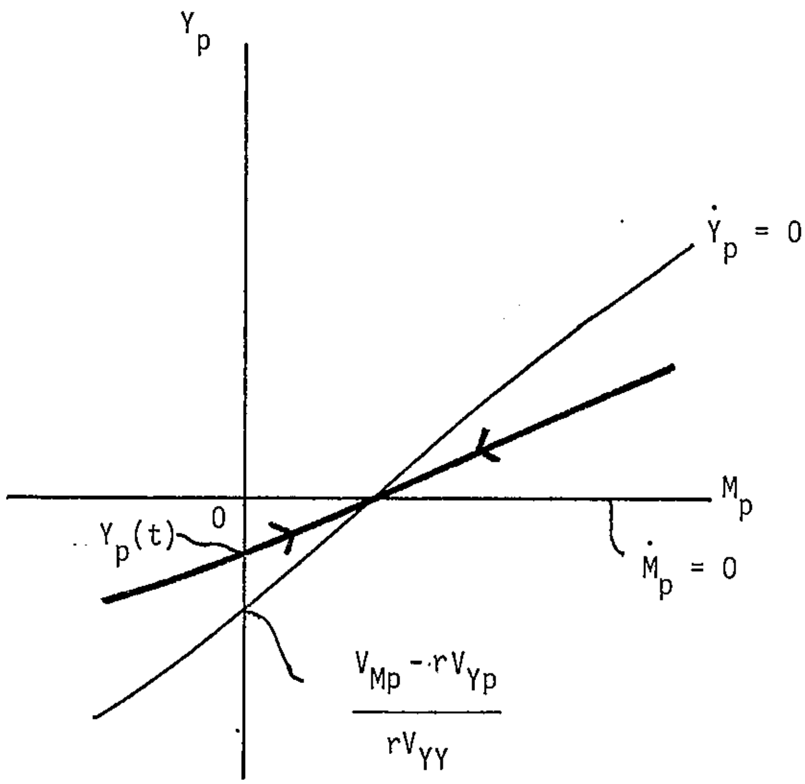

Figure 7 Comparative dynamics with respect to p(t)

$$v[Y,P,M/\bar{p}] \;=\; u[y(C(Y,P)),\, M/\bar{p}] \quad . \qquad \text{. . . .(A.5)}$$

If $p_i = p$ for all i, then from (1.10) and the linear homogeneity of y(C),

$$v[Y,P,M/\bar{p}] = u[Y/p,M/p] \quad . \qquad \text{. . . .(A.6)}$$

Now, differentiating (A.5) with respect to p_i yields

$$v_{Mp_i} \;=\; \frac{\partial}{\partial p_i}\left\{ \frac{1}{p}\, u_m[y(C(Y,P)),\, M/\bar{p}]\right\} \quad . \qquad \text{. . . .(A.7)}$$

But, if $p_i = p$ for all i, then

$$\frac{\partial}{\partial p}\left\{ \frac{1}{p}\, u_m[y(C(Y,p1_k)),\, M/p]\right\}$$

$$= \sum_i \frac{\partial}{\partial p_i}\,[\frac{1}{\bar{p}}\, u_m] \;=\; k\,\frac{\partial}{\partial p_i}\,[\frac{1}{\bar{p}}\, u_m] \quad .$$

Hence, (A.7) becomes

$$v_{Mp_i} \;=\; \frac{1}{k}\,\frac{\partial}{\partial p}\,[\frac{1}{p}\, u_m(Y/p,M/p)]$$

$$= -\,\frac{1}{kp^2}\,[u_m + yu_{my} + mu_{mm}] \quad .$$

In the same way, if $p_i = p$ for all i, then

$$v_{Yp_i} \;=\; \frac{\partial}{\partial p_i}\,[u_y\,\frac{\partial y}{\partial Y}]$$

$$= \frac{1}{k}\,\frac{\partial}{\partial p}\,[\frac{1}{p}\, u_y(Y/p,M/p)]$$

$$= -\,\frac{1}{kp^2}\,[u_y + yu_{yy} + mu_{ym}] \quad .$$

Since $v_Y = u_y/p$, $v_M = u_m/p$, (A.4) becomes

$$v_Y v_{M_{p_i}} - v_M v_{Y p_i} = \frac{1}{kp^3}[y(u_{yy} - u_{my}) + m(u_{ym} - u_{mm})] \quad .$$

$$= 0$$

where the last equality results from the homotheticity of u(y,m).

Lemma 3. In the neighbourhood of the optimal steady state, x^*, the optimal path can be approximated by

$$\dot{x} = (A + BE)(x - x^*) \quad ,$$

where $E = \{e_{ij}\}$ is a 3×3 symmetric, negative definite matrix satisfying

$$(A^T - rI)E + EA + EBE = 0 \quad .$$

Proof.

Let $v \equiv (\nu_p, \nu_n)^T$ and define

$$F(x) \equiv pc(p) - \bar{w}n$$

$$G(x,\nu) \equiv \begin{pmatrix} \nu_p \\ \nu_n \\ f(n) - c(p) - g^p(\nu_p) - g^z(z, c(p)) \end{pmatrix} \quad .$$

Then (2.4), (2.6) and (2.7) can be rewritten as

$$\dot{x} = G(x,\nu) \quad , \qquad \text{....(A.8)}$$

and the Hamiltonian (2.8) as

$$H(x,\nu,q) \;=\; F(x) + q^T G(x,\nu) \quad .$$

Define the value function:

$$J(x) \;\equiv\; \max_{\{u(s)\}} \left\{ \int_t^\infty F(x) e^{-r(s-t)} ds : \dot{x} = G(x,\nu),\ x(t) = x \right\} \quad .$$

Then along the optimal path, the vector of costate variables, q, satisfies

$$q(s) \;=\; J_x^T\,\big(x(s)\big) \quad , \qquad \ldots . (A.9)$$

where

$$J_x \;\equiv\; (J_p, J_n, J_z) \;\equiv\; (\partial J/\partial p,\ \partial J/\partial n,\ \partial J/\partial z) \quad .$$

From (2.9 d.e), $\nu_p(s)$ and $\nu_n(s)$ can be expressed as functions of q(s) and hence of $J_x\big(x(s)\big)$:

$$\nu(s) \;=\; \nu\Big[J_x\big(x(s)\big)\Big] \;\equiv\; \nu*\big(x(s)\big) \quad . \qquad \ldots . (A.10)$$

It is easy to see that $\nu_x^*(x)$, the Jacobian matrix of $\nu^*(x)$, satisfies

$$\nu_x^*(x) \;=\; BJ_{xx} \quad , \qquad \ldots . (A.11)$$

where B is defined by (2.14d) and J_{xx} is the Hessian matrix of J(x).

Substituting (A.10) into (A.8) yields the optimal path of x:

$$\dot{x} \;=\; G\big(x,\ \nu^*(x)\big) \quad . \qquad \ldots . (A.12)$$

Noting (A.11), the linear approximation of this differential equation is

$$\dot{x} = \left(A + BJ_{xx}(x^*)\right)(x - x^*) , \qquad \ldots .(A.13)$$

where A is defined by (2.14c).

Now, it is shown that $E \equiv J_{xx}(x^*)$ satisfies the algebraic Ricatti Equation (2.17). Substituting (A.9) into (2.9a-c) yields the Hamilton-Jacobi equation:

$$\frac{d}{ds} J_x(x) = -H_x\left(x, \nu^*(x), J_x(x)\right) + rJ_x(x) , \qquad \ldots .(A.14)$$

where

$$H_x = (\partial H/\partial p,\ \partial H/\partial n,\ \partial H/\partial z) .$$

Noting that the left hand side of (A.14) is equal to $J_{xx}^T G\left(x, \nu^*(x)\right)$, differentiation of (A.14) with respect to x yields

$$\frac{d}{ds} J_{xx} = rJ_{xx} - H_{xx} - H_{x\nu}\, \nu_x^* - H_{xq} J_{xx}$$

$$- J_{xx}(G_x + G_\nu\, \nu_x^*) . \qquad \ldots .(A.15)$$

At the steady state, we have $(d/ds)J_{xx} = 0$, $H_{xx} = -D$, $H_{x\nu} = 0$, $J_{xq} = A^T$, $G_x + G_\nu \nu_x^* = A + BJ_{xx}$. Hence,

$$(A^T - rI)E + EA + EBE = D .$$

Finally, it is shown that E is negative definite. Our assumptions imply that F and the right hand side of (2.4) are concave and since the right hand sides of (2.6) and (2.7) are linear and the shadow price of z, q_z, is positive at the steady state, Theorem 1 of Long (1979) can be applied to show that J(x) is concave near $x = x^*$. Using the fact that (2.4) is strictly concave in (x,ν) and F(x) is strictly concave in (p,n), Long's Theorem can be strengthened to prove the strict concavity of J(x) at $x = x^*$. This implies that $E = J_{xx}(x^*)$ is negative definite.

Lemma 4. Matrix E satisfying (2.17) is diagonal only if $\alpha_z = 0$.

Proof.

The matrix Equation (2.17) yields

$$\alpha_p e_{11}^2 + \alpha_n e_{12}^2 - re_{11} - 2c_p e_{13} = d_1 \qquad \text{....(A.16a)}$$

$$\alpha_p e_{22}^2 + \alpha_n e_{22}^2 - re_{22} + 2f' e_{23} = d_2 \qquad \text{....(A.16b)}$$

$$\alpha_p e_{13}^2 + \alpha_n e_{23}^2 - re_{33} = \alpha_z \qquad \text{....(A.16c)}$$

$$\alpha_p e_{11}e_{12} + \alpha_n e_{22}e_{12} + f' e_{13} - c_p e_{23} - re_{12} = 0 \qquad \text{....(A.16d)}$$

$$\alpha_p e_{12}e_{13} + \alpha_n e_{22}e_{23} + f' e_{33} - re_{23} = 0 \qquad \text{....(A.16e)}$$

$$\alpha_p e_{13}e_{11} + \alpha_n e_{12}e_{23} - c_p e_{33} - re_{13} = d_3 \equiv \alpha_z d \ , \qquad \text{....(A.16f)}$$

where

$$d \equiv \gamma_z \frac{c}{\eta p} \quad .$$

If E is diagonal, then $e_{12} = e_{13} = e_{23} = 0$ and (A.16e) reduces to

$$e_{33} = 0 \ .$$

Hence (A.16c) implies

$$\alpha_z = 0 \ .$$

<u>Lemma 5</u>. If $\alpha_z = 0$, then the negative semi-definite solution of (2.17) is diagonal.

<u>Proof</u>.

If E is negative semi-definite, then e_{33} is nonpositive. Hence (A.16c) yields

$$e_{13} = e_{23} = e_{33} = 0 \ ,$$

so that (A.16d) becomes

$$(\alpha_p e_{11} + \alpha_n e_{22} - r)\, e_{12} = 0 \ .$$

Since $e_{11} \leqq 0$, $e_{22} \leqq 0$, it follows that

$$e_{12} = 0 \ .$$

<u>Proposition 1</u>. Near the optimal steady state, x*, the optimal path of x can be approximated by

$$\dot{x} = G(x - x^*) \ ,$$

where

$$G = A + BE$$

$$= \begin{pmatrix} \alpha_p e_{11}, & \alpha_p e_{12}, & \alpha_p e_{13} \\ \alpha_n e_{12}, & \alpha_n e_{22}, & \alpha_n e_{23} \\ -c_p(p^*), & f'(n^*), & 0 \end{pmatrix} ,$$

and $e_{11} < 0$, $e_{22} < 0$, $(e_{12})^2 < e_{11}e_{22}$: If α_z is small, then

$$e_{ij} < 0 \qquad \text{for any } i \text{ and } j ,$$

and e_{12}, e_{13}, and e_{23} are small compared with e_{11} and e_{22} in absolute value.

Proof.

It suffices to prove the second statement.

Fix all parameters except α_z and write the solution of (2.17) as a function of α_z : $E(\alpha_z)$. Since $E(\alpha_z)$ is negative definite if $\alpha_z > 0$, the limit of E as α_z tends to zero,

$$E(0^+) = \lim_{\alpha_z \downarrow 0} E(\alpha_z) ,$$

is negative semi-definite. Observing that by Lemma 5, $e_{12} = e_{13} = e_{23} = e_{33} = 0$ at $\alpha_z = 0^+$, expansion of (A.16) shows that $dE(\alpha_z)/d\alpha_z$ at $\alpha_z = 0^+$ satisfies

$$\frac{de_{33}}{d\alpha_z} = -\frac{1}{r} < 0 ,$$

$$\frac{de_{13}}{d\alpha_z} = \frac{d + c_p de_{33}/d\alpha_z}{\alpha_p e_{11} - r} < 0 \quad ,$$

$$\frac{de_{23}}{d\alpha_z} = \frac{-f' de_{33}/d\alpha_z}{\alpha_n e_{22} - r} < 0 \quad ,$$

$$\frac{de_{12}}{d\alpha_z} = \frac{c_p de_{23}/d\alpha_z - f' de_{13}/d\alpha_z}{\alpha_p e_{11} + \alpha_n e_{22} - r} < 0 \quad ,$$

$$\frac{de_{11}}{d\alpha_z} = \frac{2c_p de_{13}/d\alpha_z}{2\alpha_p e_{11} - r} < 0 \quad ,$$

$$\frac{de_{22}}{d\alpha_z} = \frac{-2f' de_{23}/d\alpha_z}{2\alpha_n e_{22} - r} < 0 \quad .$$

The proposition now follows.

Lemma 6. The eigenvalues of Q are

(i) the eigenvalues of $G(I + L)$ with multiplicity k-1, and

(ii) the eigenvalues of $G[I - (k-1)L]$ with multiplicity 1.

Proof:[3]

Denote an eigenvalue of Q by λ. Then

$$| Q - \lambda I | = 0 ,$$

where I is an identify matrix. The matrix $Q - \lambda I$ can be partitioned into $k \times k$ blocks of 3×3 matrices, where all diagonal blocks are $G - \lambda I$ and all off-diagonal blocks are -GL. Subtract the k^{th} column blocks from other column blocks and then, to the k^{th} row block, add all other row blocks. This operation does not change the determinant and yields

$$|Q - \lambda I| = \begin{vmatrix} G(I+L)-\lambda I, & 0 & -GL \\ & G(I+L)-\lambda I, & -GL \\ \vdots & 0 \quad \ddots & \vdots \\ 0 & 0 & G(I-(k-1)L)-\lambda I \end{vmatrix}$$

$$= | G(I+L)-\lambda I|^{k-1}|G(I-(k-1)L)-\lambda I|$$

$$= 0 .$$

The Lemma now follows.

Proposition 3. The monopolistic economy is locally stable if the number of firms, k, is large and if the curvature of the inventory cost function, g^z_{zz}, is small.

Proof.

It suffices to prove that if α_z is small, the matrix GS is stable. The eigenvalues of GS satisfy the characteristic equation,

$$|GS - \lambda I| = -\lambda^3 - \beta_1\lambda^2 - \beta_2\lambda - \beta_2 = 0 ,$$

where

$$\beta_1 = -\alpha_p[e_{11}s_{11} + e_{12}s_{21} + e_{13}s_{31}] - \alpha_n e_{22}$$

$$\beta_2 = \alpha_p\alpha_n \left\{e_{11}e_{22}s_{11} - [e_{12}^2 s_{11} - (e_{22}e_{13} - e_{12}e_{23})s_{31}]\right\}$$

$$-\alpha_p e_{13}(-c_p s_{11} + f's_{21}) - \alpha_n e_{23}f'$$

$$\beta_3 = \alpha_p\alpha_n s_{11}[e_{23}(f'e_{11} + c_p e_{12}) + e_{13}(-c_p e_{22} - f'e_{12})].$$

It is easy to check that the Routh-Hurwitz condition for stability:

$$\beta_1 > 0, \quad \beta_2 > 0, \quad \beta_3 > 0$$

$$\beta_1\beta_2 > \beta_3 \quad ,$$

is satisfied if e_{12}, e_{13}, and e_{23} are small relative to e_{11} and e_{22}.