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JAPAN

Abstract

Let E be a finite set, R be the set of real numbers and $f: 2^E \to R$ be a symmetric submodular function. The pair (E,f) is called a symmetric submodular system. We examine the structures of symmetric submodular systems and provide a decomposition theory of symmetric submodular systems. The theory is a generalization of the decomposition theory of 2-connected graphs developed by Tutte.

1. Introduction

A decomposition theory of graphs is developed by Tutte [9]. A connected graph G is decomposed into a set of 2-connected subgraphs of G and the incidence relation of these 2-connected subgraphs is represented by a tree. Moreover, a 2-connected graph G is decomposed into a set of 3-connected graphs, bonds and polygons, and their structural relation is represented by a tree. Also Gomory and Hu [7] derived a tree structure of the set of minimum cuts of a capacitated undirected (or symmetric) multi-terminal network. In extracting these tree structures, symmetric submodular functions play a crucial role. Related tree representation of a collection of sets was examined by Edmonds and Giles [4].

Let E be a finite set and $f: 2^E \rightarrow R$ be a symmetric submodular function, whose precise definition will be given in Section 2. The pair (E,f) is called a symmetric submodular system. We shall consider symmetric submodular systems and provide a theory of decomposition of symmetric submodular systems, which is a generalization of the decomposition theory of 2-connected graphs by Tutte [9]. The decomposition theory can be applied to any systems with submodular functions such as graphs [9], capacitated networks [7], matroids [10], communication networks [5] etc., where if necessary the underlying submodular functions should be symmetrized (see Section 5).

2. Definitions and Assumptions

Let E be a finite set, R be the set of real numbers and $f: 2^E \to R$ be a submodular function, i.e.,

$$f(A) + f(B) \geqq f(A \cup B) + f(A \cap B) \qquad (2.1)$$

for any $A, B \subseteq E$. The pair (E,f) is called a submodular system [6] and if the submodular function f is symmetric, i.e.,

$$f(A) = f(E - A) \qquad (2.2)$$

for any $A \subseteq E$, then (E,f) is called a symmetric submodular system. If $C \subseteq E$ satisfies $|C| \geqq k$ and $|E - C| \geqq k$ for a positive integer k, we call C a k-cut of (E,f). Let $e_A \notin E$ be a new element corresponding to a nonempty subset A of E and define

$$E' = (E - A) \cup \{e_A\}, \qquad (2.3)$$

$$f'(B) = f(B) \qquad \text{if } e_A \notin B \subseteq E', \qquad (2.4a)$$

$$= f((B - \{e_A\}) \cup A) \qquad \text{if } e_A \in B \subseteq E'. \qquad (2.4b)$$

Then we call the submodular system (E',f') an aggregation of (E,f) by A and we denote it by $(E,f)//A$. A partition $P \equiv \{A_0,A_1,\cdots,A_k\}$ of E is proper if and only if, for each $i = 0, 1, \cdots, k$, A_i is nonempty. For a proper partition $P \equiv \{A_0,A_1,\cdots,A_k\}$ let us define

$$(E,f)//P = (\cdots(((E,f)//A_0)//A_1)\cdots)//A_k. \qquad (2.5)$$

Note that $(E,f)//P$ does not depend on the order of the A_i's in (2.5). If subsets C_1 and C_2 of E satisfy $C_1 \cup C_2 \neq E$, $C_1 \cap C_2 \neq \emptyset$, $C_1 - C_2 \neq \emptyset$ and $C_2 - C_1 \neq \emptyset$, then we say C_1 and C_2 cross. We define a partial order $\preceq$ on the set of partitions of E as follows. For partitions P and P^* of E, $P \preceq P^*$ if and only if for each

$A \in P$ there is an element $A^* \in P^*$ such that $A \subseteqq A^*$.

Throughout the present paper, we assume that (E,f) is a symmetric submodular system and

$$\min\{f(C) \mid C \text{ is a 1-cut of } (E,f)\} = \lambda^*. \tag{2.6}$$

We denote by $\mathcal{C}_f$ the set of 2-cuts C such that $f(C) = \lambda^*$. We shall examine the structure of the set $\mathcal{C}_f$ and decompose (E,f) based on $\mathcal{C}_f$.

3. Main Theorems

The following lemma is fundamental for the symmetric submodular system (E,f) satisfying (2.6).

Lemma 1: Suppose that C_1 and C_2 of E cross and satisfy

$$f(C_1) = f(C_2) = \lambda^*. \tag{3.1}$$

Then we have

$$f(C_1 \cup C_2) = f(C_1 \cap C_2) = f(C_1 - C_2) = f(C_2 - C_1) = \lambda^*. \tag{3.2}$$

(Proof) Since

$$f(C_1) + f(C_2) \geqq f(C_1 \cup C_2) + f(C_1 \cap C_2) \tag{3.3}$$

and C_1 and C_2 cross, we have from (2.6)

$$f(C_1 \cup C_2) = f(C_1 \cap C_2) = \lambda^*. \tag{3.4}$$

Because of the symmetry of f, Lemma 1 follows from (3.4). Q.E.D.

Lemma 2: Let e_1, e_2, e_3 and e_4 be four distinct elements of E. If $\{e_1,e_2\}$, $\{e_1,e_3\}$, $\{e_1,e_4\} \in \mathcal{C}_f$, then $\{e_2,e_3\}$, $\{e_2,e_4\}$, $\{e_3,e_4\} \in \mathcal{C}_f$.

(Proof) Since $\{e_1,e_2\}$ and $\{e_1,e_3\}$ in $\mathcal{C}_f$ cross, we have from Lemma 1

$$f(\{e_1,e_2,e_3\}) = \lambda^*. \tag{3.5}$$

If $E = \{e_1,e_2,e_3,e_4\}$, then $\{e_2,e_3\} = E - \{e_1,e_4\} \in \mathcal{C}_f$. Therefore, suppose $E \neq \{e_1,e_2,e_3,e_4\}$. Then, since $\{e_1,e_2,e_3\}$ and $\{e_1,e_4\}$ cross, we have from (3.5) and Lemma 1

$$\{e_2,e_3\} = \{e_1,e_2,e_3\} - \{e_1,e_4\} \in \mathcal{C}_f. \tag{3.6}$$

Because of the symmetry among the elements e_2, e_3 and e_4, this completes the proof of Lemma 2. Q.E.D.

Now, let us define a relation $R_f \subseteqq E \times E$ by

$$R_f = \{C \mid C \in \mathcal{C}_f, |C|=2\}. \tag{3.7}$$

<u>Theorem 1</u>: Let $G = (E,R_f)$ be a graph with the vertex set E and the edge set R_f. If G is connected, then G is a complete graph or an elementary closed path.

(Proof) By definition, connectedness of G implies that $|E| = 1$ or $|E| \geqq 4$ and thus we assume $|E| \geqq 4$. It follows from Lemma 2 that G can be a complete graph, an elementary closed path or an elementary nonclosed path. Therefore, let us assume that $E = \{e_1,e_2, \cdots ,e_n\}$ $(n \geqq 4)$ and that $\{e_i,e_{i+1}\} \in \mathcal{C}_f$ $(i=1,2,\cdots,n-1)$. Then $\{e_1,e_n\}$ must be in $\mathcal{C}_f$ because from Lemma 1 we have $\{e_2,e_3,\cdots,e_{n-1}\} \in \mathcal{C}_f$. Consequently, G cannot be an elementary nonclosed path. Q.E.D.

If the graph $G = (E,R_f)$ has at least four edges and is a complete graph (resp. an elementary closed path), then we say (E,f) is <u>of bond type</u> (resp. <u>of polygon type</u>). We call (E,f) <u>irreducible</u> if $\mathcal{C}_f$ is empty or

(E,f) is of bond type or of polygon type. In particular, if $\mathcal{C}_f$ is empty, we call (E,f) absolutely irreducible.

Suppose that, for $e^* \varepsilon E$, a proper partition $P(e^*) \equiv \{\{e^*\},A_1, A_2,\cdots,A_k\}$ of E satisfies

(i) $(E,f)//P(e^*)$ is irreducible,

(ii) for each $i = 1, 2, \cdots, k$, if $|A_i| \geqq 2$, then $A_i \varepsilon \mathcal{C}_f$.

Then $P(e^*)$ is called an irreducibility partition associated with $e^* \varepsilon E$. Let us denote by $\mathcal{P}(e^*)$ the set of all irreducibility partitions associated with $e^* \varepsilon E$. Note that $\mathcal{P}(e^*)$ is nonempty for every $e^* \varepsilon E$.

For proper partitions P and P' of E given by $P = \{A_0,A_1, \cdots,A_k\}$ and $P' = \{A_0',A_1',\cdots,A_h'\}$, let us define a proper partition $P \wedge P'$ of E by

$$P \wedge P' = \{A_i \cap A_j' \mid i=0,1,\cdots,k;j=0,1,\cdots,h;A_i \cap A_j' \neq \emptyset\}. \quad (3.8)$$

We shall show Theorems 2 - 5 from which follows the fact that, for every $e^* \varepsilon E$, $\mathcal{P}(e^*)$ is closed with respect to the operation $\wedge$ (Theorem 6). We need some preliminary lemmas.

Lemma 3: Suppose $P \equiv \{A_0,A_1,\cdots,A_k\}$ $(k \geqq 4)$ is a proper partition of E and define

$$A_\ell^* = \cup\{A_j \mid j=\ell,\ell+1,\cdots,k\} \quad (3.9)$$

and

$$P' = \{A_0,A_1,\cdots,A_{\ell-1},A_\ell^*\}, \quad (3.10)$$

where $3 \leqq \ell < k$. Then the following (i) and (ii) hold.

(i) If $(E,f)//P$ is of polygon type and $f(A_i \cup A_{i+1}) = \lambda^*$ $(i=0,1,\cdots,k)$, where $A_{k+1} = A_0$, then $(E,f)//P'$ is also of polygon type and $f(A_{\ell-1} \cup A_\ell^*) = f(A_\ell^* \cup A_0) = \lambda^*$.

(ii) If $(E,f)//P$ is of bond type, then $(E,f)//P'$ is also of bond type.

(Proof) From Lemma 1 we have $f(A_\ell^*) = \lambda^*$ and $f(A_{\ell-1} \cup A_\ell^*) = \lambda^*$. Because of the assumption and Theorem 1 this implies that $(E,f)//P'$ is of bond type or of polygon type according as $(E,f)//P$ is of bond type or of polygon type. The remaining part of (i) easily follows from Theorem 1. Q.E.D.

Lemma 4: Suppose $P \equiv \{A_0, A_1, \cdots, A_k\}$ $(k \geqq 3)$ is a proper partition of E such that $(E',f') \equiv (E,f)//P$ is of polygon type and that $f(A_i \cup A_{i+1}) = \lambda^*$ $(i=0,1,\cdots,k)$, where $A_{k+1} = A_0$. Also suppose $B \in \mathcal{C}_f$ and $A_0 \cap B = \emptyset$. Furthermore, define

$$J = \{j \mid j=1,2,\cdots,k;\ A_j \cap B \neq \emptyset\}. \tag{3.11}$$

Then, if there exists an integer i^* such that $\min J < i^* < \max J$, we have $A_{i^*} \subseteqq B$, where $\min J$ (resp. $\max J$) denotes the minimum (resp. maximum) number of J.

(Proof) Suppose there were an integer i^* such that $\min J < i^* < \max J$ and $A_{i^*} - B \neq \emptyset$. Put

$$J_1 = \{j \mid j \in J, j < i^*\}, \tag{3.12}$$

$$J_2 = \{j \mid j \in J, j > i^*\}. \tag{3.13}$$

Also define

$$A_1^* = \cup\{A_j \mid \min J_1 \leqq j \leqq \max J_2\}, \tag{3.14}$$

$$A_2^* = \cup\{A_j \mid \min J_2 \leqq j \leqq \max J_2\}. \tag{3.15}$$

It follows from Lemma 3 that the aggregation $(E',f') \equiv (E,f)//P'$ is of polygon type, where

$$P' = (P - \{A_j \mid A_j \subseteqq A_1^* \cup A_2^*;\ j=1,2,\cdots,k\}) \cup \{A_1^*, A_2^*\}. \quad (3.16)$$

Furthermore, we have $f(B^*) = \lambda^*$ for $B^* = B - A_{i^*}$ because of Lemma 1. Therefore, we have from Lemma 1 and the definition of A_1^* and A_2^*

$$f(A_1^* \cup A_2^*) = f((A_1^* \cup B^*) \cup (A_2^* \cup B^*)) = \lambda^*. \quad (3.17)$$

This contradicts the assertion that (E',f') is of polygon type and $f(A_0 \cup A_{i^*}) \neq \lambda^*$. Q.E.D.

Lemma 5: Under the assumption of Lemma 4, if B and some A_{j^*} cross and $j^* = \min J$, then $(E,f)//P'$ is of polygon type, where

$$P' = \{A_0, A_1, \cdots, A_{j^*-1}, A_{j^*} - B, A_{j^*} \cap B, A_{j^*+1}, \cdots, A_k\}. \quad (3.18)$$

Furthermore, we have

$$f(A_{j^*-1} \cup (A_{j^*} - B)) = f((A_{j^*} \cap B) \cup A_{j^*+1}) = \lambda^*. \quad (3.19)$$

(Proof) Since $A_{j^*-1} \cup A_{j^*}$ and B cross and $A_{j^*-1} \cap B = \emptyset$, we have $f(A_{j^*-1} \cup (A_{j^*} - B)) = \lambda^*$ and $f(A_{j^*} \cap B) = \lambda^*$. Therefore, from the assumption and Theorem 1 $(E,f)//P'$ must be of polygon type and the remaining part follows. Q.E.D.

Theorem 2: Suppose $P, P' \in \mathcal{P}(e^*)$ and $|P| \geqq 4$. If $(E,f)//P$ is of polygon type, then $(E,f)//P \wedge P'$ is of polygon type and, therefore, $P \wedge P' \in \mathcal{P}(e^*)$. Moreover, if $|P'| \geqq 4$, $(E,f)//P'$ is also of polygon type.

(Proof) Suppose $P = \{\{e^*\} \equiv A_0, A_1, \cdots, A_k\}$ $(k \geqq 3)$ and $P' = \{\{e^*\} \equiv A_0', A_1', \cdots, A_h'\}$. If $A_i \in P$ and $A_j' \in P'$ cross, then for the proper partition P_1 obtained from P by dividing A_i into $A_i - A_j'$ and

$A_i \cap A_j'$, $(E,f)//P_1$ is irreducible and of polygon type due to Lemma 5. By repeating this process we obtain a proper partition $P^* \equiv \{\{e^*\} \equiv A_0^*, A_1^*, \cdots, A_{k^*}^*\}$ which is maximal, with respect to $\leqq$, with the property: " $P^* \leqq P$ and A_i^* and A_j' do not cross for any $A_i^* \in P^*$ and $A_j' \in P'$." The obtained $(E,f)//P^*$ is irreducible and of polygon type.

If there is no A_i^* in P^* such that A_i^* contains at least two A_j' 's, then $P^* = P \wedge P'$ and this completes the proof. Therefore, suppose that some $A_{i_0}^*$ is expressed as $A_{i_0}^* = \bigcup \{A_j' \mid j = t_1, t_2, \cdots, t_p\}$ $(p \geqq 2)$. Because of the way of constructing P^*, $A_{i_0}^*$ belongs to P, so that $f(A_{i_0}^*) = \lambda^*$. It follows that $(E,f)//P'$ must be of polygon type or of bond type. In either case, from Theorem 1, for some $j^* \in \{t_1, t_2, \cdots, t_p\}$ and some $j' \in \{0,1,\cdots,h\} - \{t_1, t_2, \cdots, t_p\}$ there hold $f(A_{j^*}' \cup A_{j'}') = \lambda^*$. Therefore, since $A_{i_0}^*$ and $A_{j^*}' \cup A_{j'}'$ cross, we see from Lemma 5 that $(E,f)//P_1^*$ is of polygon type, where P_1^* is the proper partition of E obtained from P^* by dividing $A_{i_0}^*$ into $A_{i_0}^* \cap (A_{j^*}' \cup A_{j'}') = A_{j^*}'$ and $A_{i_0}^* - (A_{j^*}' \cup A_{j'}') = A_{i_0}^* - A_{j^*}'$. By repeating this process we reach the partition $P \wedge P'$ for which $(E,f)//P \wedge P'$ is of polygon type.

Moreover, since $P \wedge P' \leqq P'$, if $|P'| \geqq 4$, then $(E,f)//P'$ is of polygon type due to Lemma 3. Q.E.D.

<u>Lemma 6</u>: Suppose $P \equiv \{A_0, A_1, \cdots, A_k\}$ $(k \geqq 3)$ is a proper partition of E and $(E',f') \equiv (E,f)//P$ is of bond type. Also suppose $B \in \mathcal{C}_f$ and $A_{j^*} \in P$ cross and $A_0 \cap B = \emptyset$. Then $(E,f)//P'$ is of bond type, where $P' = \{A_0, A_1, \cdots, A_{j^*-1}, A_{j^*} - B, A_{j^*} \cap B, A_{j^*+1}, \cdots, A_k\}$.

(Proof) Since B and A_{j*} cross, there is an $A_{i*} \varepsilon P$ such that $A_{i*} \cap B \neq \emptyset$ and $i^* \neq 0, j^*$. Put $B^* = A_{i*} \cup B$. Then we have $f(A_{i*} \cup B) = \lambda^*$. Since B and A_{j*} cross and B^* and $A_{i*} \cup A_{j*}$ cross, we get

$$f(A_{j*} \cap B) = f(A_{j*} - B) = f(A_{i*} \cup (A_{j*} \cap B)) = \lambda^*. \qquad (3.20)$$

From (3.20) and Theorem 1 we see that $(E,f)//P'$ is of bond type.

Q.E.D.

<u>Theorem 3</u>: Suppose $P, P' \varepsilon \mathcal{P}(e^*)$ and $|P| \geqq 4$. If $(E,f)//P$ is of bond type, then $(E,f)//P \wedge P'$ is of bond type and, therefore, $P \wedge P' \varepsilon \mathcal{P}(e^*)$. Moreover, if $|P'| \geqq 4$, $(E,f)//P'$ is also of bond type.

(Proof) Theorem 3 can be shown by using Lemmas 3 and 6 and Theorem 1 in a way similar to the proof of Theorem 2. Q.E.D.

<u>Theorem 4</u>: Suppose $e^* \varepsilon E$, $P = \{\{e^*\}, A_1, A_2\} \varepsilon \mathcal{P}(e^*)$ and $P' = \{\{e^*\}, A_1', A_2'\} \varepsilon \mathcal{P}(e^*)$. Then $P \wedge P' \varepsilon \mathcal{P}(e^*)$. If $|P| = 3$ for any $P \varepsilon \mathcal{P}(e^*)$, then $|\mathcal{P}(e^*)| = 1$.

(Proof) Suppose $P \neq P'$.

First, suppose $A_1 \subsetneqq A_1'$. Then $|A_2| \geqq 2$ and $f(\{e^*\} \cup A_1) = f(E - A_2) = \lambda^*$. Therefore, for the partition $P \wedge P' = \{\{e^*\}, A_1, A_2 \cap A_1', A_2 - A_1'\}$, $(E,f)//P \wedge P'$ is of bond type or of polygon type and $P \wedge P' \varepsilon \mathcal{P}(e^*)$.

Next, suppose A_1' and A_1 cross and A_1' and A_2 cross. Then $f(\{e^*\} \cup (A_1 - A_1')) = f(A_1 \cap A_1') = f(A_2 \cap A_1') = f(A_2 - A_1') = \lambda^*$. It follows that, for $P \wedge P' = \{\{e^*\}, A_1 - A_1', A_1 \cap A_1', A_2 \cap A_1', A_2 - A_1'\}$, $(E,f)//P \wedge P'$ is of bond type or of polygon type and $P \wedge P' \varepsilon \mathcal{P}(e^*)$.

The remaining part of the theorem follows from the fact that, if $P, P' \varepsilon \mathcal{P}(e^*)$, $P \neq P'$ and $|P| = |P'| = 3$, then $|P \wedge P'| \geqq 4$. Q.E.D.

Lemma 7: Suppose that $P \equiv \{A_0, A_1, \cdots, A_k\}$ $(k \geqq 3)$ is a proper partition of E and that $(E,f)//P$ is absolutely irreducible. Then, for any $B \in \mathcal{C}_f$ such that $A_0 \cap B = \emptyset$, B and any of $A_1, \cdots, A_k$ do not cross.

(Proof) Suppose B and A_1 cross. Let us define

$$I = \{i \mid A_i \cap B \neq \emptyset,\ i=1,2,\cdots,k\}. \tag{3.21}$$

Then $|I| \geqq 2$ and, from Lemma 1, $A^* \equiv \bigcup\{A_i \mid i \in I\}$ satisfies $f(A^*) = \lambda^*$. It follows that $I = \{1,2,\cdots,k\}$, since $(E,f)//P$ is absolutely irreducible. Put

$$B^* = (B \cup (\bigcup\{A_i \mid i=2,\cdots,k\})) - A_1. \tag{3.22}$$

From Lemma 1 we have $f(B^*) = \lambda^*$. Consequently, $f(A_0 \cup A_1) = \lambda^*$, since $B^* = E - (A_0 \cup A_1)$. This contradicts the absolute irreducibility of $(E,f)//P$. Q.E.D.

Theorem 5: Suppose that, for some $P \in \mathcal{P}(e^*)$ such that $|P| \geqq 4$, $(E,f)//P$ is absolutely irreducible. Then $|\mathcal{P}(e^*)| = 1$.

(Proof) Suppose $P = \{\{e^*\}, A_1, \cdots, A_k\}$ and there is another $P' = \{\{e^*\}, A_1', \cdots, A_h'\}$ in $\mathcal{P}(e^*)$. It follows from Lemma 7 and the absolute irreducibility of $(E,f)//P$ that each $A_j' \in P'$ is included in some $A_i \in P$. Suppose that, for some distinct indices $j_1, j_2 \in \{1,2,\cdots,h\}$, $A_{j_1}' \cup A_{j_2}'$ is included in some A_i. Then $(E,f)//P'$ must be of bond type or of polygon type. This contradicts Theorem 2 or 3. Therefore, $P = P'$. Q.E.D.

It should be noted that, if $|E| \leqq 3$, (E,f) is absolutely irreducible. Therefore, from Theorems 2 - 5 we have the following.

Theorem 6: For any $e^* \in E$, there is a unique minimal element of the partially ordered set $(\mathcal{P}(e^*), \leqq)$.

Because of Theorem 6, for each $e^* \in E$, we call the unique minimal element of $\mathcal{P}(e^*)$ the minimal irreducibility partition of E associated with e^* and denote it by $\hat{P}(e^*)$. Moreover, we call $A \in \hat{P}(e^*)$ a minimal irreducibility component of (E,f) associated with e^*.

Lemma 8: For $e^*, e \in E$, if the set $\{e\}$ is a minimal irreducibility component of (E,f) associated with e^*, then $\hat{P}(e^*) = \hat{P}(e)$.

(Proof) From the assumption, $\hat{P}(e^*) \in \mathcal{P}(e)$. Therefore, $\hat{P}(e) \leqq \hat{P}(e^*)$ and $\hat{P}(e) \in \mathcal{P}(e^*)$. By the minimality of $\hat{P}(e^*)$, we have $\hat{P}(e^*) = \hat{P}(e)$.

Q.E.D.

Theorem 7: Suppose a set $D \subseteqq E$ is a minimal irreducibility component of (E,f) associated with $e^* \in E$ such that $|D| \geqq 2$. Then, for any $e \in D$, $E - D$ is included in a minimal irreducibility component of (E,f) associated with e.

(Proof) Let $\hat{P}(e^*) = \{\{e^*\}, A_1, \cdots, A_k\}$ and $\hat{P}(e) = \{\{e\}, A_1', \cdots, A_h'\}$, where $e \in A_1 = D$ and $e^* \in A_1'$. Suppose that $A_1 \cup A_1' \neq E$. Then, since from Lemma 8 we have $\{e^*\} \subsetneqq A_1'$ and since from Lemmas 5, 6 and 7 for each $A_j' \in P(e)$ A_j' and any of $A_1, \cdots, A_k$ do not cross, both A_1' and $E - A_1'$ are the union of at least two A_i's of $\hat{P}(e^*)$. Therefore, $(E,f)//\hat{P}(e^*)$ is of bond type or of polygon type. Similarly as the proof of Theorem 2, this contradicts the minimality of $\hat{P}(e^*)$ and $\hat{P}(e)$.

Q.E.D.

4. Canonical Decomposition

Let us define an equivalence relation $\hat{R} \subsetneqq E \times E$ as follows: for e^*, $e \in E$, $\{e^*, e\} \in \hat{R}$ if and only if $\hat{P}(e^*) = \hat{P}(e)$. Let $\Pi \equiv \{S_1, S_2, \cdots, S_p\}$ be the proper partition of E composed of the equivalence classes of E relative to $\hat{R}$. The partition Π is called the canonical 2-cut partition, of level 1, of E. For any $S_j \in \Pi$, define

$$\hat{P}(S_j) = \hat{P}(e) \tag{4.1}$$

for any $e \in S_j$, where note that $\hat{P}(e) = \hat{P}(e')$ for any e, $e' \in S_j$. Each $A \in \hat{P}(S_j)$ such that $|A| \geqq 2$ is called a minimal irreducibility component of (E,f) associated with S_j.

Suppose that, for each $i = 1, 2, \cdots, k$ $(k \geqq 3)$, A_i is a minimal irreducibility component of (E,f) associated with $S_{j(i)} \in \Pi$ and that $P^* \equiv \{E-A_1, E-A_2, \cdots, E-A_k\}$ is a proper partition of E. Then we call the partition P^* a 2-cut aggregation partition, of level 1, of E. Denote by $\mathcal{A}$ the set of 2-cut aggregation partitions, of level 1, of E. Moreover, we call the aggregation $(E,f)//P^*$ a 2-cut aggregation, of level 1, of (E,f) by P^*.

Let $G_1^* = (V_1^*, E_1^*)$ be a graph with a vertex set V_1^* and an edge set E_1^* defined as follows:

$$V_1^* = V_\Pi \cup V_{\mathcal{A}}\ , \tag{4.2}$$

where $V_\Pi = \{v_S \mid S \in \Pi\}$ and $V_{\mathcal{A}} = \{v_P \mid P \in \mathcal{A}\}$, and

$$E_1^* = A_1^* \dot{\cup} B_1^*, \tag{4.3}$$

where

(i) $A_1^* \subsetneqq V_\Pi \times V_\Pi$ and $(v_S, v_{S'}) \in A_1^*$ if and only if $S, S' \in \Pi$ and $E - A = A'$ for minimal irreducibility components A and A' associated with S and S', respectively,

and

(ii) $B_1^* \subsetneqq V_\Pi \times V_{\mathcal{A}}$ and $(v_S, v_P) \in B_1^*$ if and only if $S \in \Pi$, $P \in \mathcal{A}$ and $E - A = B$ for a minimal irreducibility component A associated with S and a component B of the 2-cut aggregation partition P.

We can easily see from Theorem 7 that the graph $G_1^* = (V_1^*, A_1^*)$ is a tree. We call the tree G_1^* the <u>canonical decomposition tree, of level 1, of</u> (E,f). It should be noted that for each vertex v of G_1^*, if v corresponds to an $S_j \in \Pi$, then the vertex v is associated with $(E,f)//\hat{P}(S_j)$ and, if v corresponds to a 2-cut aggregation partition P^*, then v is associated with the 2-cut aggregation $(E,f)//P^*$. Also note that there may be more than one 2-cut aggregation partitions of E of (E,f).

If a 2-cut aggregation $(E,f)//P^*$ of (E,f) is reducible, then further construct the canonical decomposition tree, of level 1, of $(E,f)//P^*$ and repeat this decomposition process until the constructed canonical decomposition tree does not contain any vertex which corresponds to a reducible 2-cut aggregation. If a canonical decomposition tree is obtained after k-1 2-cut aggregations, then we call the tree the canonical decomposition tree, <u>of level k</u>, of (E,f).

In this way we can decompose (E,f) into irreducible aggregations of (E,f) and extract the tree structures of these aggregations of all levels and, at the same time, the hierarchical structure of the reducible 2-cut aggregations.

A canonical decomposition tree of level k+1 can be embedded into a canonical decomposition tree of level k as follows. Let G_{k+1}^* and G_k^* be the canonical decomposition trees, of level 1, of $(E^{(k)},f^{(k)})$ and $(E^{(k-1)},f^{(k-1)})$, respectively, and

$$(E^{(k)},f^{(k)}) = (E^{(k-1)},f^{(k-1)})//P^{(k-1)}, \qquad (4.4)$$

where $P^{(k-1)}$ is a 2-cut aggregation partition of $E^{(k-1)}$ of $(E^{(k-1)}, f^{(k-1)})$. Note that $E^{(k)} = \{e_A \mid A \varepsilon P^{(k-1)}\}$. Let v^* be the vertex in G_k^* which corresponds to $P^{(k-1)}$. Also let $v_S^{(k)}$ be the vertex in G_k^* which corresponds to a component S of the canonical 2-cut partition of $E^{(k-1)}$ such that $v_S^{(k)}$ is adjacent to v^* and $E - A = B$ for a minimal irreducibility component A associated with S and a component B of $P^{(k-1)}$. Furthermore, let S^* be a component of the canonical 2-cut partition of $E^{(k)}$ containing the element e_B. Then replace the edge $(v_S^{(k)},v^*)$ by $(v_S^{(k)},v_{S^*}^{(k+1)})$, where $v_{S^*}^{(k+1)}$ is the vertex in G_{k+1}^* which corresponds to S^*. In this way replace all the edges, in G_k^*, incident to v^* and then delete v^*, which gives a tree composed of G_k^* and G_{k+1}^*.

All the canonical decomposition trees can thus be embedded into the canonical decomposition tree, of level 1, of (E,f) by repeatedly embedding canonical decomposition trees into canonical decomposition trees of lower levels. We call the tree composed of all the canonical decomposition trees the _total decomposition tree of_ (E,f).

5. Examples of Symmetric Submodular Systems and Their Decompositions

Now, let us show some examples.

Example 1: Let $G = (V,E)$ be a connected but not 2-connected graph with a vertex set V and an edge set E. Replace each edge $e \in E$ by parallel edges e' and e'' and let $G' = (V,E')$ be the graph obtained by such replacement of edges of G. Then define

$$f'(A) = |V(A)| + |V(E - A)| - |V| \tag{5.1}$$

for any $A \subseteq E'$. where for $B \subseteq E'$ $V(B)$ is the set of end-vertices of edges in B. Then (E',f') is a symmetric submodular system and satisfies (2.6) with $\lambda^* = 1$. Any 2-cut aggregations, of level 1, of (E',f') are of bond type, so that (E',f') is decomposed up to level 1.

The canonical decomposition tree, of level 1, of (E',f') is essentially the same as the tree representing the incidence relation of 2-connected subgraphs of G which is described in [9].

Remark 1: The decomposition of a connected graph into 2-connected subgraphs is determined by the structure of minimum 1-cuts of a symmetric submodular system (E,f), where f is defined similarly as (5.1) for the original graph $G = (V,E)$. The argument in Example 1 is that we duplicate each edge $e \in E$, appropriately define a symmetric submodular system (E',f') and express the minimum 1-cuts of (E,f) equivalently as the minimum 2-cuts of (E',f') to which the decomposition theory developed in the present paper is directly applicable.

Remark 2: We can of course develop a decomposition theory based on the structure of minimum 1-cuts of symmetric submodular systems, which is similar to the theory, by Gomory and Hu [7], for representing the structure of the set of minimum cuts in a symmetric network by a tree. Though the tree described in [7] is not unique, we can define a unique "canonical" decomposition tree similarly as the theory developed in the present paper.

Example 2: Let $G = (V,E)$ be a 2-connected graph and define

$$f(A) = |V(A)| + |V(E - A)| - |V| \tag{5.2}$$

for any $A \subsetneqq E$. Then (E,f) is a symmetric submodular system and satisfies (2.6) with $\lambda^* = 2$. The total decomposition tree of (E,f) is the same as the tree representing the structure of the set of two-terminal subgraphs of G described by Tutte [9], where the hierarchical structure of the set of two-terminal subgraphs is implicit.

Example 3: Let $M = (E,\rho)$ be a 2-connected matroid with a rank function ρ. Let us define

$$f(A) = \rho(A) + \rho(E - A) - \rho(E) + 1 \tag{5.3}$$

for any $A \subsetneqq E$. Then (E,f) is a symmetric submodular system and satisfies (2.6) with $\lambda^* = 2$ (cf. [10], [11]). Therefore, we can obtain the canonical decomposition trees of (E,f). Note that f defined by (5.3) is a symmetrization of the rank function ρ. It may also be noted that, if E is a circuit of the matroid (E,ρ), the corresponding (E,f) is not of polygon type but of bond type. Related works on matroid decompositions were made by Bixby [1] and Cunningham [3].

Remark 3: We have not discussed the algorithmic aspects of decompositions of symmetric submodular systems. Whether or not there exists an efficient algorithm for decomposing a symmetric submodular system depends on how the submodular system is represented. See [8] for decompositions of 2-connected graphs and [2] and [3] for decompositions of 2-connected matroids.

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