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On the Skiadas "Conditional Preference Approach" to Choice Under Uncertainty

by

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Abstract

We compare the Skiadas approach with the standard Savage framework of choice under uncertainty. At first glance, properties of Skiadas "conditional preferences" such as coherence and disappointment seem analogous to similarly motivated notions of decomposibility and disappointment aversion defined on Savage "ex ante preferences". We show, however, that coherence per se places almost no restriction on the structure of ex ante preferences. Coherence is an 'external' restriction across preferences whereas notions of decomposibility in the Savage framework are 'internal' to the particular preference relation. Similarly, standard notions of disappointment aversion refer to 'within act' disappointments. Skiadas's notion of disappointment aversion for families of conditional preference relations neither implies nor is implied by standard notions of disappointment aversion for ex ante preferences.

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1 Introduction

Skiadas [8] introduces an interesting new way to think about choice under uncertainty in terms of families of conditional preferences. At first sight, both Skiadas's approach and his suggested axioms seem very closely related to work in the more standard framework where agents have only one primitive ex ante preference relation over acts. For example, Skiadas's re-interpretation of Savage's sure thing principle using his coherence axiom appears close to the decomposition axiom, or weak decomposability, recently used by Grant, Kajii & Polak [4]. Skiadas's notion of disappointment aversion appears close to the disappointment aversion notions of Gul [2] or of Grant & Kajii [3]. In each case, the underlying motivation seems similar. These resemblances, however, are somewhat misleading. In particular, both Skiadas's coherence and disappointment properties are restrictions across preference relations. The seemingly analogous properties of ex ante preferences are restrictions internal to one preference relation. This paper examines relations between similarly motivated notions from the old 'Savage' and new Skiadas frameworks.

In section 2, we introduce the two frameworks, and we formally define and informally discuss coherence, decomposability and the different notions of disappointment aversion. In section 3, we show that coherence places almost no internal restriction at all on preferences. Indeed, given any two preference relations over acts, we can construct a set of coherent Skiadas preference relations such that the first of the pair is the conditional relation for some event and the second is the ex ante preference (the conditional preference for the universal event). Also, even if the conditional preference relation for each individual state satisfies some internal property such as the sure-thing principle or weak decomposability, we can construct a coherent set of Skiadas preferences such that the ex ante preferences violate this internal property. Thus, starting with Skiadas conditional preferences on events smaller than the universal event, coherence places no restriction on ex ante preferences.

In section 4, taking coherence as given, we examine Skiadas's notion of disappointment aversion. If a coherent family of Skiadas conditional preference relations satisfies his formalization of disappointment aversion for the conditional preference framework, then the associated ex ante preference relation satisfies weak decomposability. (Ben: sentence deleted as paragraph in section 4 has also been deleted). Moreover, this does *not* imply that the associated ex ante preference relation satisfies disappointment aversion in the sense used by Gul (or Grant & Kajii). Conversely, if an ex ante preference relation associated with a coherent family of Skiadas conditional preference relations satisfies Gul's notion of disappointment aversion, this does *not* imply that the family satisfies disappointment aversion in the sense used by Skiadas.

In short, then, the axioms Skiadas considers in his new framework are almost completely independent of what at first might appear to be their natural counterparts in the standard single-preference-relation framework. These results should not be thought of as criticism of Skiadas's approach; rather, they underline the novelty and potential interest of that approach.

2 Two Frameworks

Denote by $\mathcal{S} = \{\ldots, s, \ldots\}$ a set of states, by $\mathcal{E} = \{\ldots, A, B, \ldots, E, \ldots\}$ the set of events which is a given σ-field on the universal event $\mathcal{S}$, and by $\mathcal{X} = \{\ldots, x, y, z, \ldots\}$ a set of outcomes (or 'objective' consequences). An act is a (measurable) function $f : \mathcal{S} \to \mathcal{X}$. Let $f(\mathcal{S}) = \{f(s) \,|\, s \in \mathcal{S}\}$ be the outcome set associated with the act f, and let $\mathcal{F} = \{\ldots, f, g, h, \ldots\}$ denote the set of acts on $\mathcal{S}$ with finite outcome sets. We will abuse notation and use x to denote both the outcome x in $\mathcal{X}$ and the constant act $f(\mathcal{S}) = \{x\}$. Let $\succeq$ be a binary relation over ordered pairs of acts in $\mathcal{F}$, representing the individual's preferences. Let $\succ$ and $\sim$ correspond to strict preference and indifference, respectively.

The following notation to describe an act will be convenient. For an event E in $\mathcal{E}$, and any two acts f and g in $\mathcal{F}$, let $f_E g$ be the act which gives, for each state s, the outcome $f(s)$ if s is in E and the outcome $g(s)$ if s is in the complement of E (denoted $\mathcal{S} \backslash E$). Using this notation we can define the set of *null* events, $\mathcal{N} \subset \mathcal{E}$, as follows: $E \in \mathcal{N}$ if and only if, for all acts f, g and $h \in \mathcal{F}$, $f_E h \sim g_E h$.

In the following, we will assume that any preference relation over acts satisfies Savage's ordering axiom, P1. Other assumptions will be introduced as needed. Perhaps the most controversial of Savage's axioms is his sure thing principle, P2.

P2 (Sure Thing Principle): For all events E and acts f, g, h and h', if $f_E h \succeq g_E h$ then $f_E h' \succeq g_E h'$.

The sure thing principle imposes separability across events. In Grant, Kajii & Polak [4] we considered the following weakening of P2 that still captures much of the intuition behind Savage's original motivation for the sure-thing principle.

Weak Decomposability. For any pair of acts f and g in $\mathcal{F}$, and any event A in $\mathcal{E}$: $g_A f \succ f$ and $f_A g \succ f$ implies $g \succ f$.

The idea of this weaker axiom is to allow complex choices to be decomposed and considered one event at a time, without imposing the full separability of Savage's P2.

In Skiadas's framework (see [7] and [8]), a decision maker is endowed with a richer set of preferences than in Savage's. To facilitate comparison, we translate his framework as follows.[1] An individual is characterized not simply by the unconditional Savage preference relation $\succeq$, but by a whole family of "conditional preferences relations", $\{\succeq^E : E \in \mathcal{E}\}$, one for each event. Each conditional preference statement $f \succeq^E g$ can be interpreted as the ability of the agent to compare what it would be like to have chosen f if E occurs with what it would be like to have chosen g if E occurs. This preference can take into account more than the outcomes on E. Thus Skiadas conditional preference relations are over entire acts not just over sub-acts confined to the outcomes induced on a particular event. In Skiadas's framework, Savage's unconditional or ex ante preference relation, $\succeq$, is identified with the conditional preference relation, $\succeq^S$,associated with the universal event S.

Skiadas defines the following property.

(Strict) Coherence A family of conditional preference relations $\{\succeq^E : E \in \mathcal{E}\}$ is (*strictly*) *coherent* if for any pair of disjoint events A and B in $\mathcal{E}$, and any pair of acts f and g in $\mathcal{F}$:

1. $f \succeq^A g$ and $f \succeq^B g$ implies $f \succeq^{A \cup B} g$.
2. $f \succ^A g$ and $f \succeq^B g$ implies $f \succ^{A \cup B} g$.

Like weak decomposability, coherence[2] can be thought of as a property that allows decision problems to be considered one event at a time. These two similarly motivated properties turn out, however, to be independent.

Weak decomposability invites a thought experiment when comparing acts f and g, where the agent applies her given ex ante preference relation to new (but related) acts. She first compares two acts, $f_E g$ and f, that differ in the outcomes they yield on the event E, but which both yield the outcomes from act f on the complementary event $S \backslash E$. Then, she repeats this exercise for two acts, $g_E f$ and f, that differ on $S \backslash E$, but which both yield the outcomes from f on E. Each of these comparisons uses the same (ex ante) preference relation over acts. There is no need to define conditional preference relations. But, in all, there are three acts considered, f, $g_E f$, and $f_E g$, and the conclusion involves a fourth act, g.

Coherence, on the other hand, invites a different thought experiment when comparing acts f and g, where the agent applies new conditional preference relations to the given original acts. She first compares how she would feel if she chose f should E occur with how she will feel if she chose g if E occurs. Then repeats this exercise for $S \backslash E$. These comparison involve

[1] Skiadis does not define outcomes as Savage does, and so in principle his framework is more general than what follows. But this is enough for our purposes.

[2] Skiadas [8] defines both 'coherence' and 'strict coherence'. In this paper we only use the strict version, so the use of the term 'coherence' shall refer to this strict version.

two different (conditional) preference relations over acts, $\succeq^E$ and $\succeq^{S\backslash E}$, and the conclusion involves a third, $\succeq^S$. But, each comparison only involves the same initial two acts, f and g. There is no need to consider compound acts such as $f_E g$ or $g_E f$.

In addition to coherence, Skiadas defines the following analogue to P2.

Separability A family of conditional preference relations $\{\succeq^E : E \in \mathcal{E}\}$ is *separable* if for any pair of acts f and g in $\mathcal{F}$, and any event $E \in \mathcal{E}$,

$$(f(s) = g(s) \text{ for every } s \text{ in } E) \text{ implies } f \sim^E g$$

Separability looks a lot like P2. The following lemma (due to Skiadas) shows they are indeed formally related.

Lemma (Skiadas) *If preferences are separable and coherent then for any triple of acts f, g and h in $\mathcal{F}$, and any event $E \in \mathcal{S}$,*

$$f_E h \succeq g_E h \text{ if and only if } f_E h \succeq^E g_E h$$

Moreover, the unconditional preference relation $\succeq$ satisfies P2.

Proof. This is immediate since $f_E h$ and $g_E h$ yield the same outcomes on $\mathcal{S}\backslash E$. ■

It is also readily apparent that if we start with an unconditional preference relation $\succeq$ that satisfies P2 we can always *construct* a family conditional preference relations $\{\succeq^E : E \in \mathcal{E}\}$ that is separable and coherent simply by defining

$$f \succeq^E g \text{ if } f_E h \succeq g_E h \text{ for some (and hence by P2 for any) } h \text{ in } \mathcal{F}$$

So in this sense subjective expected utility can be understood as a special case of a set of conditional preferences that are both separable and coherent.

Skiadas [8] also defines the following weakening of separability.

Disappointment Aversion A family of conditional preference relations $\{\succeq^E : E \in \mathcal{E}\}$ exhibits *weak* (respectively, *strict*) *disappointment aversion* if for any pair of acts f and g in $\mathcal{F}$, and any event $E \in \mathcal{E}\backslash\mathcal{N}$,

$$(f(s) = g(s) \text{ for every } s \text{ in } E \text{ and } g \succeq f \text{ (resp. } g \succ f))$$

$$\text{implies } f \succeq^E g \text{ (resp. } f \succ^E g).$$

The idea is as follows. If act g is preferred to act f overall but their outcomes on the event E are the same, then E is a more 'disappointing' event for the act g than for the act f. If the agent dislikes disappointment, then (since all else is equal by construction) she will be less unhappy in the event E if she chose f than if she chose g. This notion of disappointment, however, is different from that used by other writers. Though their motivations are similar, the different notions turn out to be independent.

Both Gul [2] and Grant & Kajii [3] use the term disappointment to describe outcomes within specific lotteries.[3] For Gul, a outcome x in the support of the lottery p is disappointing if x is worse than the certainty equivalent of p. For Grant & Kajii, an outcome x in the lottery p is disappointing if x is worse than the best outcome in p. Loosely speaking, both Gul's and Grant & Kajii's agents exhibit disappointment aversion by 'weighing' the "disappointment outcomes" in a lottery more heavily than the "elation outcomes". Their notions of disappointment can easily be extended from lotteries to acts. For example, the outcome x is a disappointment outcome of the act f in the Gul sense if x is in the range of f and $f \succ x$. Both Gul's and Grant & Kajii's ideas are 'within-act' notions of disappointment. In Skiadas's definition of disappointment, however, consider the outcomes of the acts g and f on the critical event E (the event on which f and g agree, and on which g is supposed to be disappointing). These outcomes need not be bad outcomes compared to the outcomes of g or f on E complement. Indeed, there is nothing in Skiadas's definition to prevent the outcomes of g (and hence f) on E from being better than any outcomes of g or f on E complement. In this sense, Skiadas's is not a 'within-act' notion of disappointment.

3 Implications of Coherence

Let $\succeq$ be a preference relation and consider the trivial family of Skiadas conditional preferences $\{\succeq^E\}$ where, for all events E, $\succeq^E := \succeq$. This family is (trivially) coherent. So, coherence per se imposes no restriction on the ex ante preference relation. In this subsection, we argue more generally that coherence requires hardly any restriction on the ex ante preference relation.

We start with the following definition and observation due to Skiadas [8].

Additive Aggregation *A family of conditional preference relations, $\{\succeq^E : E \in \mathcal{E}\}$, admits an* additive aggregation *if there is an additive probability measure P on S, and a function $v : \mathcal{F} \times S \to \mathbb{R}$ such that $\succeq^E$ is defined by the utility function $V^E(f) = \int_E v(f,s)P(ds)$.*

[3] See also Mas-Colell, Whinston & Green [6], pp 180-1.

Observation 1 *A family of conditional preference relations* $\{\succeq^E : E \in \mathcal{E}\}$ *that admits an additive aggregation is coherent. Indeed, if* $V^A(f) \geq V^A(g)$ *and* $V^B(f) \geq V^B(g)$*, then* $\int_A v(f,s)P(ds) \geq \int_A v(g,s)P(ds)$ *and* $\int_B v(f,s)P(ds) \geq \int_B v(g,s)P(ds)$*, hence if* A *and* B *are disjoint,* $V^{A\cup B}(f) = \int_{A\cup B} v(f,s)P(ds) \geq \int_{A\cup B} v(g,s)P(ds) = V^{A\cup B}(g)$.

Skiadas [8] Theorem 2 also provides conditions under which the form $\int_E v(f,s)P(ds)$, is not only sufficient for coherence but also necessary. Notice that this form is much more general than the separable form $\int_E v(f(s),s)P(ds)$. Loosely speaking, given an act f, the aggregation in Observation 1 is not over the 'conditional utilities', associated with each state s, from the outcome $f(s)$ yielded in that state. Rather, it is over the 'conditional utilities', associated with each state s, from the entire act f. Each function $v(.,s)$ is a representation of the conditional preference relation over entire acts given 'event' $\{s\}$. This, then, is an aggregation over different preference relations.

There is an analogy here to social choice. We can regard each s in S as an individual with her own preference relation. Each E in $\mathcal{E}$ is then a collection of individuals. The preference relation $\succeq^E$ can be regarded as a group preference and $\succeq^S$ can be regarded as the social preference relation. In this analogy, coherence is like the strong Pareto principle. In social choice, however, the Pareto principle only links social and individual preferences. Coherence links each group preference to its sub-groups. For example, suppose society consists of three individuals E_1, E_2 and E_3. Coherence requires not only that the social preference relation $\succeq^S$ obey the Pareto principle with respect to $\succeq^{E_1}$, $\succeq^{E_2}$ and $\succeq^{E_3}$, but also that the group preference $\succeq^{E_1\cup E_2}$ obey the Pareto principle with respect to its members preferences, $\succeq^{E_1}$ and $\succeq^{E_2}$. Skiadas's observation above shows that this principle implies that each group preference can be given an additive representation.

There is still, however, a large degree of freedom in constructing a coherent family of preference relations using the rule above. Here is a useful special case.

Observation 2 *Let* P *be an additive probability measure* P *on* S*. Let* $h_k : S \to \mathbb{R}$, $k = 1, ..., K$*, be a family of integrable functions and* V_k, $k = 1, ..., K$*, be a family of utility functions on* $\mathcal{F}$*. Set* $v(f,s) := \sum_{k=1}^{K} h_k(s)\, V_k(f)$*, and for each* E *in* $\mathcal{E}$*, set* $V^E(f) := \int_E v(f,s)\, P(ds)$*. Then, by the previous observation, it follows that this forms a coherent family of preference relations with the ex ante preference relation represented by* $\sum_{k=1}^{K} V_k(f) \int_S h_k(s)\, P(ds)$.

We can interpret this form using our social choice analogy. The population is infinite. The preferences of each 'individual' s can be represented by a weighted sum of the 'base' utility functions, V_k, where the weights (which we allow to be any real numbers) are given by h_k. Each 'group' preference can then also be represented as a weighted sum of the 'base' utility functions, with the weights formed by integrating over the measure of individuals in the group.

As we saw with the trivial example that began the section, however, coherence does not itself restrict the shape of ex ante preferences. This conclusion does not rely on the trivial example. In fact, given any pair of preference relations over acts, $\succeq_1$ and $\succeq_2$, and any event $E \subset \mathcal{S}$, one can construct a coherent family of conditional preference relations for which $\succeq_1$ is the unconditional preference relation for this family and $\succeq_2$ is the Skiadas conditional preference relation given E.

Proposition 1 *Fix V and W, functions on $\mathcal{F}$, and fix an event $E \subset S$. Let P be an additive probability measure on S with $0 < P(E) < 1$. Then there is a coherent family of conditional preferences where the ex ante preferences are represented by V and the conditional preferences given E is represented by W.*

Proof. Let $F = \mathcal{S} \backslash E$. Set $V_1 := V$, $V_2 := W/P(E)$, and set $h_1 := \frac{1}{P(F)} 1_F$ and $h_2 := 1 - h_1$. Set $v(f,s) := \sum_{k=1}^{2} h_k(s) V_k(f)$, and for each A in $\mathcal{E}$, set $V^A(f) := \int_A v(f,s) P(ds)$. Then, by Observation 2, this forms a coherent family. Noting that $\int_S h_1(s) P(ds) = 1$ and $\int_E h_1(s) P(ds) = 0$, we get $\int_S v(f,s) P(ds) = V(f)$ and $\int_E v(f,s) P(ds) = W(f)$ as required. ■

A consequence of Proposition 1 is that coherence on its own does not imply that either the conditional or the ex ante preferences satisfies any particular editing or substitution property. We could choose V and W such as the sure thing principle or weak decomposability applies to one, to both or to neither.[4] The following example takes this a step further. Even if, for every state s, the conditional preference relation associated with that state, $\succeq^{\{s\}}$, belongs to the same class of preferences, a coherent family of conditional preferences built up from these 'state' conditional preferences need not all belong to this particular class. In particular, the unconditional preference relation may be very different in form and nature to the underlying 'state' conditional preferences. That is, editing or substitution properties, such as the sure thing principle or weak decomposability need not be preserved under coherent aggregation.

Example 1 *Fix the universal set $\mathcal{S} = [0,1]$ and the outcomes $\mathcal{X} = \{x, y\}$. Let P be the uniform probability measure over $\mathcal{S}$. For each act f in $\mathcal{F}$, set $v(f,s) := \ln\left[1 + P\left(f^{-1}(x)\right)\right]$ if s is in $[0, 0.5)$, $v(f,s) := \ln[1 + P(f^{-1}(y))]$ if s is in $[0.5, 1]$. For each event A in $\mathcal{E}$, let $\succeq^A$ be represented by the functional $V^A(f) := \int_A v(f,s) P(ds)$.*

[4] Since it is straightforward to construct preferences that do not satisfy coherence, the converse — that particular internal substitution properties do not imply coherence — is trivial.

Applying monotone transformations to these representations, we see that, for each state s in $[0, 0.5)$ (respectively, in $[0.5, 1]$), the state preference relation, $\succeq^{\{s\}}$, can be represented by $P(f^{-1}(x))$ (respectively, $P(f^{-1}(y))$). This is a subjective expected utility functional, so the underlying state preferences each satisfy all the Savage axioms (and hence also weak decomposability). The ex ante preference relation, $\succeq^{S}$, however, can be represented by $[1 + P(f^{-1}(x))][1 + P(f^{-1}(y))]$. This is not unlike a Nash social welfare function. These ex ante preferences do not satisfy either the sure thing principle, P2, or weak decomposability (or even Savage's monotonicity axiom P3).

Proposition 1 above also demonstrates that for a given unconditional preference relation the 'number' of coherent families of conditional preference relations that are consistent with this unconditional preference relation is at least the cardinality of the set of events.

4 Implications of Disappointment Aversion

In this section we explore the relation between Skiadas's notion of disappointment aversion and both (a) weak decomposability; and (b) Gul's earlier notion of disappointment aversion defined on a (single) ex ante preference relation. Skiadas [8] shows that under some regularity conditions (for example, an appropriate notion of continuity) a coherent family of conditional preference relations, $\{\succeq^{E}: E \in \mathcal{E}\}$, satisfies disappointment aversion if and only if there exists a probability measure P on $\mathcal{E}$, and a function $v : \mathcal{X} \times S \times \mathbb{R} \to \mathbb{R}$ that is nonincreasing in its last argument, such that for each E in $\mathcal{E}$, $V^{E}(f) = \int_{E} v(f(s), s, V(f))P(ds)$ represents $\succeq^{E}$where $V(f)$ ($\equiv V^{S}(f)$) uniquely solves the equation

$$V(f) = \int_{S} v(f(s), s, V(f))P(ds) \text{ and represents } \succeq . \tag{1}$$

But this in turn implies that the unconditional preference relation associated with this family of conditional preference relations must satisfy weak decomposability. To see this, set $\varphi(x, s, w) := v(x, s, w) - w$. We can then rewrite the implicit representation in (??) as $\int_{S} \varphi(f(s), s, V(f)) P(ds) = 0$. Since $v(x, s, w)$ is non-increasing in its third argument it follows that $\varphi(x, s, w)$ is strictly decreasing in its third argument. Hence, for any pair of acts f and g, we have $g \succ f$ if and only if $\int_{S} \varphi(g(s), s, V(f)) P(ds) > 0$. Weak decomposability holds since: if $g_A f \succ f$ and $f_A g \succ f$, then (writing $\bar{v} = V(f)$) $\int_{A} \varphi(g(s), s, \bar{v})P(ds)$ $+ \int_{S\setminus A} \varphi(f(s), s, \bar{v}) P(ds) > 0$ and $\int_{S\setminus A} \varphi(g(s), s, \bar{v})P(ds) + \int_{A} \varphi(f(s), s, \bar{v}) P(ds) > 0$. Hence $\int_{A} \varphi(g(s), s, \bar{v})P(ds) + \int_{S\setminus A} \varphi(g(s), s, \bar{v})P(ds) > 0$; that is, $g \succ f$ as required.

Grant, Kajii & Polak [4] show that, given probabilistic sophistication, weak decomposability is equivalent to betweenness. Thus, a corollary of the above result is that, given probabilistic sophistication, if a Skiadas family of conditional preferences is both coherent and satisfies his notion of disappointment aversion, then the ex ante preference relation satisfies betweenness.[5] Grant & Kajii's [3] notion of disappointment applies to rank-dependent preferences. In general, such preferences do not satisfy betweenness. So, the ex ante preference relation associated with a family of coherent, Skiadas-disappointment averse preference relations will not satisfy Grant-Kajii-disappointment aversion.

Gul's notion of disappointment aversion, however, is defined on preferences over lotteries that satisfy betweenness. It turns out, however, that the Gul and Skiadas notions are formally independent. In particular, the next example shows we can find a coherent family of preference relations such that the associated ex ante unconditional preference relation is Gul-disappointment averse but the family as a whole is not Skiadas-disappointment averse. And we can find a coherent family of preference relations which are Skiadas-disappointment averse but such that the associated ex ante unconditional preference relation is not Gul-disappointment averse.

Example 2 *Fix* $\mathcal{X} = \mathcal{S} = [0,1]$. *Let* μ *be the Lebesgue measure on* $\mathcal{S}$, *let*

$$\varphi_b(x,s,w) := \begin{cases} b(x-w) & \text{if } x \leq w \\ (x-w) & \text{if } x > w \end{cases} \quad \text{where } b > 0,$$

and let $v_{a,b}(x,s,w) := a\varphi_b(x,s,w) + w$ *where* $a > 0$. *Let* $\{\succeq^E_{a,b}: E \in \mathcal{E}\}$ *be the family of preference relations such that, for each* E *in* $\mathcal{E}$, *the conditional preference relation* $\succeq^E_{a,b}$ *is represented by the functional* $V^E_{a,b}$ *defined by* $V^E_{a,b}(f) = \int_{s\in E} v_{a,b}(f(s),s,V_b(f))\,\mu(ds)$, *where* $a > 0$ *and* $V_b(f)$ $(\equiv V^S_{a,b}(f))$ *solves*

$$V_b(f) = \int_S v_{a,b}(f(s),s,V_b(f))\,\mu(ds). \tag{2}$$

By construction this family is coherent. Moreover,

1. *the associated ex ante preference relation* $\succeq^S_{a,b} = \succeq_b$ *(which is represented by the functional* $V_b(f)$*) is Gul-disappointment averse, if and only if* $b \geq 1$*; and*
2. *the family as a whole is Skiadas-disappointment averse if and only if* $a \geq 1$ *and* $ab \geq 1$.

First notice that, from expression (2), $V_b(f)$ solves $0 = \int_S \varphi_b(f(s),s,V_b(f))\,\mu(ds)$. Since φ_b is decreasing in its third argument, $V_b(f)$ is well-defined. To prove the first claim, compare the representation of preferences over lotteries induced by V_b to such representations that satisfy Gul's notion of disappointment aversion. Let $u : [0,1] \to \mathbb{R}$ be a strictly increasing (utility) function. Let ψ_β be given by

[5] This result is Skiadas's [7] Proposition 3.

$$\psi_\beta(x,w) := \begin{cases} (1+\beta)(u(x)-w) & \text{if } u(x) \leq w \\ u(x)-w & \text{if } u(x) > w \end{cases} \quad \text{where } \beta > -1.$$

And let p denote a (typical) lottery over the unit interval. Gul [2] shows that a preference relation over lotteries over the unit interval satisfies Gul-disappointment aversion if and only if it can be represented by a functional V_β defined implicitly by $0 = \int_0^1 \psi_\beta(x, V_\beta(p))p(dx)$ with the parameter $\beta \geq 0$. Each act f corresponds to a lottery p_f given by $p_f(x) := \mu \circ f^{-1}(x)$. Thus V_b induces a preference relation over lotteries over the unit interval, V_β, given by setting $u(x) := x$ and setting $\beta := b - 1$, so that $\psi_\beta(x,v) := \varphi_b(x,s,v)$. As required, this preference relation is Gul-disappointment averse if and only if $b \geq 1$.

To show the second claim notice that

$$\frac{\partial v_{a,b}(x,s,w)}{\partial w} = \begin{cases} -ab+1 & \text{if } x < w \\ -a+1 & \text{if } x > w \end{cases}.$$

The claim thus follows directly from Skiadas's result quoted above, since $a \geq 1$ and $ab \geq 1$ is necessary and sufficient for $v_{a,b}$ to be nonincreasing in its last argument.

Hence it follows if $a < 1$ and $b > 1$, then we have a coherent family of preference relations such that the ex ante unconditional preference relation is Gul-disappointment averse but the family as a whole is not Skiadas-disappointment averse. Similarly, if $ab > 1$ and $b < 1$, then we have a coherent family of preference relations such that the family as a whole is Skiadas-disappointment averse but the ex ante unconditional is not Gul-disappointment averse.

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