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A BICRITERIA EUCLIDEAN LOCATION ASSOCIATED WITH MAXIMIN AND MINIMAX CRITERIA

by

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Abstract

This article is devoted to the problem of locating a facility within a bounded polygon in continuous space with the objectives of maximizing the Euclidean distance to the nearest population center and minimizing the Euclidean distance to the farthest demand. An $O(kmn \log kmn)$ *algorithm for finding the analytical expressions of the efficient set and the trade-off curve for the bicriteria model, based on the nearest- and farthest-point Voronoi diagrams, is given, where* m*,* n *and* k *are the numbers of the population centers, the demands and the edges of the polygon, respectively. Some geometrical features of the efficient set and the trade-off curve are also presented.*

Keywords: *Location; undesirable facility; efficient set; trade-off curve; Voronoi diagram*

1. INTRODUCTION

In the last two decades, many articles have been devoted to the study of single facility bicriteria planar location models. For example, McGinnis and White(1978), Hansen and Thisse(1981), Hansen et al.(1981), Hamacher and Nickel(1996), Ohsawa(1998) analyzed the models associated with the minisum and minimax criteria, and Current et al.(1990) wrote a survey paper on multiobjective location models. In their formulations, the facility may be considered as *desirable*. On the other hand, some facilities have both desirable and *undesirable* characteristics, as indicated by Hansen and Thisse(1981), Erkut and Neuman(1989), Daskin(1995). Typical examples of these facilities are airports. An airport is desirable in the sense that inhabitants want to increase their accessibility to the airport. However, it is undesirable in the sense that the planners want it to be as far as possible from any inhabitant because of the noise. Another example is the radar base. The radar bases are desirable in the sense that the power required by the station is minimized such that all the points within a region can be monitored by the station. However, they are undesirable in the sense that these bases need to be located far from their neighborhood to maintain secrecy and to be well-defended. Fire stations, police stations and hospitals can also be considered as such facilities because they contribute to heavy traffic jam with much noise. Thus the planners locating these facilities frequently face the trade-off between minimizing the distance to the farthest inhabitant and maximizing the distance to the nearest inhabitant. Hansen and Thisse(1981) formulated a single facility bicriteria model associated with maximin and minisum criteria in Euclidean space and proposed the Big Square-Small Square method for finding the efficient set.

We are concerned with a single facility bicriteria model associated with maximin and minimax criteria in Euclidean space. The purpose of this paper is to present a procedure for generating the analytical expressions of the efficient set and the trade-off curve between the conflicting goals, and to characterize the efficient set and the trade-offs. It will be shown that the efficient set can be analytically expressed with the help of the nearest- and farthest-point Voronoi diagrams. Indeed the planners would be interested in the efficient set and the trade-off curve because by finding all the efficient locations, they can reduce the number of alternatives. The trade-off curve enables them to compare these alternatives graphically in terms of these two criteria.

When dealing with the bicriteria models associated with the minisum and minimax criteria in Euclidean space, the corresponding objective functions of these criteria are both convex. Accordingly, the efficient set is given by the set of minimizers of the convex combinations of these two criteria: see Geoffrion(1968). Hansen et al.(1981), Ohsawa(1998). On the other hand, the main difficulty here in finding the efficient set for our formulation arises from the fact that the objective function corresponding to the maximin criterion is neither quasi-convex nor quasi-concave. Therefore, the standard mathematical programming approach and the Geoffrion's result are useless for deriving the efficient set for our formulation.

The numerical solutions such as the straightforward modification of the Big Square-Small Square method and the method by Drezner and Wesolowsky(1980) may generate efficient locations for our model. However, as compared to the numerical methods, the analytical expression has at least two advantages. First, we will show that in general, the efficient set of our bicriteria model often consists of some discontinuous segments. This means that it is difficult to interpolate the efficient set based on the efficient locations generated by the numerical methods. Thus, only the analytic approaches enable us to find the efficient set, which considers the trade-offs between two criteria of resulting from the alternative locations. Second, analytical expressions will also enable the planner to carry out some sensitivity analyses both on the location of the facility in geographical space and on the corresponding cost in criterion space. Nevertheless, no analytical approaches to the subject have been reported in the literature.

The organization of this paper is as follows. The formulation and the solution of two single objective location models and the two types of Voronoi diagrams are presented in Section 2. Section 3 describes the analytical procedures for identifying the efficient set and the trade-off curve, based entirely on the geometrical analyses. Finally, we present our conclusions in Section 4.

2. LOCATION MODELS

Single Objective Models

Let Ω be a bounded, non-empty polygon in R^2. Let $\mathrm{p}_1, \cdots, \mathrm{p}_m$ represent the locations of m population centers on a plane. In the *maximin* model, a facility is established within Ω in order to maximize the Euclidean distance from the facility to its nearest population center:

see Hansen and Thisse(1983). Mathematically, this is

$$\max_{\mathrm{x}\in\Omega} \qquad F(\mathrm{x}) = \min_{i\in\{1,\cdots,m\}} \|\mathrm{x}-\mathrm{p}_i\|. \tag{1}$$

The minimizer, denoted by a^*, is called an *anti-center.* Let $\mathrm{q}_1, \cdots, \mathrm{q}_n$ represent the locations of n demand points on the plane. In the *minimax* model, a facility is set up in order to minimize the farthest Euclidean distances from the facility to these demand points: see Hansen and Thisse(1983). Mathematically, this is

$$\min_{\mathrm{x}\in\Omega} \qquad G(\mathrm{x}) = \max_{i\in\{1,\cdots,n\}} \|\mathrm{x}-\mathrm{q}_i\|. \tag{2}$$

The minimizer of $G(\mathrm{x})$, denoted by c^*, is called a *center.* The center is unique.

The location models (1) and (2) are called the *largest empty circle problem* and the *smallest enclosing circle problem*, respectively in Computational Geometry: see Shamos and Hoey(1975), Okabe and Suzuki(1997). For a specified value λ, the corresponding level set $L_F(\lambda)$ (resp. $L_G(\lambda)$) is defined as the set $\{\mathrm{x}|F(\mathrm{x}) \le \lambda\}$ (resp. $\{\mathrm{x}|G(\mathrm{x}) \le \lambda\}$). It is evident from the definition that the level set $L_F(\lambda)$ (resp. $L_G(\lambda)$) is the union (resp. intersection) of the region inside the disks around the population centers with a radius of λ. Thus, the level set $L_F(\lambda)$ is neither convex nor concave for $\forall\lambda > 0$, and the level set $L_G(\lambda)$ is convex for $\forall\lambda > 0$: see Hansen and Thisse(1983). We denote their boundaries by $\partial L_F(\lambda)$ and $\partial L_G(\lambda)$.

Voronoi Diagrams

In basic Voronoi diagram literature, for the set of population centers $\mathrm{p}_1, \cdots, \mathrm{p}_m$ (resp. demands $\mathrm{q}_1, \cdots, \mathrm{q}_n$), the *nearest-point* (resp. *farthest-point*) *Voronoi polygon* associated with the ith population center (resp. jth demand) is denoted by V_i (resp. W_i) and is defined as follows:

$$V_i = \bigcap_{k\in\{1,\cdots,m\}\setminus\{i\}} \{\mathrm{x} \mid \|\mathrm{x}-\mathrm{p}_i\| \le \|\mathrm{x}-\mathrm{p}_k\|\}, \tag{3}$$

$$W_j = \bigcap_{k\in\{1,\cdots,n\}\setminus\{j\}} \{\mathrm{x} \mid \|\mathrm{x}-\mathrm{q}_j\| \ge \|\mathrm{x}-\mathrm{q}_k\|\}. \tag{4}$$

It is evident from these definitions that both V_i and W_j are closed and convex. We denote their boundaries by ∂V_i and ∂W_j, respectively. The union of $V_1, \cdots, V_m$ (resp. $W_1, \cdots, W_n$) is called a *nearest-point* (resp. *farthest-point*) *Voronoi diagram*: see Shamos and Hoey(1975). The edges and vertices of these diagrams are called the *nearest-point* (resp. *farthest-point*) *Voronoi edges* and the *nearest-point* (resp. *farthest-point*) *Voronoi vertices*, respectively.

When the feasible region Ω is the convex hull of $p_1, \cdots, p_m$, the maximin model (1) can be solved in $O(m \log m)$ time geometrically with the help of the nearest-point Voronoi diagram: see Shamos and Hoey(1975). Indeed, the anti-center a^* is defined either by a nearest-point Voronoi vertex or by the intersection of a nearest-point Voronoi edge and the boundary of that convex hull. On the other hand, when the feasible region Ω is the entire plane, i.e., $\Omega = R^2$, the minimax model (2) can also be solved in $O(n \log n)$ time geometrically with the help of the farthest-point Voronoi diagram: see Shamos and Hoey(1975). Indeed, the center c^* is defined either by a farthest-point Voronoi vertex, or by the intersection of a farthest-point Voronoi edge and the line segment connecting the two points generating this edge. Ohsawa and Imai(1997) analyzed the near-optimality of the minimax model by use of a farthest-point Voronoi diagram. Thus, these two location models can be dealt with as discrete optimization problems. An example of the nearest- and farthest-point Voronoi diagrams on the squared feasible region is shown in Figure 1, where $m = n = 5$, $p_i = q_i$ for $i = 1, \cdots, 5$. In this figure, the edges of the nearest- and farthest-point Voronoi edges are indicated by the faintest and the second faintest segments respectively, and the anti-center a^* and the center c^* are plotted by asterisks.

Dominance and Efficiency

For $x, y \in \Omega$, we say x dominates y and define $x \succ y$ if and only if $F(y) \leq F(x)$ and $G(y) \geq G(x)$, with strict inequality for at least one inequality. The point x is *efficient* (*Pareto-optimal*) if and only if there does not exist some $y (\in \Omega \setminus x)$ such that $y \succ x$. Let us call the set of all efficient points the *efficient set* and denote it by S^*. Since c^* optimizes the minisum model (2) uniquely, $c^* \in S^*$. When we use actual data, they are located in general positions, so that a^* is unique. In this case, $a^* \in S^*$.

3. BICRITERIA MODEL

Properties for Constructing the Efficient Set

Suppose that the feasible region Ω consists of k edges. Before stating the method for generating the efficient set, two properties will be described to justify the procedure for constructing S^*.

First, for a fixed λ, consider a constrained minimax model in which the objective is to minimize $G(x)$ subject to the constraint $x \in \Omega \setminus L_F(\lambda)$. The minimizer of this constrained minimax model is denoted by $s^*(\lambda)$. Clearly, as λ increases, the region $\Omega \setminus L_F(\lambda)$ will shrink, and will disappear for $\lambda > F(a^*)$. Since $G(x)$ is convex, $s^*(\lambda) \in \partial L_F(\lambda)$, i.e., $s^*(\lambda) = \mathrm{argmin}_{x \in \partial L_F(\lambda)} G(x)$. As implicitly shown by Hansen et al.(1981), $S^* = \{s^*(\lambda) | \lambda \in [F(c^*), F(a^*)]\}$.

Second, for $F(c^*) < \forall\lambda \leq F(a^*)$, $s^*(\lambda)$ lies on a nearest-point Voronoi edge associated with $p_1, \cdots, p_m$ or on a farthest-point Voronoi edge associated with $q_1, \cdots, q_n$ or on an edge within $\partial\Omega$. Thus, we see that unless $s^*(\lambda)$ lies on $\partial\Omega$, it is equidistant from either two population centers or two demands. Let us show this important property. Divide the feasible region Ω into mn subregions $V_i \cap W_j \cap \Omega$ $(i = 1, \cdots, m; j = 1, \cdots, n)$. For $\forall x \in V_i \cap W_j \cap \Omega$, define the point x' as a minimizer of $G(y)$ with $y \in V_i \cap W_j \cap \Omega$ and $F(y) = F(x)$. It follows from the definition of V_i and W_j in (3) and (4) that $x \prec x'$, and x' lies on the intersection point of the circle around p_i with a radius of $F(x)$ and either $\partial(V_i \cap W_j \cap \Omega)$ or the line segment connecting p_i and q_j. These situations are illustrated in Figure 2. Unless x' lies on $\partial(V_i \cap W_j \cap \Omega)$, $x \prec x'$, so $x \prec z$, where z is an intersection point of the line segment connecting p_i and q_j, and $\partial(V_i \cap W_j \cap \Omega)$. This completes the proof.

Method for Constructing the Efficient Set

Let E be the set consisting of all the nearest-Voronoi edges associated with $p_1, \cdots, p_m$ and all the farthest-point Voronoi edge associated $q_1, \cdots, q_n$, and all the edges of $\partial\Omega$. As we have seen, our problem is reduced to finding the efficient set from E. For $e \in \partial(V_i \cap W_j)$, let p'_i and q'_j denote the foot of the perpendicular from p_i and q_j to e, respectively. For $\forall x \in e$, three cases must be considered separately: *Case 1:* p'_i is between q'_j and x; *Case 2:* x is between p'_i and q'_j; *Case 3:* q'_j is between p'_i and x. These situation are given in Figure 3. From this Figure, we can obtain the following relationship between $F(x)$ and $G(x)$ based on the Pythagorean theorem:

$$G(x)^2 = \left(\sqrt{F(x)^2 - \alpha^2} + \gamma\right)^2 + \beta^2 \qquad \text{if } p'_i \text{ is between } q'_j \text{ and } x, \tag{5}$$

$$G(x)^2 = \left(\sqrt{F(x)^2 - \alpha^2} - \gamma\right)^2 + \beta^2 \qquad \text{otherwise.} \tag{6}$$

where $\alpha = \|p_i - p'_i\|$, $\beta = \|q_j - q'_j\|$ and $\gamma = \|p'_i - q'_j\|$. Thus, for $\forall e \in E$, the relationship between $F(x)$ and $G(x)$ for all the points within e can be expressed analytically by both or either of (5) and (6). This enables us to compare the alternatives analytically in criterion space.

Figure 4 shows the graph of $(F(\mathrm{x}), G(\mathrm{x}))$ for all the points within E, with the horizontal and vertical axes measuring the values of $F(\mathrm{x})$ and $G(\mathrm{x})$, respectively. In this Figure, as in the case of Figure 1, the edges of the nearest- and farthest-point Voronoi edges are indicated by the faintest and the second faintest segments, respectively. As we moves along a edge, $F(\mathrm{x})$ and $G(\mathrm{x})$ changes continuously, so this graph in criterion space has the same network structure as the graph consisting of nearest- and farthest-point Voronoi edges and the edges of $\partial\Omega$ in geographical space. In this figure, an efficient point has no alternatives northwest of it. In addition, $S^* = \{\mathrm{s}^*(\lambda) | \lambda \in [F(\mathrm{c}^*), F(\mathrm{a}^*)]\}$ as we have seen, and $G(\mathrm{s}^*(\lambda))$ is increasing with respect to λ for $F(\mathrm{c}^*) < \lambda < F(\mathrm{a}^*)$. This is because as λ increases, the region $\Omega \setminus L_F(\lambda)$ will shrink, and $\mathrm{s}^*(\lambda)$ lies on $\partial L_F(\lambda)$, as we have noted. To avoid misunderstanding, we call the set of the point $(F(\mathrm{x}), G(\mathrm{x}))$ for all $\mathrm{x} \in S^*$ the *trade-off curve*. Therefore, we see that the trade-off curve coincides with the lower envelope of the collection of these curves for $F(\mathrm{c}^*) \leq F(\mathrm{x}) \leq F(\mathrm{a}^*)$.

The algorithm to find the efficient set and the trade-off curve can be stated in a compact form as follows:

1. In geographical space, construct the nearest-point Voronoi diagram associated with $\mathrm{p}_1, \cdots, \mathrm{p}_m$, and the farthest-point Voronoi diagram associated with $\mathrm{q}_1, \cdots, \mathrm{q}_n$ on the entire plane.

2. Let E be the set of the nearest- and farthest-point Voronoi edges and the edges of $\partial\Omega$.

3. In criterion space, plot the curves (5) and (6) corresponding to all the edges within E for $F(\mathrm{c}^*) \leq F(\mathrm{x}) \leq F(\mathrm{a}^*)$.

4. Find the lower envelope of these curves.

5. Let S^* be the set of edges or subedges (which are obtained by splitting up a edge within E) corresponding to the curves on the lower envelope.

The efficient set S^* is shown in Figure 5 as the thick segments. As shown in this Figure, the efficient set consists of four parts which are numbered according to the increasing order of the maximin value, i.e., $F(\mathrm{x})$. Thus, $\mathrm{s}^*(\lambda)$ moves from c^* to u_2 through u_1 along two Voronoi edges, then jumps to u_3 and moves to u_4 along a Voronoi edge, then jumps to u_5 and moves to u_7 through u_6 along a Voronoi edge and an edge of $\partial\Omega$, and then jumps to u_8 and moves to a^* along an edge of $\partial\Omega$. Let us now study the complexity of this method. Step

1 can be done in $O(m \log m) + O(n \log n)$ time by means of a divide-and-conquer technique: see Shamos and Hoey(1975). Since there are $O(m)$ nearest-point Voronoi edges and $O(n)$ farthest-point Voronoi edges in the entire plane, respectively: see Shamos and Hoey(1975), the cardinality of the set E is $O(kmn)$. So, Steps 2 and 3 requires $O(kmn)$ operations. Elementary manipulations show that for two edges within E, their corresponding curves (5) or (6) intersect each other at most two times. This means that the lower envelope consists of $O(kmn)$ curves, thus Step 4 requires $O(kmn \log kmn)$ time in the worst case with a divide-and-conquer method: see Boissonnat and Yvinec(1998). Since the number of the curves on the lower envelope is $O(kmn)$, Step 5 can be carried out in $O(kmn)$ time. Therefore, the total time complexity is $O(kmn \log kmn)$.

The following remarks should now be stated. First, when the feasible region Ω is convex with $O(k) \leq O(m)$ and $O(k) \leq O(n)$, the total time complexity reduces to $O(mn \log mn)$ because $\partial\Omega$ intersects each Voronoi edge at most two times.

Second, instead of the Euclidean distance, consider the model with the transportation cost function increasing and continuous in the Euclidean distance. It should be noted that the efficient set of a bicriteria model using this cost function coincides with that using a simple Euclidean distance. This is because this cost function does not affect the shapes of the contours of both $F(\mathrm{x})$ and $G(\mathrm{x})$. Therefore, our algorithm can also be applied directly to the models with the cost function.

Efficient Set and Trade-Off Curve

The following three properties can be defined. First, the efficient set in geographical space consists of only $O(kmn)$ line segments, and the trade-off curve in criterion space consists of only $O(kmn)$ curves (5) and (6). Thus, the delineation of the efficient set and the construction for the trade-off curve may easily be accomplished by using some software of GIS. Also, the geometrical expressions of the efficient set and the trade-off curve may be of value in terms of intuitively understanding.

Second, the inspection of Figures 4 and 5 confirms the fact that jumps of the trade-off curve in criterion space occur at some nearest-point Voronoi vertices in geographical space. In any case, such jumps occur only at nearest-point Voronoi vertices, the intersections of nearest-point Voronoi edges and $\partial\Omega$, and vertices of $\partial\Omega$. An intuitive explanation of this is that as λ increases, $\Omega \setminus L_F(\lambda)$ will shrink into these vertices and intersections, and some of the subregions of $\Omega \setminus L_F(\lambda)$ will disappear. So, if $\mathrm{s}^*(\lambda)$ was on such subregion, it must jump towards the new feasible region. This jump induces the minimax value, i.e., $G(\mathrm{x})$ to discontinuously go up. Therefore, we can conclude that these vertices and intersections tend to be optimal for the scalarized location problems in which the objective is to minimize a convex combination of the maximin and minimax objective functions with suitable weights assigned to each of the two objectives.

Third, an observation of Figure 4 shows that the trade-off curve corresponding to either the nearest-point Voronoi edges or the edges of $\partial\Omega$ becomes concave. This means that interior points of these edges cannot be optimal for the scalarized location problem. Indeed, we see from Figure 4 that only the points between c^* and u_1, and two points u_7, a^* can be optimal for each scalarized problem. Let us show the property that in any case, the trade-off curve corresponding to the nearest-point Voronoi edges is concave, and when Ω is convex, the trade-off curve corresponding to the edges of $\partial\Omega$ is also concave. For any $e \in S^* \cap (\partial(V_i \cap W_j \cap \Omega) \setminus \partial W_j)$, we may assume that p_i lies on the same side of the line including e from q_j. (Otherwise, since $e \in \partial V_i$, there exists $k(\neq i)$ with $e \in \partial V_i \cap \partial V_k$, so we can replace i by k). For $\forall \mathrm{x} \in e$, we again consider the three cases defined in the previous discussion. It should be noted that in the last two cases and the first case with $\alpha/F(\mathrm{x}) \leq \beta/G(\mathrm{x})$, x cannot be efficient, so we can exclude these cases from consideration. This can be seen in Figure 6, where for each case, there exists the point $\mathrm{x}' \in V_i \cap W_j \cap \Omega$ such that $\|\mathrm{x}' - \mathrm{p}_i\| = \|\mathrm{x} - \mathrm{p}_i\|$ and $\|\mathrm{x}' - \mathrm{q}_j\| < \|\mathrm{x} - \mathrm{q}_j\|$, so x' dominates x. Therefore, it suffices to examine only the first case with $\alpha/F(\mathrm{x}) > \beta/G(\mathrm{x})$, i.e., the function (5) with $\alpha G(\mathrm{x}) > \beta F(\mathrm{x})$. Differentiating $G(\mathrm{x})$ two times with respect to $F(\mathrm{x})$ yields

$$\frac{d^2G(\mathrm{x})}{dF(\mathrm{x})^2} = \frac{1}{G(\mathrm{x})^3(F(\mathrm{x})^2 - \alpha^2)} \left(-\alpha^2 G(\mathrm{x})^2 \left(1 + \frac{\gamma}{\sqrt{F(\mathrm{x})^2 - \alpha^2}} \right) + \beta^2 F(\mathrm{x})^2 \right). \quad (7)$$

Since $F(\mathrm{x}) \geq \alpha$ and $\alpha^2 G(\mathrm{x})^2 > \beta^2 F(\mathrm{x})^2$, we have $\frac{d^2G(\mathrm{x})}{dF(\mathrm{x})^2} < \frac{-\alpha^2 G(\mathrm{x})^2 + \beta^2 F(\mathrm{x})^2}{G(\mathrm{x})^3(F(\mathrm{x})^2 - \alpha^2)} < 0$, as required.

4. CONCLUSIONS AND EXTENSIONS

In this paper, we have presented an analytical procedure of delineating the efficient set and the trade-off curve of the single facility bicriteria planar location model with maximin and minimax criteria. In addition, we have characterized geometrically the efficient set and the trade-off curves corresponding to the efficient locations.

One possible extension of our methods is to apply it to the bicriteria model associated with the generalized maximin and minimax criteria. Instead of maximizing the nearest distance as defined in (1) (resp. minimizing the farthest distance as defined in (2)), consider the optimization problem in which the sth nearest distance is maximized (resp. the tth farthest distance is minimized). This corresponds to the case where the planner makes the facility planning such that the minimum distance to $(m+1-s)$ population centers is maximized (resp. the maximum distance to $(n+1-t)$ demands is minimized) without taking $s-1$ population centers (resp. $(t-1)$ demands) into account. For example, to construct a new airport, some population centers in the neighborhood of the airport may be transferred, and some demands may be ignored because their locations are too far from the airport. We shall call this type of model the *generalized* maximin model (resp. *generalized* minimax model). For each model, instead of using an ordinary Voronoi diagram, a Voronoi diagram of order s (resp. of order $n+1-t$) can be used. More detail on this diagram can be found in Shamos and Hoey (1975), Lee(1982). Since Voronoi diagrams of any order consist of some line segments, we can derive the efficient set and the trade-off curve of the bicriteria model associated with generalized maximin and minimax criteria by similar procedures.

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References

[1] Boissonnat, J.T. and M. Yvinec(1998): *Algorithmic Geometry.* Cambridge University Press, Cambridge.

[2] Current, J., H. Min and D. Schlling(1990): Multiobjective analysis of facility location decisions, *European Journal of Operational Research, 49*, 295-307.

[3] Daskin, M.S.(1995): *Network and Discrete Location.* Wiley, New York.

[4] Drezner, Z. and G.O. Wesolowsky (1980): A maximin location problem with maximum distance constraints, *AIIE Transactions, 12*, 49-252.

[5] Erkut, E. and S. Neuman (1989): Analytical models for locating undesirable facilities, *European Journal of Operational Research, 40*, 275-291.

[6] Geoffrion, A.M.(1968): Proper efficiency and the theory of vector maximization, *Journal of Mathematical Analysis and Applications, 22*, 618-630.

[7] Hamacher, H.W. and S. Nickel (1996): Multicriteria planar location problems, *European Journal of Operational Research, 94*, 66-86.

[8] Hansen, P. and J.F. Thisse (1981): The generalized Weber-Rawls problem in *Operational Research '81*, ed. by J.P. Brans. North-Holland, Amsterdam.

[9] Hansen, P. and J.F. Thisse (1983): Recent advances in continuous location theory, *Sistemi Urbani, 5*, 33-54.

[10] Hansen, P., D. Peeters and J.F. Thisse (1981): Constrained location and the Weber-Rawls problem, *Annals of Discrete Mathematics, 11*, 147-166.

[11] Lee, D.T. (1982): On k-nearest neighbor Voronoi diagrams in the plane, *IEEE Transactions on Computers, 31*, 478-487.

[12] McGinnis, L.F. and J.A. White(1978): Single facility rectilinear location problem with multiple criteria, *Transportation Science, 12*, 217-231.

[13] Ohsawa, Y. and A. Imai (1997): Degree of locational freedom in a single facility Euclidean minimax location model, *Location Science, 5*, 29-45.

[14] Ohsawa, Y. (1998): A geometric solution for quadratic bicriteria location models, *European Journal of Operational Research*, to appear.

[15] Okabe, A. and A. Suzuki (1997): Locational Optimization problems solved through Voronoi diagrams, *European Journal of Operational Research, 98*, 445-456.

[16] Shamos, M.I., and D. Hoey (1975): Closest-point problems. Proceedings of the 16th Annual IEEE Symposium on Foundations of Computer Science, 151-162.

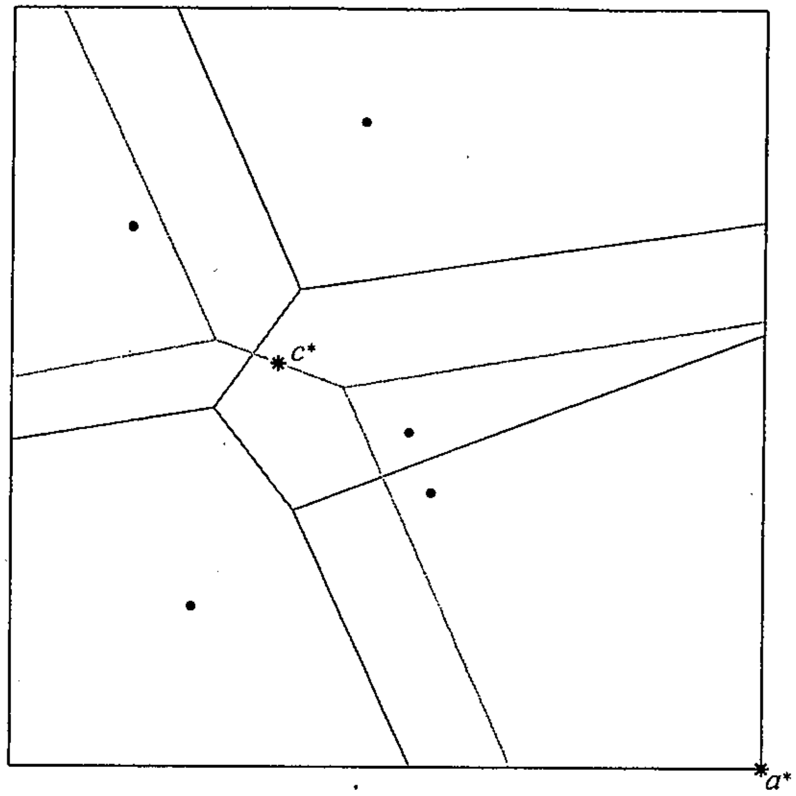

Figure 1: Nearest- and farthest-point Voronoi diagrams

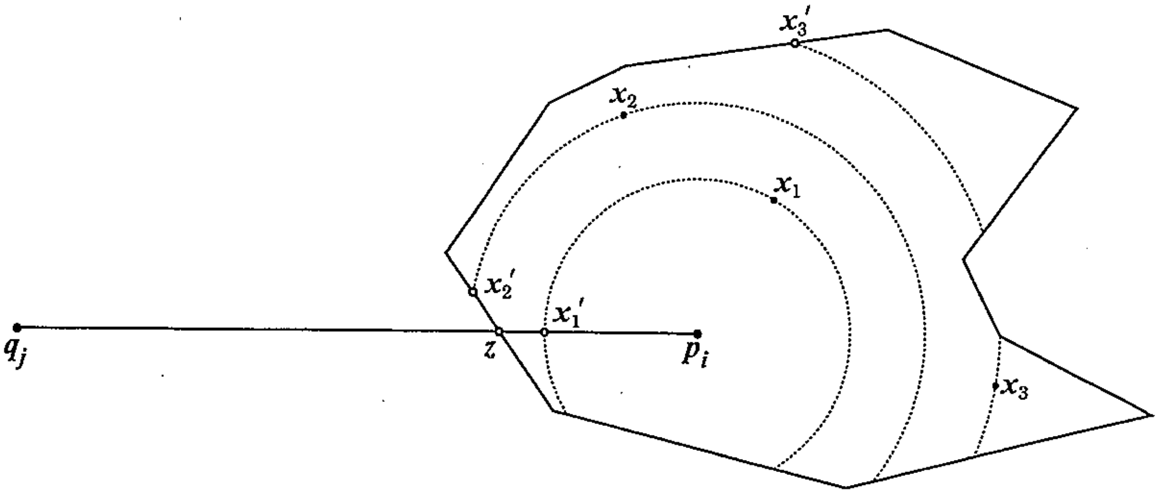

Figure 2: $x_1 \prec x_1' \prec z$, $x_2 \prec x_2'$ and $x_3 \prec x_3'$

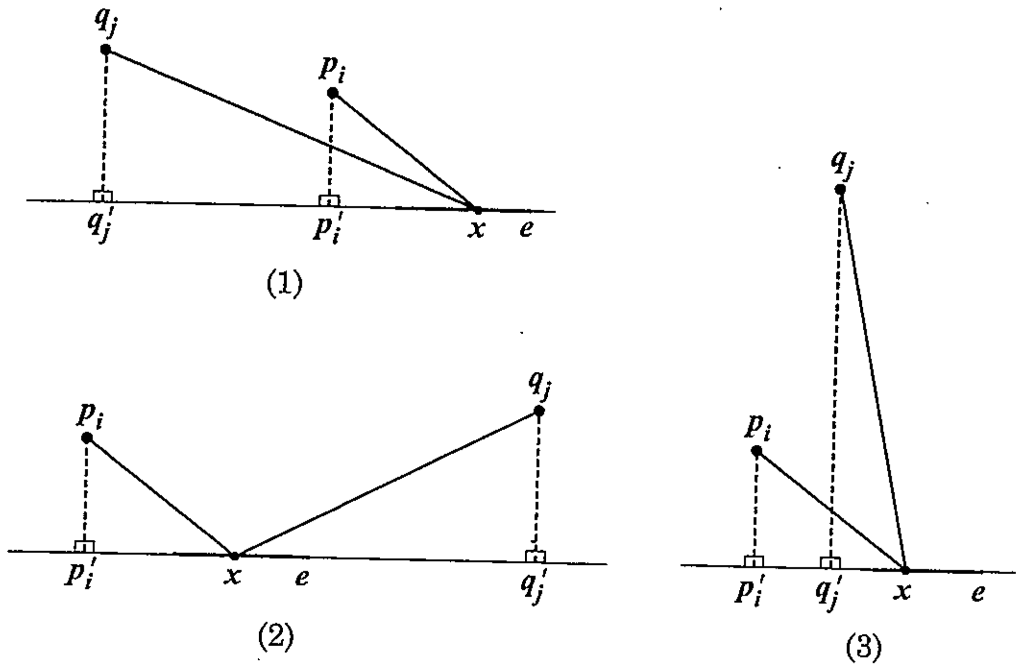

Figure 3: Relationship between $F(\mathrm{x})$ and $G(\mathrm{x})$

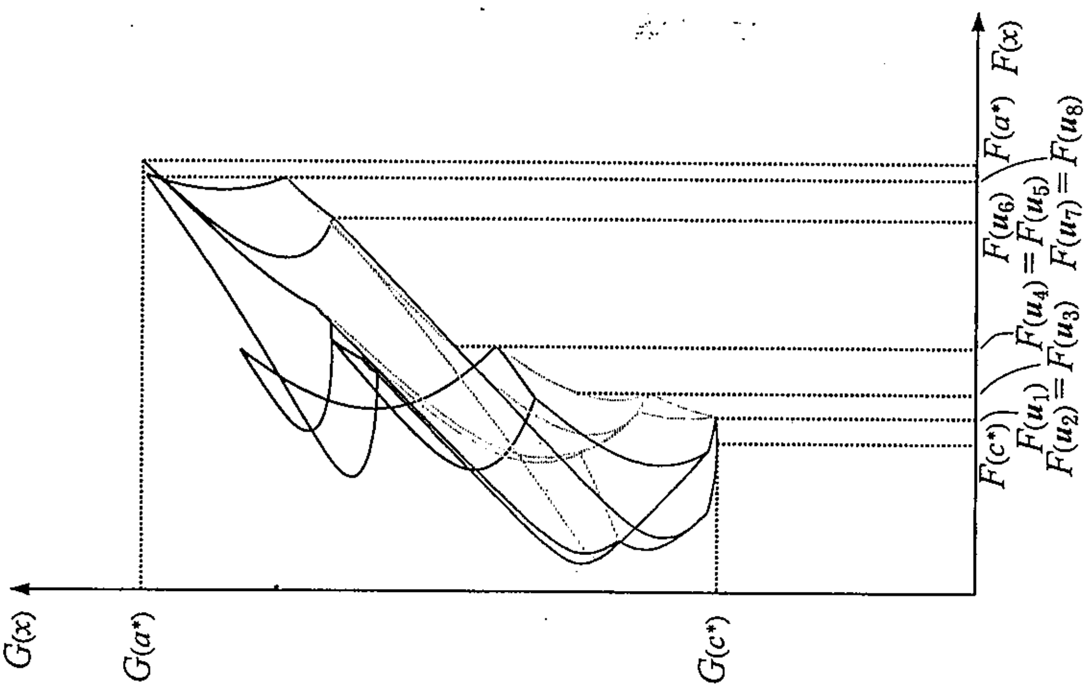

Figure 4: Trade-off curve in criterion space

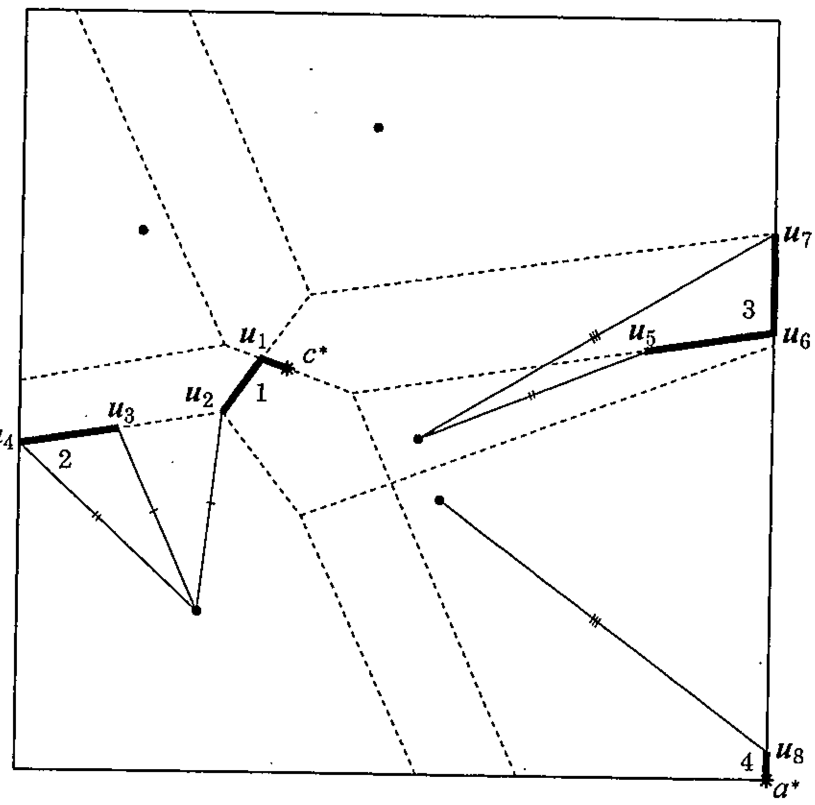

Figure 5: Efficient set in geographical space

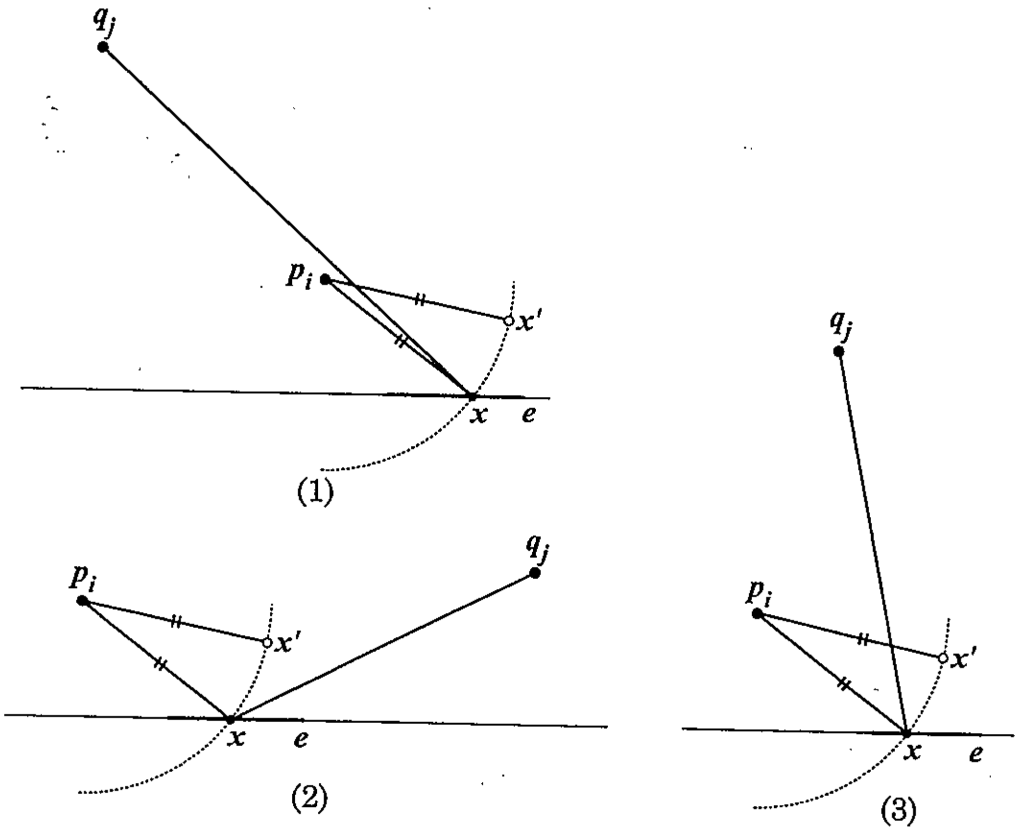

Figure 6: In any case, y dominates x