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Efficient Patterns of Linear Facilities

by

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In this paper, we formulate two types of simple location models of linear facilities such as expressways and railways, and analyze the optimal location of these facilities and the corresponding benefits. We first prove that sequential optimizations and the simultaneous optimizations lead to very similar results. Moreover we show that there are various Figures of expressways, for example, the Figures concerning loops or no loop, in close of maximum benefit, so people can plan considering other real conditions. Furthermore, we show the benefit for finite velocity is almost proportional to that for infinite velocity.

INTRODUCTION

A movement has an origin and a destination and these two points are distributed in two-dimensional space. Linear rapid transport systems, in short, expressways and railroad, are also arranged in two-dimensional space. In this paper, we can use these rapid transport systems as the same theory. So we use the word "expressway" in return for "rapid transport systems". Unlike many studies, we will not use network theory to express the reduction of travel time by expressways, but depend on mathematical programming technique. So we apply simultaneous optimization and sequential optimization of two models in this paper and argue the property of linear facility in two-dimensional space. In addition, we argue that these optimizations are essentialy equivalent.

First, let us imagine that people move from an origin P_0 to a destination P_1 in a subject region D. The minimum time without expressways is $t_0(P_0, P_1)$. The minimum time utilizing them is $t^*(P_0, P_1)$ as long as $t^*(P_0, P_1) \leq t_0(P_0, P_1)$. This movement of population density is $\mu(P_0, P_1)$. We regard the benefit as the total reduction of travel time and use this benefit as the measure in this paper, because people use expressways for reducing moving time. This benefit is the difference of total time between the travel time not utilizing expressways and the one utilizing expressways. In other word, the benefit is the total reduction of travel time due to an existing expressways and this benefit is expressed by

$$B = \int\int_{D \in P_0, P_1} \{t_0(P_0, P_1) - t^*(P_0, P_1)\} \cdot \mu(P_0, P_1) \ \mathrm{d}P_0 \mathrm{d}P_1.$$

Integrand function, $t_0(P_0, P_1) - t^*(P_0, P_1)$ represent the reduction of travel time from P_0 to P_1 when using expressways. The benefit is the sum of last expression for all the points in the subject region. The estimation of this expression requires four integrations in four dimensions and is quite complex (Arii and Koshizuka:1997).

Ordinarily P_0 and P_1 can be any point in the subject region. In this paper we reduce the travels to commuting to the city center O in the morning. Moreover we make a hypothesis that population is ditributed homogenously over the all region to avoid the problem of people relocating to take advantage of travel time reductions, therefore $\mu(P_0, \mathrm{O}) = 1$. We use rectilinear distances because they represent well movements in a city, and because they ease computation. A typical travel starts from P_0 at coordinates (x, y) and ends at the center. Then the benefit is given by

$$B = \iint_{D \in P_0} \{t_0(P_0, \mathrm{O}) - t^*(P_0, \mathrm{O})\} \cdot \mu(P_0, \mathrm{O}) \mathrm{d}P_0.$$

When a subject region is a rectangle a wide and b long, the domain of integration is defined by inequalities of polynomial of one degree with respect to x, y, a, b. Integrand functions are also polynomial of one degree with respect to x, y, a, b. Therefore the benefit B is polynomial of the third degree on the length. Using the expression of this benefit, we optimize two models in this paper:(1) the case of a crosswise expressway, (2) the case of longer expressways. These optimizations leads to very similar results concerning the maximum benefits and the optimal expressways' configrations. Until now, therelation between the distance from the center and the velocity has been considered a key point (Branston:1974, Hutchinson:1974), but this paper shows that the benefit when velocity v on expressways is constant is $(1 - 1/v)$ times the benefit when velocity v is infinit. We discuss the property of benefit of the expressways and the optimal configrations using the following simple models.

1.THE CASE OF A CROSSWISE EXPRESSWAY

A rectangle with sides a and b is a subject region. The unique city center is the middle of the rectangle (Figure 1). The crosswise expressway p by q passes through the city center and the length of expressways is $l = p + q$. People move at a velocity of v on expressway and move at a velocity of 1 on the other space. We assume people can get on expressways everywhere. The reason of this assumption is that the benefits where intervals of interchanges are zero look like constant times of the benefits where they are constant (Arii and Koshizuka,1997).

The origin is at the center and the axes correspond to the expressways. And people go to the center from the origin (x, y). Distances are rectilinear and population density is uniform.

Benefit B is the total reduced time using these expressways to travel to the center. This benefit is the difference between total movement time not using expressways and total movement time using expressways. This benefit corresponds to the total reduced time induced by the expressways in the subject region. We calculate the lengh of expressways maximizing the benefit B. Figure 2 show the route of various origins. A minimum time from the origin (x, y) to the center is $t_0 = |x| + |y|$ if not using the expressways, $t_1 = |x|/v + |y|$ if using the horizontal expressway where $|x| \geq p/2$, $t_2 = |x| - p/2 + p/(2v) + |y|$ is where $p/2 \geq |x| \geq a/2$, is $t_3 = |x| + |y|/v$ using vertical expressways where $|y| \leq q/2$, is $t_4 = |x| + |y| - q/2 + q/(2v)$ where $q/2 \leq |y| \leq b/2$. Moreover if we adopt the convention: $c = 1 - 1/v$, the benefit B is represented by

$$B = \begin{cases} 4\{\int_0^{\frac{q}{2}}\int_0^{x}(t_0 - t_1)\mathrm{d}y\mathrm{d}x + \int_{\frac{q}{2}}^{\frac{p}{2}}\int_0^{\frac{b}{2}}(t_0 - t_1)\mathrm{d}y\mathrm{d}x + \int_{\frac{p}{2}}^{\frac{a}{2}}\int_0^{\frac{b}{2}}(t_0 - t_2)\mathrm{d}y\mathrm{d}x \\ + \int_0^{\frac{q}{2}}\int_x^{\frac{q}{2}}(t_0 - t_3)\mathrm{d}y\mathrm{d}x + \int_0^{\frac{q}{2}}\int_{\frac{q}{2}}^{\frac{b}{2}}(t_0 - t_4)\mathrm{d}y\mathrm{d}x\} \\ = c\{\frac{1}{6}p^3 - \frac{1}{2}lp^2 + \frac{1}{2}(l^2 - bl + ab)p - \frac{1}{6}l^3 + \frac{1}{4}bl^2\}, & \text{if } p \geq q, \\ \\ c\{-\frac{1}{6}p^3 + \frac{1}{2}a(l - b)p + \frac{1}{2}abl - \frac{1}{4}al^2\}, & \text{otherwise,} \end{cases}$$

subject to

$$0 \leq b \leq a, 0 \leq p \leq a, 0 \leq q \leq b, l = p + q.$$

When we optimize the proportion of the expressways which maximize benefit B, we see that the solution is independent of velocity v. Therfore Figures 3, 4 and 5 use $v = \infty$. The solutions are $p \geq q$ at all times,

$$p^* = l, q^* = 0, B^* = \frac{1}{2}abl - \frac{1}{4}bl^2, \quad \text{for } 0 \leq l \leq a\ .$$

$$p^* = l - \sqrt{b(l - a)}, q^* = \sqrt{b(l - a)},$$
$$B^* = \frac{1}{2}abl - \frac{1}{4}bl^2 + \frac{1}{3}\{b(l - a)\}^{\frac{3}{2}}, \quad \text{for } a \leq l \leq a + b\ .$$

The equation which replaces p^* with q^* is a solutions only if $a = b$.

From this result, the optimal proportion of expressways is dependent on total length l. For example, where $a = b = 1$ and $l = 3/2$, Figure 3 shows the relation between benefit B and horizontal length of the expressway p. This Figure has two tops and then $p^* = \sqrt{2}/2, (3 - \sqrt{2})/2$. For $a = b$, the benefits obtained replacing a by b and p by q is also maximum. Thick curves in Figures 4 and 5 show the relation among B, p^* and l. The benefit is zero at the origin and the distance between benefit's contours is 0.02. Thick curves represent the route of maximum benefit in relation to various value of p, l. Therefore minimum p^* in $l \geq a$ and l is follows :

$$p^* = a - \frac{b}{4}, l = a + \frac{b}{4}.$$

That is, p needs to be longer than $a - b/4$ to maximize the benefit B when l is such that $a - b/4 \leq l$. The length of horizontal expressway needs $a - b/4$ all the time to maximize the benefit when $l \geq a - b/4$.

Now we calculate the case of sequential optimization. We consider building Δl one after another with a initial length l. First, we treat the case when $l < a$, the length of horizontal expressway $p = l$ and the length of vertical expressway $q = 0$. An expressway of a lengh Δl is added. The benefit can be expressed $B(p, q) = B(l + \Delta l, 0)$ if Δl is added to the horizontal expressway and $B(l, \Delta l)$ if it is added to the vertical expressway. That is,

$$\begin{aligned} \Delta B(l+\Delta l, 0) &= B(l+\Delta l, 0) - B(l, 0) \\ &= \frac{1}{2}(a-l)b\Delta l - \frac{1}{4}b\Delta l^2, \\ \Delta B(l, \Delta l) &= B(l, \Delta l) - B(l, 0) \\ &= -\frac{1}{6}\Delta l^3 + \frac{1}{4}b\Delta l^2. \end{aligned}$$

Therefore the difference of these benefits is

$$\begin{aligned} &\Delta B(l+\Delta l, 0) - \Delta B(l, \Delta l) \\ &\quad = \frac{1}{2}\{a - (l+\Delta l)\}b\Delta l + \frac{1}{6}\Delta l^3 > 0. \end{aligned}$$

As a result of this, to build parallel to the long side is $(a - l)b\Delta l/2$ better than to build parallel to short side when $0 \leq l < a$. Expressways should be built parallel to short side after it is filled up parallel to the long side.

The benefits and the patterns are different if optimization is sequential or simultaneous. Expressways are built following simultaneous optimization if the final length is known. But Figures 4 and 5 show that the difference of benefits is small (the simultaneous solution, p^* correspond to the thick line and the sequential solution in $l \geq a$ is $p = a$). Benefits are little change arround optimal solutions. And changes of patterns of expressways doesn't influence on benefit so much. Accordingly the benefit is enough high using sequential optimization in relation to a crosswise expressway when planner cannot use even simultaneous optimization.

2.THE CASE OF LONGER EXPRESSWAYS

We estimate the benefit when the length of the expressways respect the relation $l \geq a + b$. The subject region is a square with a side $2a$ and the unique center is the middle of the square (Figure 6).

The length of an expressway (a thick line) is $2a$. And there are t expressways in the subject region. At this time we calculate the maximum benefit B of t expressways. And we discuss about the maximum benefit and the pattern of expressways and the property of expressways.

At first we estimate three expressions of the total benefit B: $B_L(p,q,s)$, $B_U(p,q,s)$, $B_O(p,q,s)$. Type L ,type U and type O in Figure 7 are dependent on p, q and s. Subject regions are rectangles with sides p and q in these Figures. So, origins of movements are in these rectangles and destinations are the centers. The functions $B_L(p,q,s)$, $B_U(p,q,s)$, $B_O(p,q,s)$ estimate their benefits. The origins are the left and lower vertexes of these rectangles in Figure 7. The x axes are horizontal and the y axes are vertical. The type L benefit is:

$$B_L(p,q,s)=\begin{cases} B_{L_1}(p,q,s), & \text{for } p \leq q \\ B_{L_2}(p,q,s), & \text{for } p \geq q \ , \end{cases}$$

where

$$\begin{aligned} B_{L_1}(p,q,s) &= \int_0^p \int_0^x (cx+cs)\mathrm{d}y\mathrm{d}x + \int_0^p \int_x^q (cy+cs)\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{2}cpq^2 + cpqs + \frac{1}{6}cp^3, \\ B_{L_2}(p,q,s) &= BL_1(q,p,s) \\ &= \frac{1}{2}cp^2q + cpqs + \frac{1}{6}cq^3. \end{aligned}$$

Type U benefit is:

$$B_U(p,q,s)=\begin{cases} B_{U_1}(p,q,s), & \text{for } p \leq \frac{2}{2-c}q \\ B_{U_2}(p,q,s), & \text{for } p \geq \frac{2}{2-c}q \ , \end{cases}$$

where

$$\begin{aligned} B_{U_1}(p,q,s) &= \int_0^{\frac{1}{2}(2-c)p} \int_x^q (cy+cs)\mathrm{d}y\mathrm{d}x + \int_{\frac{1}{2}(2-c)p}^{p} \int_{-\frac{2-c}{c}x+\frac{2-c}{c}p}^{q} \{2x+cy-(2-c)p+cs\}\mathrm{d}y\mathrm{d}x \\ &\quad + \int_0^{\frac{1}{2}(2-c)p} \int_0^x (cx+cs)\mathrm{d}y\mathrm{d}x + \int_{\frac{1}{2}(2-c)p}^{p} \int_0^{-\frac{2-c}{c}x+\frac{2-c}{c}p} (cx+cs)\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{24}cp\{(2-c)^2p^2 + 6cpq + 12q^2 + 24qs\}, \\ B_{U_2}(p,q,s) &= \int_0^q \int_x^q (cy+cs)\mathrm{d}y\mathrm{d}x + \int_0^q \int_0^x (cx+cs)\mathrm{d}y\mathrm{d}x \\ &\quad + \int_q^{p-\frac{c}{2-c}q} \int_0^q (cx+cs)\mathrm{d}y\mathrm{d}x + \int_{p-\frac{c}{2-c}q}^{p} \int_0^{-\frac{2-c}{c}x+\frac{2-c}{c}p} (cx+cs)\mathrm{d}y\mathrm{d}x \\ &\quad + \int_{p-\frac{c}{2-c}q}^{p} \int_{-\frac{2-c}{c}x+\frac{2-c}{c}p}^{q} \{2x+cy-(2-c)p+cs\}\mathrm{d}y\mathrm{d}x \\ &= \frac{cq}{6(2-c)}\{3(2-c)p^2 + 2q^2 + 6(2-c)ps\}. \end{aligned}$$

Type O benefit is:

$$B_O(p,q,s) = \begin{cases} B_{O_1}(p,q,s), & \text{for } p \le q \\ B_{O_2}(p,q,s), & \text{for } p \ge q \ , \end{cases}$$

where

$$\begin{aligned} B_{O_1}(p,q,s) &= \int_0^{\frac{1}{2}(2-c)p} \int_x^{-\frac{c}{2-c}x+q} (cy+cs)\mathrm{d}y\mathrm{d}x + \int_0^{\frac{1}{2}(2-c)p} \int_0^{x} (cx+cs)\mathrm{d}y\mathrm{d}x \\ &\quad + \int_{\frac{1}{2}(2-c)p}^{p} \int_0^{-\frac{2-c}{c}x+\frac{2-c}{c}p} (cx+cs)\mathrm{d}y\mathrm{d}x + \int_{\frac{1}{2}(2-c)p}^{p} \int_{-\frac{2-c}{c}x+\frac{2-c}{c}p}^{x-p+q} \{2x+cy-(2-c)p+cs\}\mathrm{d}y\mathrm{d}x \\ &\quad + \int_0^{\frac{1}{2}(2-c)p} \int_{-\frac{c}{2-c}x+q}^{q} \{cx+2y-(2-c)q+cs\}\mathrm{d}y\mathrm{d}x + \int_{\frac{1}{2}(2-c)p}^{p} \int_{x-p+q}^{q} \{cx+2y-(2-c)q+cs\}\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{12}cp\{(2-c)p^2+3cpq+6q^2+12qs\}, \\ B_{O_2}(p,q,s) &= BO_1(q,p,s) \\ &= \frac{1}{12}cq\{(2-c)q^2+3cpq+6p^2+12ps\}. \end{aligned}$$

3.THE SEQUENTIAL OPTIMIZATION OF EXPRESSWAYS

We calculate the benefit using sequential optimization when an expressway of length $2a$ is added. At first we show four necessary conditions for the optimal solution. The optimal solutions to both simultaneous and sequential models has to meet the following necessary conditions:

(i) The maximum benefit is in $0 \le p, 0 \le q, 0 \le r \le p$ in case 1 of Figure 8 when $r = 2p/(4-c)$. The increase of this benefit is

$$B = BL(\frac{1}{2}p, q, \frac{1}{2}p) + BU(\frac{1}{2}p, q, 0) - BL(p, q, 0)$$

and we always have $\partial B/\partial p > 0$.

(ii) The maximum benefit is in $0 \le p, 0 \le q, 0 \le r \le p$ in case 2 of Figure 8 when $r = 2p/(4-c)$.

(iii) The maximum benefit is in $0 \le p, 0 \le q, 0 \le r \le p$ in case 3 of Figure 8 when $r = p/2$. The increase of this benefit is

$$B = BU(\frac{1}{2}p, q, \frac{1}{2}p) + BU(\frac{1}{2}p, q, 0) - BU(p, q, 0)$$

and we always have $\partial B/\partial p > 0$.

(iv) The maximum benefit is in $0 \leq p, 0 \leq q, 0 \leq r \leq p$ in case 4 of Figure 8 when $r = p/2$.

The explanation for the necessary conditions are as follows. The first and second necessary conditions have been shown before. When a vertical (or horizontal) expressway is built between vertical (or horizontal) boundary and the closest vertical (or horizontal) expressway to maximize benefit B, it has to be on the ratio of (2-c)/2 between the boundary and the expressway. This result is independent of the other vertical and horizontal expressways and the location of the center. Moreover, it is built where the distance between the expressway and the boundary is maximum. The third and fourth necessary condition have also been shown before. When a vertical (or horizontal) expressway is built between a vertical (or horizontal) expressway and an adjacent expressway to maximize benefit B, it has to be in the middle of these expressways. This result is independent of the other vertical and horizontal expressways and the location of the center. Besides when an expressways is built between two expressways to maximize benefit B, it is built between the two more distant adjacent expressways. From the four necessary conditions, locations of vertical expressways are independent of locations of horizontal expressways and the location of the center to maximize benefit. Distance between two expressways should be the same as the distance between the boundary and its closest expressway.

If these conditions are met, Be_t is the maximum benefit with t expressways using sequential optimization. The origin is the center, x axis is horizontal on the center and y axis is vertical on the center for calculating Be_t. If there are one expressways, the expressway is horizontal or vertical on the center. If there are two expressways, the both expressways are horizontal and vertical on the center. If there are more three expressways, they have to lie at one of next three places.

1. From the necessary conditions (i) and (ii), the best location of a new expressway is at space of the ratio of (2-c)/2 between the boundary and the expressway because it is independent from the horizontal expressways.

2. From the necessary conditions (iii) and (iv), the best location of a new expressway is in the middle

between two adjacent expressways because it is independent from the horizontal expressways.

3. The best location of a new expressway is $y = 2a/(4-c)$.

However the optimal location of expressways cannot be at $y = 2/(4-c)$ for any t. Simple calculation shows that.

The analytical expression of Be_t is omitted for brevity but the maximum benefit with t expressways are in proportion to a^3. And the optimal configrations corresponding benefits of Be_t are shown in Figure 9. Moreover an example of relation between Be_t and t in case of $a = 1, v = \infty$ is also shown in Figure 10.

4.THE SIMULTANEOUS OPTIMIZATION OF EXPRESSWAYS

In this chapter we calculate the maximum benefit using simultaneous optimization. In case of $t \geq 3$, we assume the two expressways pass through the center. Of course, one expressway passes through the center when $t = 1$ and both expressways pass through the center when $t = 2$ as in chapter 4. Here, $Bq(n, m)$ is the maximum benefit in Figure 11. Figure 11 shows the situation when there are n vertical expressways and m horizontal expressways. The subject region is the square with side a. This subject region is a quarter of the one of Figure 6. And all movement go to the center O which is the left lower vertex. Then $i = a/(2n + 2 - c)$ is the distance between two vertical expressways side by side deduced from necessary conditions (1)-(4). And $j = a/(2m + 2 - c)$ is the distance between two horizontal ones. So the maximum benefit in Figure 11 is

$$\begin{aligned} Bq(n,m) &= nmBO(i,j,0) + nBU(i, \frac{2-c}{2}j, 0) + mBU(j, \frac{2-c}{2}i, 0) + BL(\frac{2-c}{2}i, \frac{2-c}{2}j, 0) \\ &\quad + \frac{1}{2}cijnm\{(n-1)i + (m-1)j\} + \frac{2-c}{2}cij\{nmj + \frac{n(n-1)i}{2} \\ &\quad + nmi + \frac{m(m-1)j}{2}\} + (\frac{2-c}{2})^2 cij(ni + mj). \end{aligned} \tag{1}$$

Moreover there is the necessary condition (v).

(v) The function $Bq(n, m)$ is monotonous increasing with respect to n and m because $\partial Bq(n, m)/\partial n > 0$ and $\partial Bq(n, m)/\partial m > 0$ if n and m are independent variables. Moreover $Bq(n, m)$ is a concave function about n and m because of $\partial^2 Bq(n, m)/\partial n^2 < 0$ and $\partial^2 Bq(n, m)/\partial m^2 < 0$.

From the necessary condition (v), we deduce that the more the expressways' patterns is symmetric with respect to the center, the higher the benefit. The benefit Bi_t is the maximum benefit with simultaneous optimization in Figure 6. When we think t is an even number or an odd number, the benefit Bi_t is, from the equation (1) and the necessary conditions (i)-(v) in $t \geq 3$ and $n \geq m \geq 0$,

$$Bi_{(2n+2m+3)} = \max_{n,m} \begin{bmatrix} 2Bq(n+1,m) + 2Bq(n,m), \\ 2Bq(n,m+1) + 2Bq(n,m) \end{bmatrix}, \tag{2}$$

$$Bi_{(2n+2m+4)} = \max_{n,m} \begin{bmatrix} 4Bq(n+1,m), \\ 4Bq(n,m+1), \\ Bq(n,m) + Bq(n,m+1) + Bq(n+1,m) + Bq(n+1,m+1) \end{bmatrix}. \tag{3}$$

And where $m = 0$, $t = 2n+3$ when t is an odd numver, and $m = 0$, $t = 2n+2$ when t is an even number,equations (2) and (3) are

$$Bi_t = \begin{cases} 2ca^3, \quad (\text{if } t = 1) \\ 4Bq(\frac{t-2}{2}, 0) = \frac{2c}{3(t-c)^2}\{4 + 4c^2 + c(2-9t) - 6t + 6t^2\}a^3, \quad (\text{if } t \geq 2,\ t \text{ is an even number}), \\ 2Bq(\frac{t-3}{2}, 0) + 2Bq(\frac{t-3}{2}+1, 0) \\ = \frac{c}{3}\{\frac{16+4c^2+c(11-9t)-18t+6t^2}{(t-1-c)^2} + \frac{4+4c^2-c(7+9t)+6t+6t^2}{(t+1-c)^2}\}a^3, \quad (\text{if } t \geq 2,\ t \text{ is an odd number}). \end{cases} \tag{4}$$

Here t is $2n+2m+3$ if t is an odd number and t is $2n+2m+4$ if t is an even number. The equations (2) and (3) have maximum values if $m = 0$. Because $Bq(n+1, m-1) > Bq(n,m), n \geq m \geq 1$. So these equations have minimum values if m is a maximum value in $n \geq m \geq 0$, that is , $n = m$ if $n+m$ is an even number and $n+1 = m$ if $n+m$ is an odd number. Figure 12 show expressways' patterns for the maximum benefit, in short $m = 0$, and of the minimum benefit, in short m is maximum, in equations (2) and (3). The benefits, $Bi_{(2n+2m+3)}$ and $Bi_{(2n+2m+4)}$, in relation to every n and m are represented in Figure 13 in the case $a = 1$ and $v = \infty$. The equation (4) is the one which substitute $m = 0$ for equations (2) and (3). That is, equation (4) is the maximum benefit and the patterns of these expressways have no loop pattern.

5.THE PROPERTY OF EXPRESSWAYS

In this paper we discuss details implementations but the foundation of expressways and the property of the benefit and the effective patterns by using some simple models. They are special features to analyse linear facilities in a plane mathematically and to make clear the property of expressways.

They are argued that location problems setting the facilities represented as points in the subject region. In general, the property of the problem lead to the fact that sequential solutions and simultaneous solutions are almost equivalent. Location problems setting the linear facilities to the lines in the subject region in this paper are of the same property. The benefit of sequential optimization differ from the benefits of simultaneous optimization as seen in chapters 2,4 and 5. But these benefits are nearly equal. The ratio of the difference between these benefits is $(Bi_t - Be_t)/Bi_t$. Figure 14 shows the relation between this ratio and t, for the case $a = 1$ and $c = 1$ because the benefits are proportional to the cube of the side lengh of the subject region. The ratio is 0.0094 even when $t = 6$.

Furthermore the patterns of sequential optimization differ from the one of simultaneous optimizations where expressways are side by side. The benefit of sequential optimization is more than 99% of the simultaneous optimization's benefit. It is very difficult to plan using simultaneous optimization because of uncertainty of the future. Besides there are cases where planners must alter old plans. Therefore this result is very important when people plan expressways.

Figure 15 show the ratio of difference between the maximum value when $m = 0$, and the minimum value when m is maximum, for equations (2) and (3) where $a = 1$ and $c = 1$. The patterns in Figure 12 repesent the minimium and maximum cases. We think the loop pattern of $t = 4$ is an exception because this pattern is under construction where $t > 4$. If so, the benefits of the loop patterns are about 98% of the benefits of maximum patterns. When a loop pattern is necessary for another reason, the benefit of loop pattern is only about 2% less than the maximum benefit. Therefore, the benefit of expressways is almost maximum if the necessary conditions (i)-(v) are satisfied. These necessary conditions mean that expressways should be built as uniform as possible. From this, the patterns of almost maximum benefit are various.

The optimal solution is very convenient for mathematical programming. But urban planning prefers near optimal solutions because there are many cases in which planner can't put a facility on the optimal solution. This paper shows there are many near optimal patterns of approximate solutions satisfying the necessary conditions from (i)-(v). Planners can implement new expressways without departing much from the optimal solution and taking into account other requirements. This is an important result for planners.

The equation (4) represents the maximum benefit of this model. This function increases monotonously with respect to a, c, t and it is concave with respect to t because of the fifth necessary condition. The benefit has a little increase for putting on another expressway if there are a certain total length of expressways. This reason is that the effect of a new expressway is limited to the space between the two formerly adjacent expressways and do not extend to the all region.

On the other hand, the benefit of reduced travel time is a function of velocity v. We plot every Be_t and every equation (2) and (3) where $t \leq 10$ to show the relation between velocity and benefit. Figure 16 shows the relation between the benefit of $v = \infty$ and $v = 4$. Figure17 shows the relation between it of $v = \infty$ and $v = 2$. These lines have a slope of $1 - 1/v, (= c)$ and pass through the origin. The supremum of the ratio, benefit of $v =$ constant / the benefit of $v = \infty$, is $(1 - 1/v)$. Because this ratio is $(1 - 1/v)$ if the route to the center is same for any movement and any v. But these two values are nearly equal. The benefit of $v = \infty$ times $(1 - 1/v)$ is an approximate function of the benefit when v is constant.

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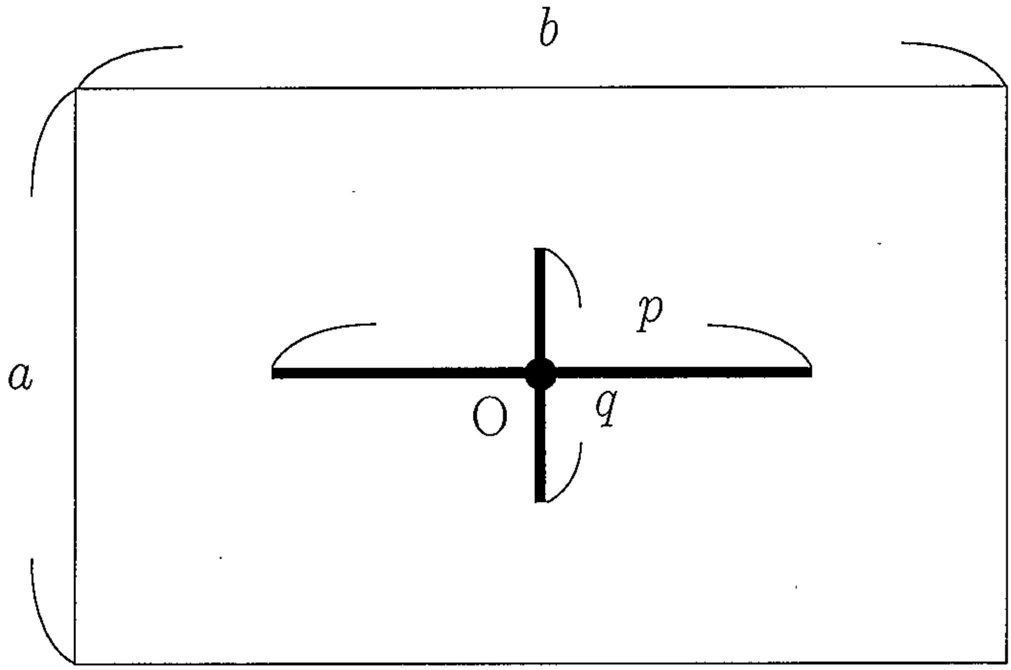

Figure 1: Model 1

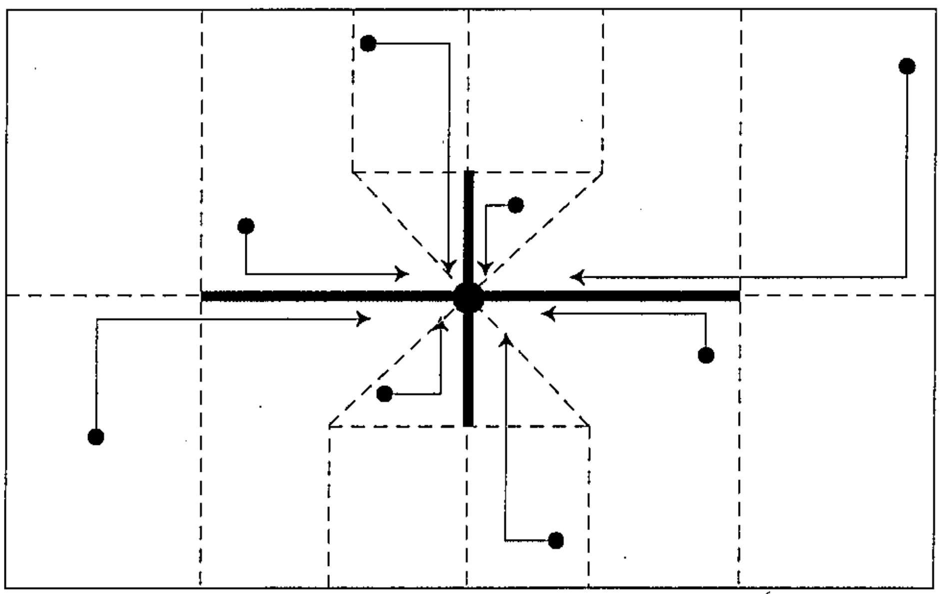

Figure 2: Routes from various origins

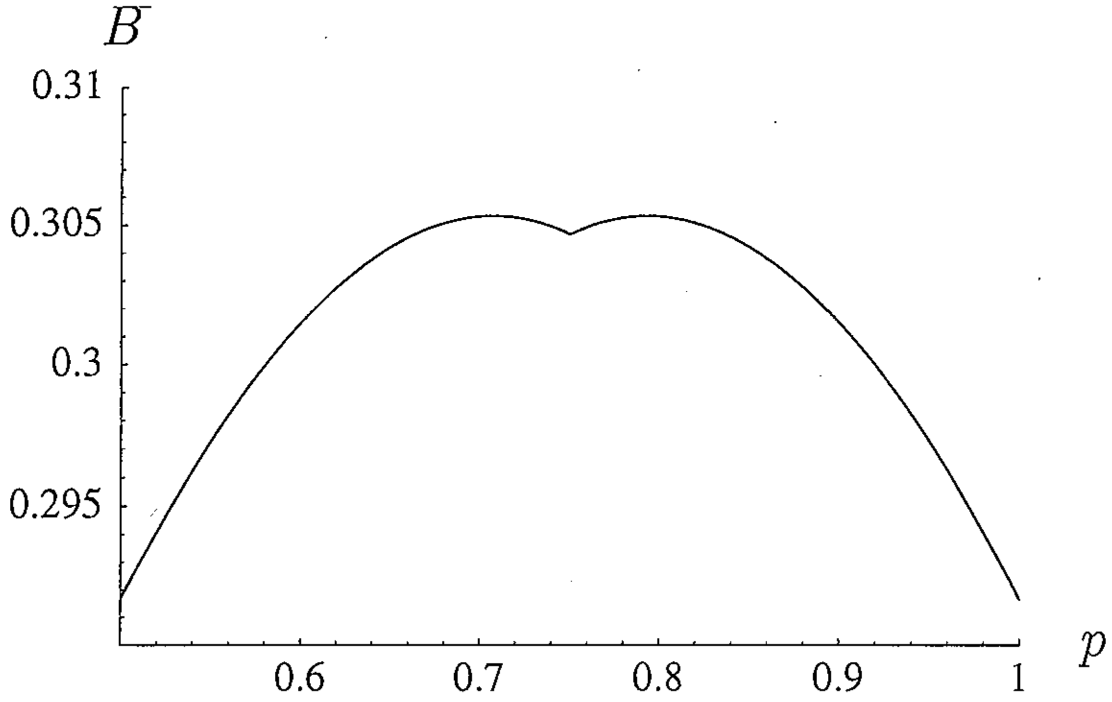

Figure 3: Relation between benefit to p if $a = b = 1, l = \frac{3}{2}$

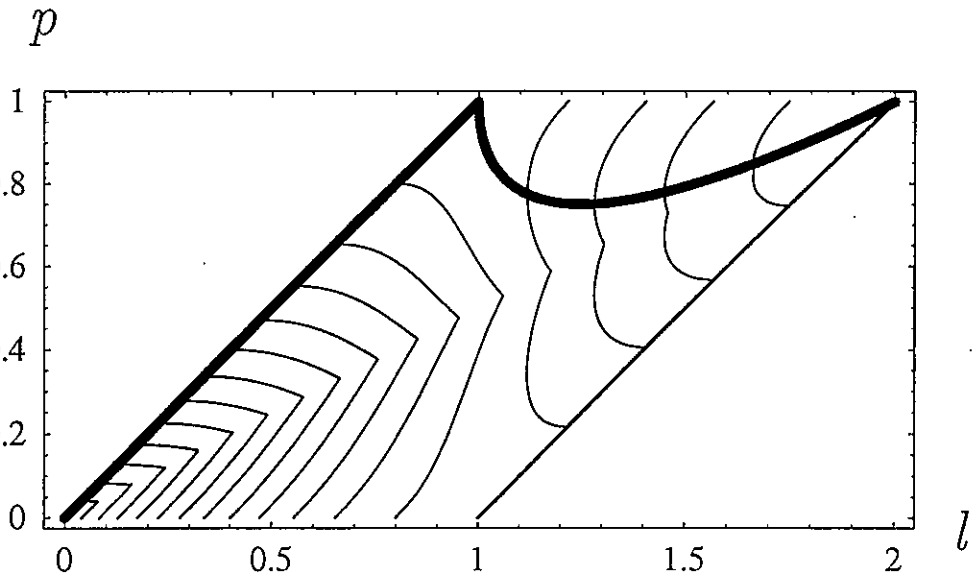

Figure 4: Relation among benefit, p and l if $a = b = 1$

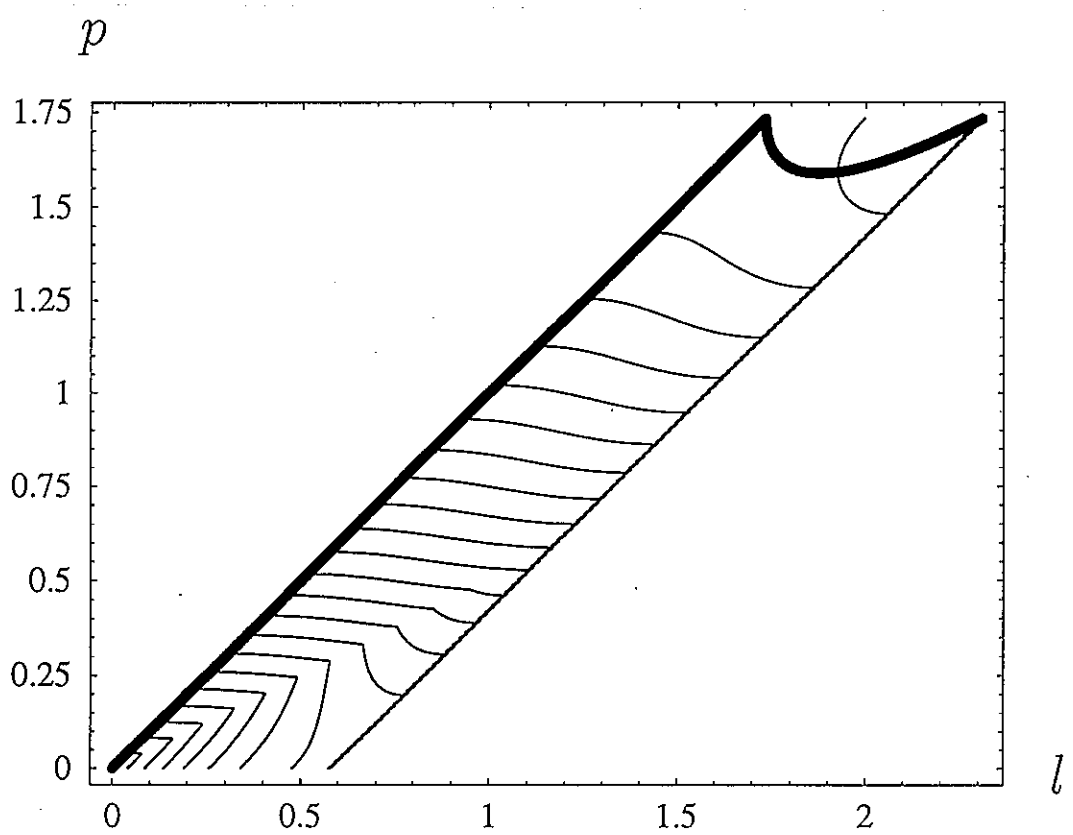

Figure 5: Relation among benefit, p and l if $a = \sqrt{3}, b = \frac{1}{\sqrt{3}}$

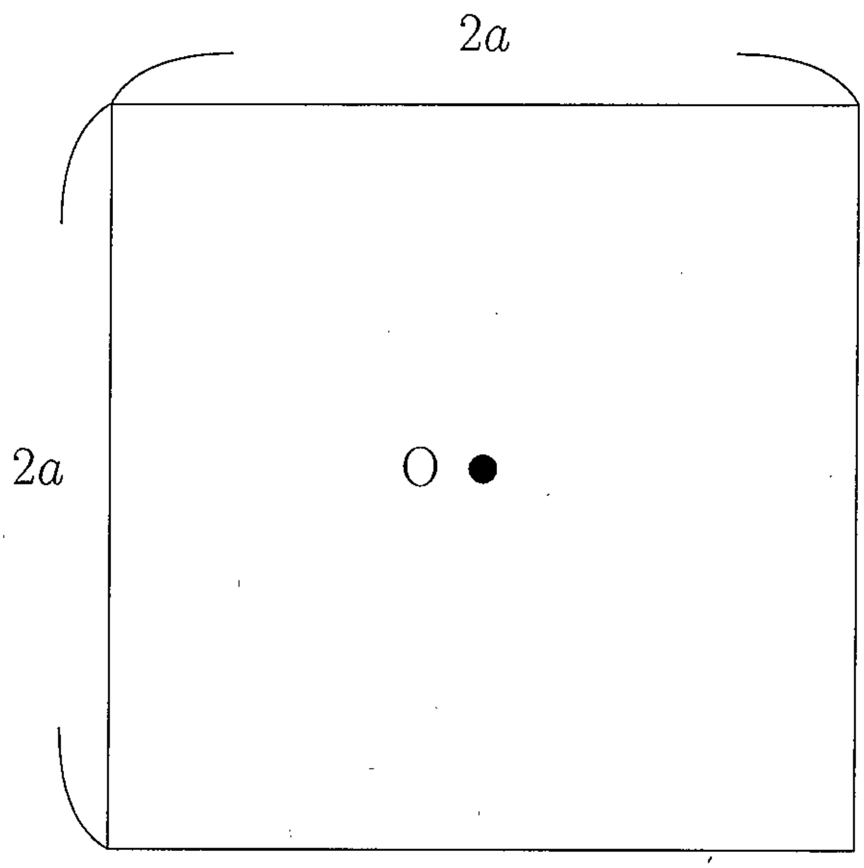

Figure 6: Model 2

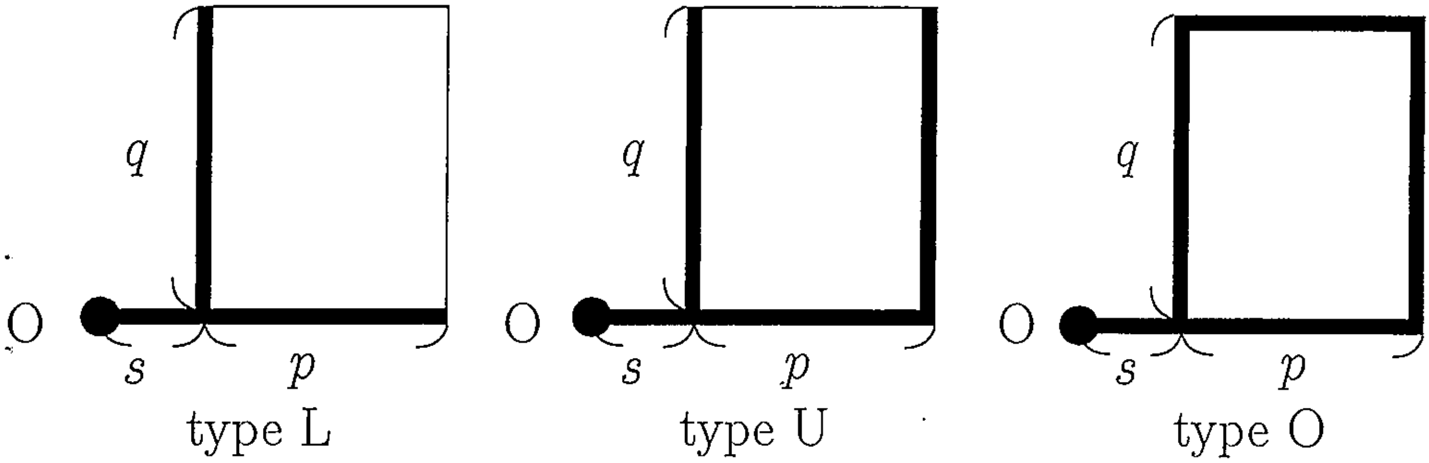

Figure 7: Model 3

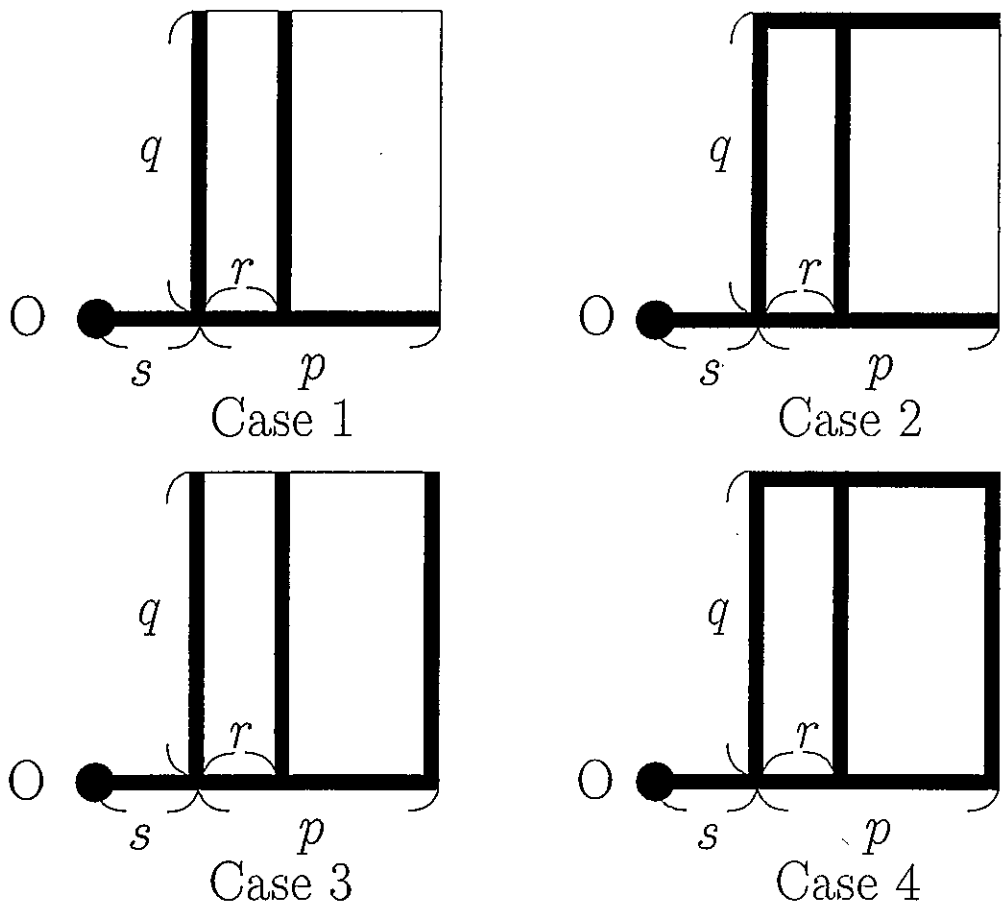

Figure 8: Model 4

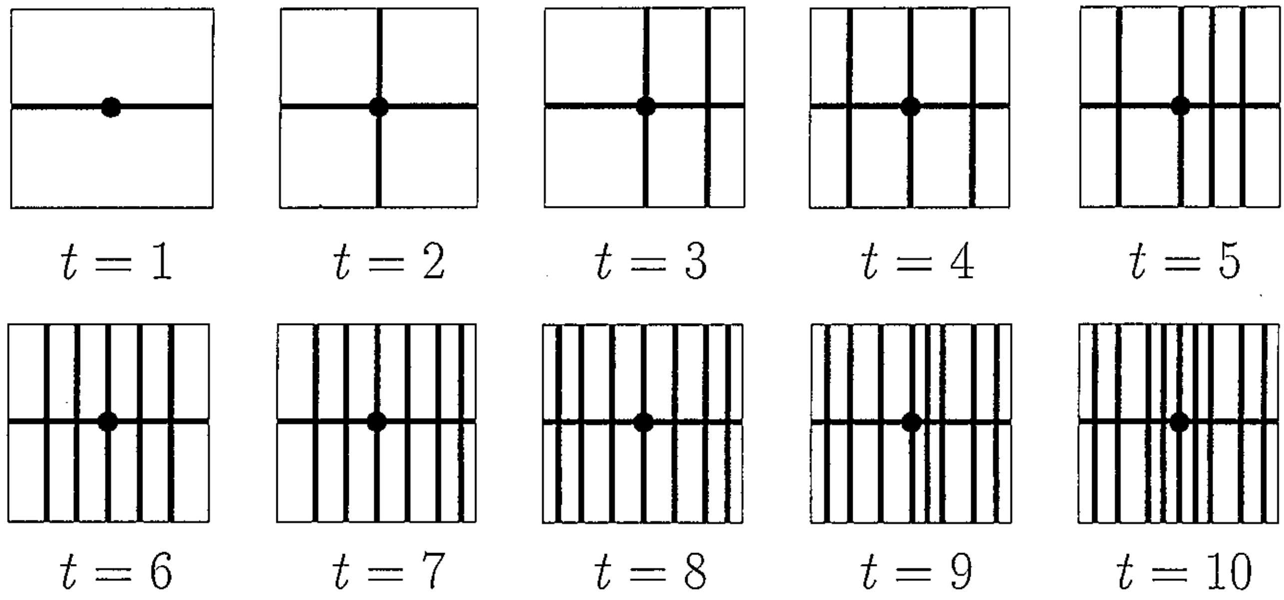

Figure 9: Patterns of Be_t

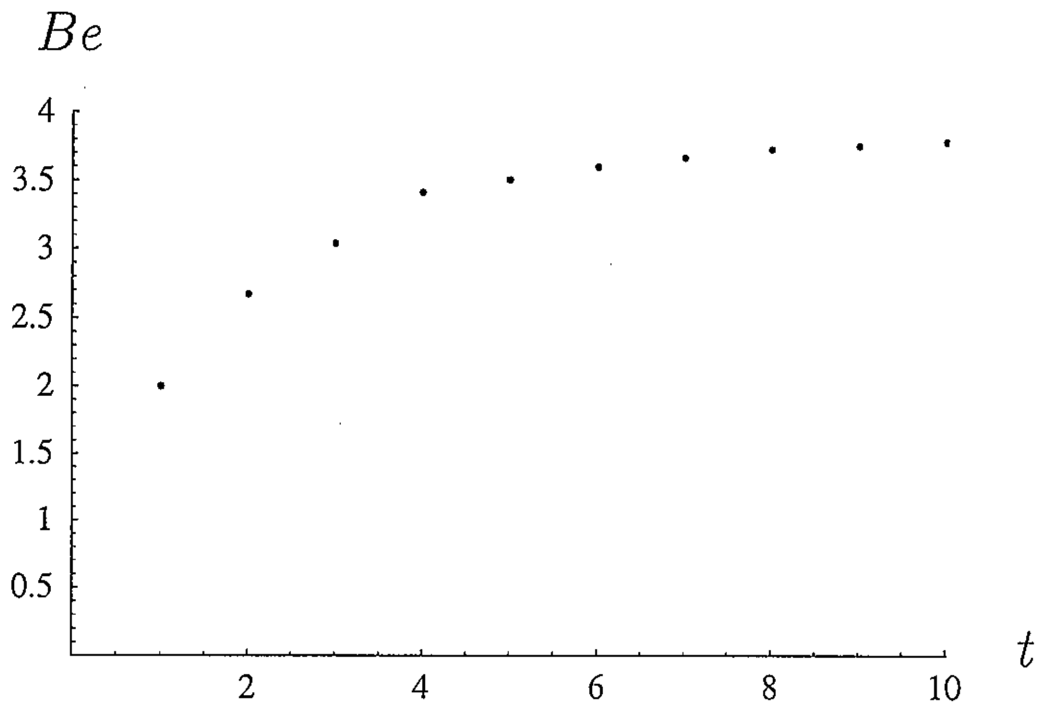

Figure 10: Benefit of sequential optimization

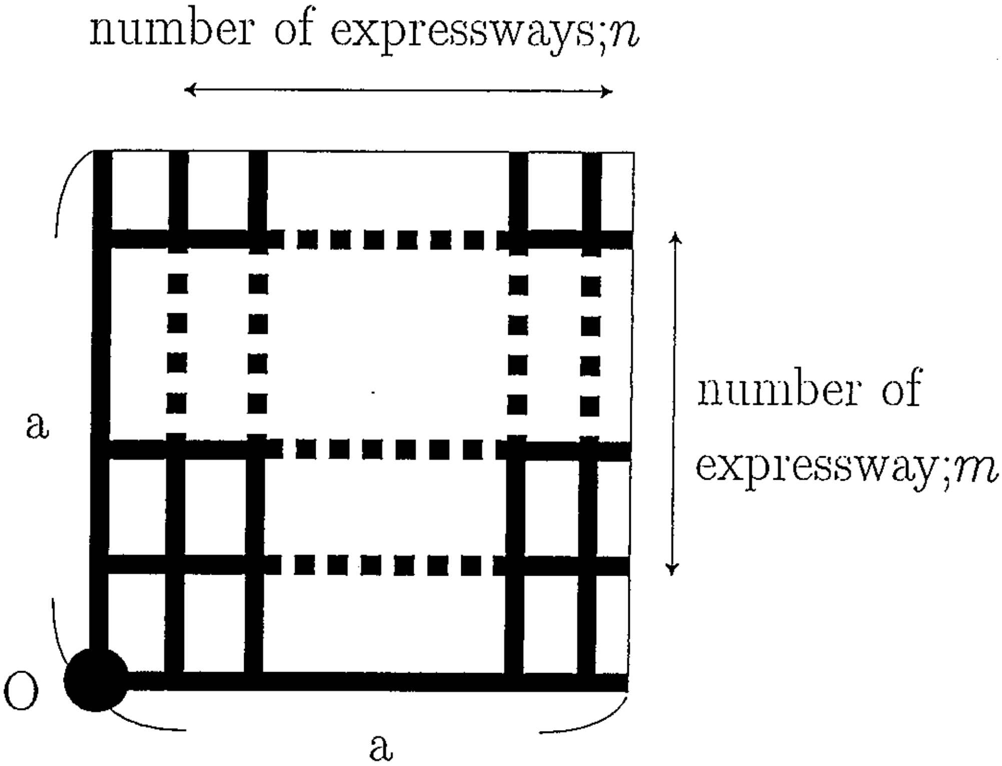

Figure 11: Model 5

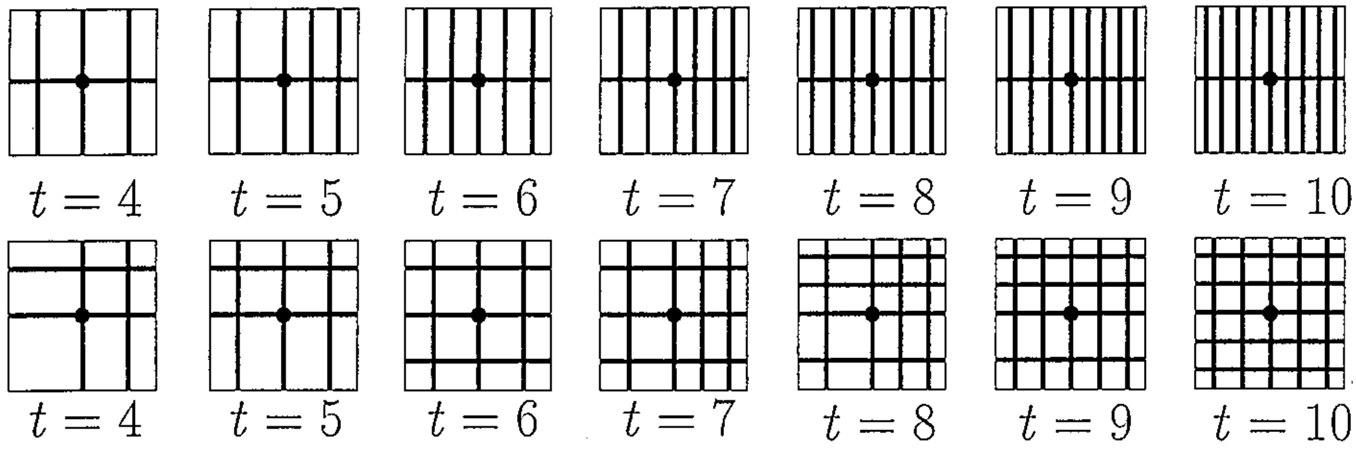

Figure 12: Patterns of maximum benefit(above) and mimimum benefit(below) in equations(7),(8) for $4 \leq t \leq 10$

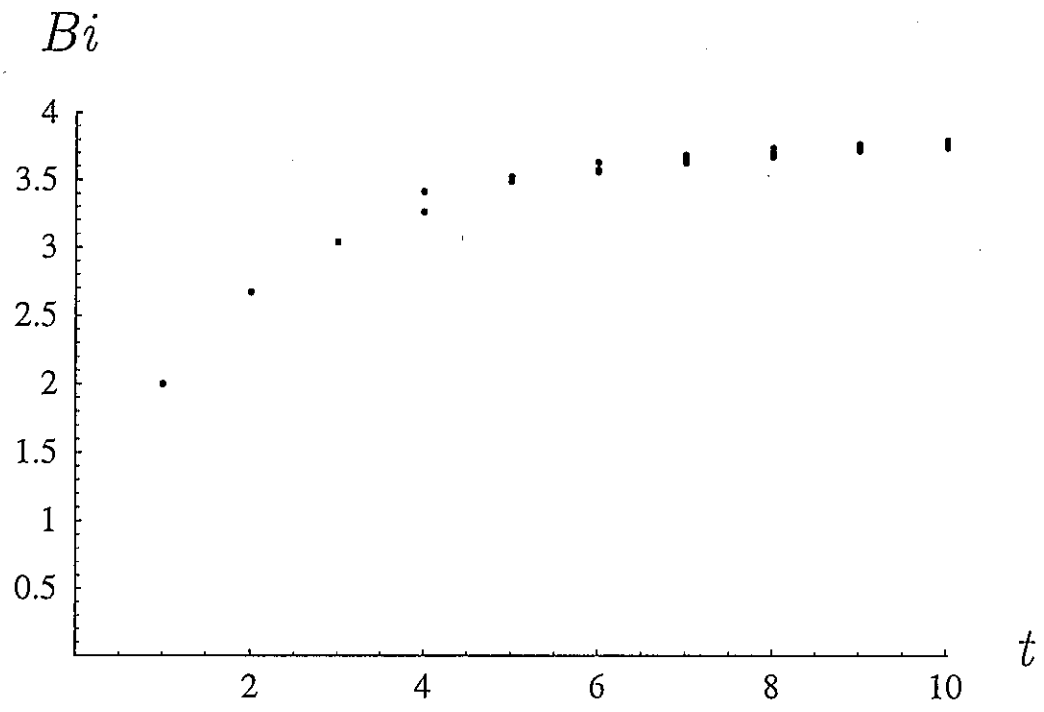

Figure 13: Benefit of simultaneous optimization

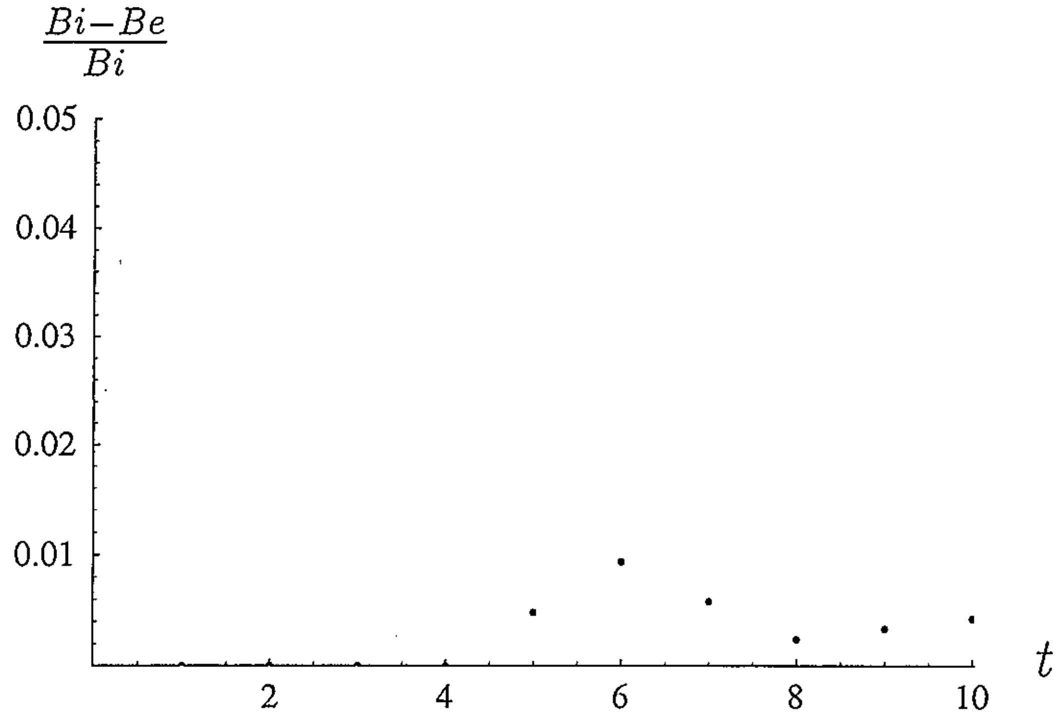

Figure 14: Ratio of difference between Bi_t and Be_t

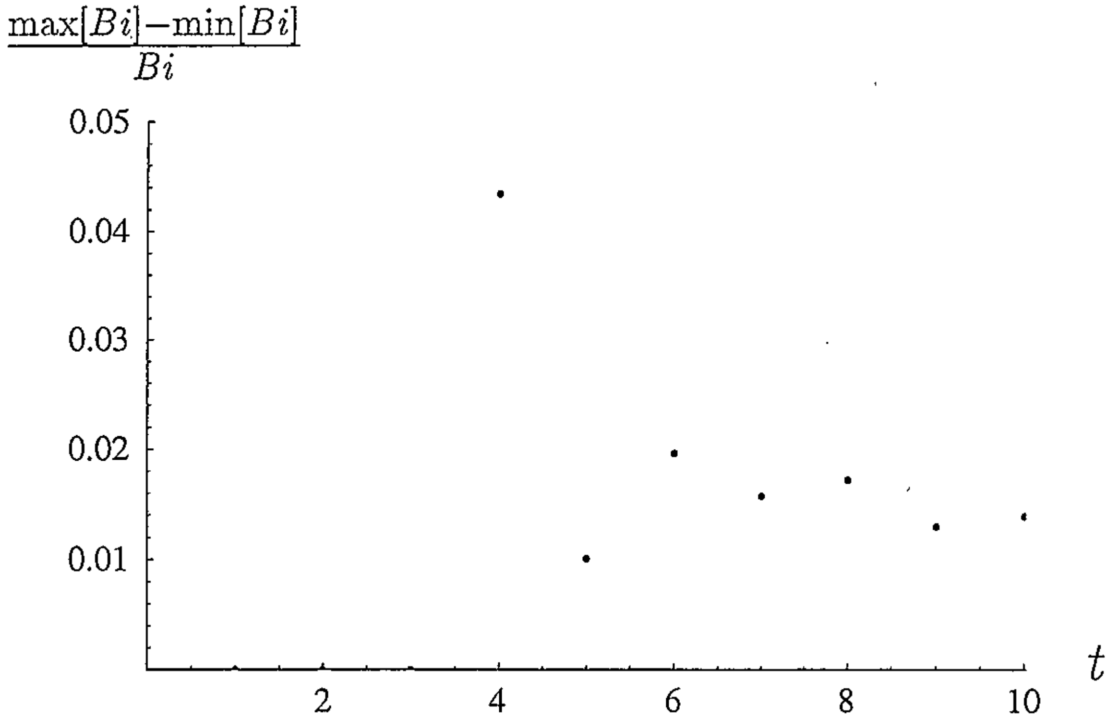

Figure 15: Ratio of difference between equations(7),(8) if $t' = n + m$

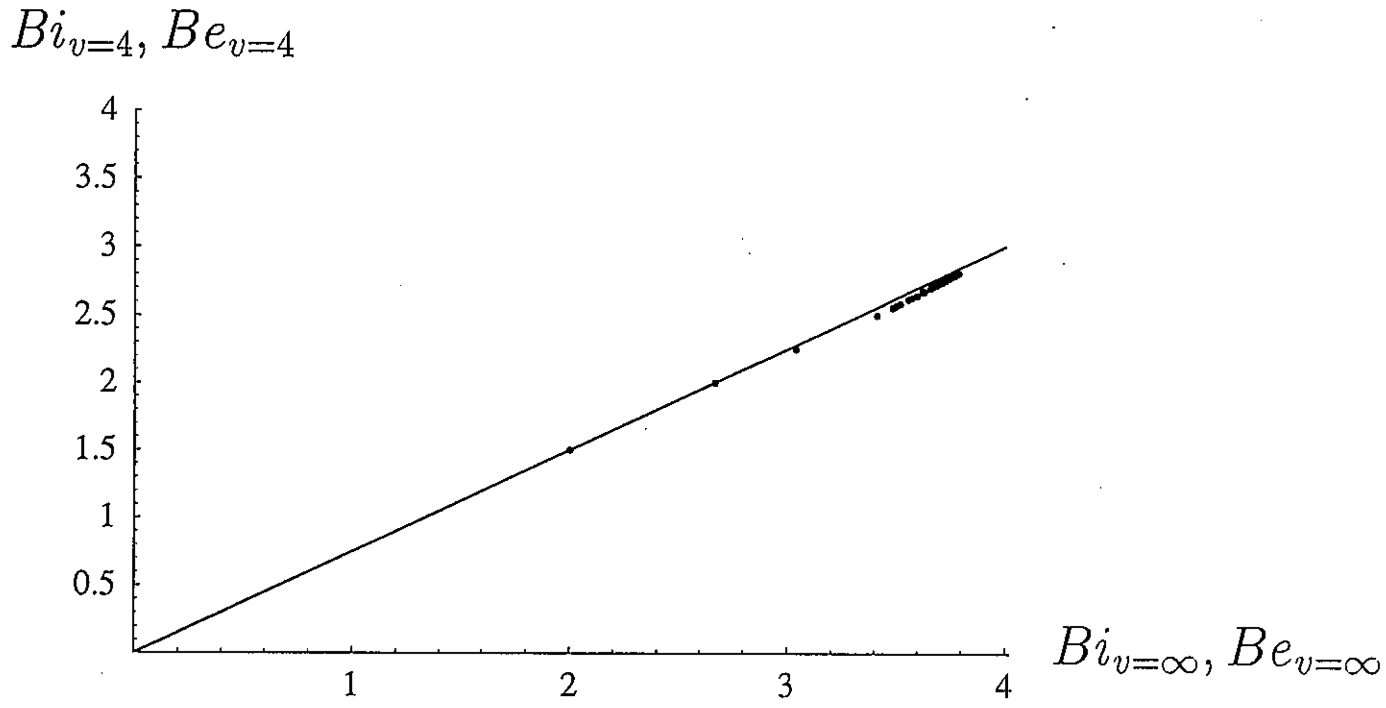

Figure 16: Relation between benefit of $v = \infty$ to benefit of $v = 4$

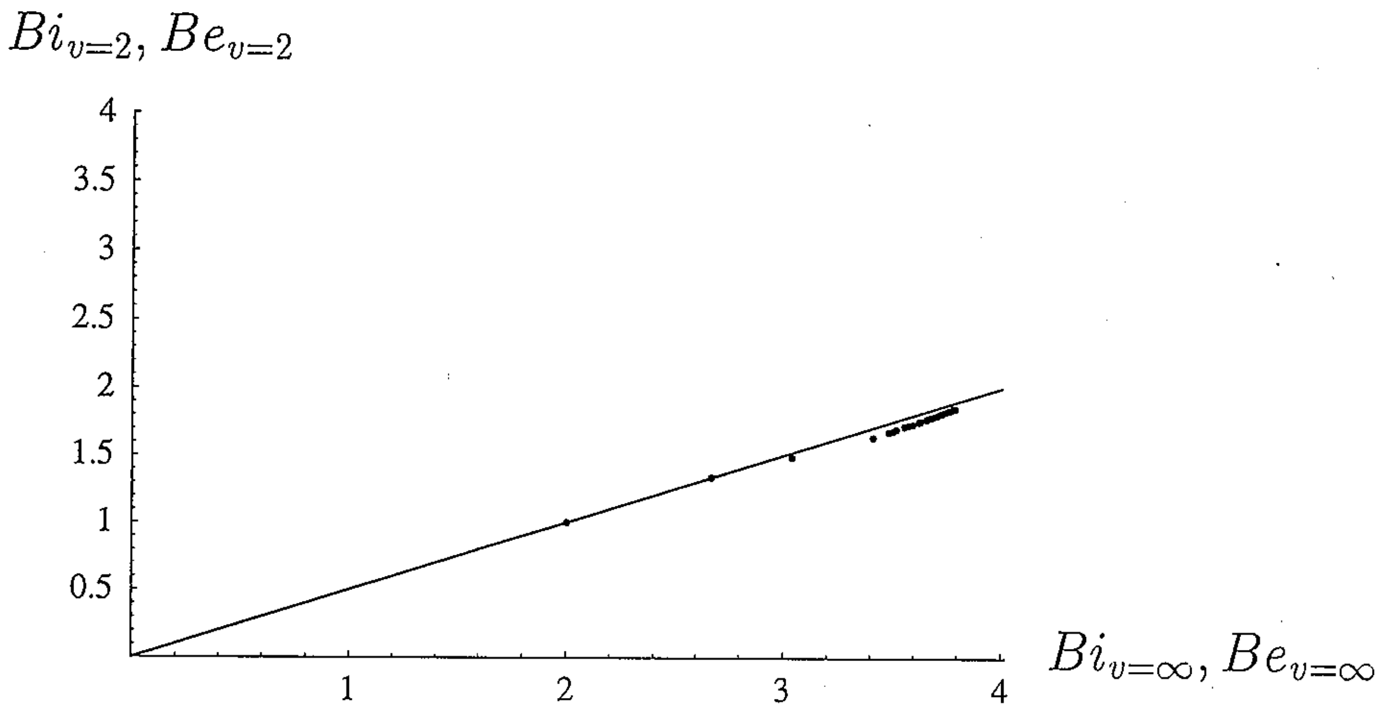

Figure 17: Relation between benefit of $v = \infty$ to benefit of $v = 2$