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Finite-Dimensional Utilities

by

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Abstract

Sufficient axioms are identified for the existence of a finite-dimensional quasi-linear utility function whose lexicographically ordered vectors preserve a decision maker's preference order on a mixture set $\mathcal{M}$. It is shown that those axioms are also necessary for the linear lexicographic representation when the underlying set $\mathcal{M}$ is a mixture space.

1 Introduction

The aim of this paper is to identify sufficient (but also preferably necessary) axioms for the existence of a finite-dimensional utility function whose lexicographically ordered vectors preserve a decision maker's preference order on a mixture set $\mathcal{M}$. Such a lexicographic representation was first axiomatized by Hausner (1954) who obtained an infinite-dimensional utility function on a mixture space. His derivation involves embedding a utility space (i.e., a mixture space with an order relation) in an ordered vector space and showing by a nonconstructive method (see Hausner and Wendel, 1952) that every ordered vector space is isomorphic to a subspace of a lexicographic function space.

Hausner also noted that the dimension of the lexicographic representation is finite if the dimension of the underlying mixture space is finite. This fact was recently utilized in deriving a lexicographic probability system for decision making under uncertainty (see Blume, Brandenburger and Dekel, 1991). However, Hausner's method may be somewhat undesirable for practical reason, since it is not constructive. To cope with such a drawback, Fishburn (1971, 1982) presented a direct and constructive derivation of a finite-dimensional lexicographic representation on a mixture set. He made it clear that a hierarchical structure with respect to the set-inclusion for mixture subsets generated by a certain equivalence relation on the set of all non-degenerated preference intervals reflects multidimensionality of the lexicographic representation.

Finite-dimensionality and its constructive derivation are important for practical tractability. In this respect, LaValle and Fishburn (1991, 1992) recently developed

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lexicographic generalizations of subjective expected utility and derived the notion of matrix probabilities. Fishburn and LaValle (1992) examined decomposition structures of lexicographically ordered multiattribute expected utility.

Fishburn's hierarchical axiom, sufficient but not necessary for finite-dimensional lexicographic representations, consists of several auxiliary relations constructed by a decision maker's preference order, and then prescribes the number of levels in the hierarchical structure of mixture subsets, which corresponds to dimension of the utility function. Therefore, his axiom may be undesirable in the sense that it is not preference-based, i.e., it is not described directly by the preference order. We shall develop such a preferece-based axiom that is sufficient for finite-dimensionalilty when the underlying set $\mathcal{M}$ is a mixture set. However, it is shown that the axiom is also necessary when $\mathcal{M}$ is a mixture space.

The paper is organized as follows. Section 2 introduces a sufficeint axiom system for finite-dimensionality and present the main theorem, where the necessity of the axiom system is also proved when $\mathcal{M}$ is a mixture space. Then Section 3 provides the sufficiency proof of the main theorem.

2 Finite Dimensionality

Let I and I^0 be respectively the closed and open unit intervals. A set $\mathcal{M}$ is a *mixture set* if for any $\lambda \in I$ and any ordered pair $(x, y) \in \mathcal{M} \times \mathcal{M}$, there is a unique element $x\lambda y$ in$\mathcal{M}$ such that, for all $x, y \in \mathcal{M}$ and all $\lambda, \mu \in I$,

M1. $x1y = x$,
M2. $x\lambda y = y(1-\lambda)x$,
M3. $(x\mu y)\lambda y = x(\lambda\mu)y$.

Following Hausner (1954), a set $\mathcal{M}$ is said to be a *mixture space* if, in addition, the following condition is imposed:

M4. $$(x\mu y)\lambda z = x(\lambda\mu)\left(y\frac{\lambda(1-\mu)}{1-\lambda\mu}z\right)$$

for all $x, y, z \in \mathcal{M}$ and all $\lambda, \mu \in I$ for which $\lambda\mu \neq 1$.

Let $\succ$ signify a binary relation on a mixture set $\mathcal{M}$ with $\sim$ and $\succsim$ defined in the usual way: for all $x, y \in \mathcal{M}$, $x \sim y$ iff not$(x \succ y)$ and not$(y \succ x)$, and $x \succsim y$ iff $x \succ y$ or $x \sim y$. The relation $\succ$ is *asymmetric* if $x \succ y$ implies not$(y \succ x)$ for all $x, y \in \mathcal{M}$, and *negatively transitive* if $x \succ y$ implies $x \succ z$ or $z \succ y$ for all $x, y, z \in \mathcal{M}$. We say that $\succ$ is a *weak order* if it is asymmetric and negatively transitive.

We are concerned with a binary relation $\succ$ on a mixture set $\mathcal{M}$, which satisfies the following three axioms, understood as applying to all $x, y, z \in \mathcal{M}$ and all $\lambda \in I$.

A1. *$\succ$ on $\mathcal{M}$ is a weak order.*
A2. *If $x \succ y$ then $x\lambda z \succ y\lambda z$.*
A3. *If $x \sim y$ then $x\lambda z \sim y\lambda z$.*

The meanings of those axioms are familiar from the expected utility theory. The representational implication of the axioms proved by Hausner (1954) is that $\succ$ on $\mathcal{M}$ has a lexicographic representation, i.e., there is a multidimensional function on $\mathcal{M}$ whose lexicographically ordered vectors preserve a binary relation $\succ$ when $\mathcal{M}$ is a mixture space. Furthermore, any function u on $\mathcal{M}$ that constitutes the lexicographic representation is shown to be *linear* in the following sense: for all $x, y \in \mathcal{M}$ and all $\lambda \in I$,

$$u(x\lambda y) = \lambda u(x) + (1 - \lambda)u(y).$$

There are several versions of the Archimedean axiom that are employed in expected utility theories, but omitted or relaxed in the lexicographic representation. The usual version is stated as follows: for all $x, y, z \in \mathcal{M}$,

Archimedean Axiom AA1. *If $x \succ y$ and $y \succ z$, then there are $\alpha, \beta \in I^0$ such that $x\alpha z \succ y$ and $y \succ x\beta z$.*

As is well known, axioms A1, A2, and AA1 are necessary and sufficient for the existence of a linear function u on a mixture set $\mathcal{M}$ such that, for all $x, y \in \mathcal{M}$, $x \succ y \Longleftrightarrow u(x) > u(y)$.

The second version that we shall relax in our finite dimensional representations is stated as follows: for all $x, y, z, w \in \mathcal{M}$,

Archimedean Axiom AA2. *If $x \succ y$ and $x\lambda z \succ y\lambda w$ for all $\lambda \in I^0$, then not($w \succ z$).*

This is the weakest Archimedean axiom that will suffice for the one-way representation, i.e., there is a linear function u on a mixture set $\mathcal{M}$ such that, for all $x, y \in \mathcal{M}$, $x \succ y \Longrightarrow u(x) > u(y)$ (see Aumann (1962) and Fishburn (1971b)).

Let $\Re^n$ be an n-dimensional Euclidean space. Then we define a binary lexicographic relation $>_L$ on $\Re^n$ as follows: for all n-dimensional real vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\Re^n$,

$$a >_L b \iff a \neq b \text{ and } a_k > b_k \text{ for the smallest } k \text{ for which } a_k \neq b_k.$$

We say that $(\mathcal{M}, \succ)$ has an at most n-dimensional *quasilinear* lexicographic representation $(u_1, \ldots, u_n)$ if $u_1, \ldots, u_n$ are real valued functions on $\mathcal{M}$ such that, for all $x, y \in \mathcal{M}$,

$$x \succ y \Longleftrightarrow (u_1(x), \ldots, u_n(x)) >_L (u_1(y), \ldots, u_n(y))$$

with each u_k on $\mathcal{M}$ quasilinear, i.e., if $u_k(x) \neq u_k(y)$ and $u_j(x) = u_j(y)$ for each $j < k$, then for all $\lambda \in I$,

$$u_k(x\lambda y) = \lambda u_k(x) + (1 - \lambda)u_k(y).$$

When each u_k on $\mathcal{M}$ is linear, $(\mathcal{M}, \succ)$ is said to have an at most n dimensional *linear* lexicographic representation.

Now we state a relaxation of the Archimedean axiom AA2 below, which is understood as applying to all positive integers n and all $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in \mathcal{M}$.

A4(n). *If, for all* $\lambda \in I^0$,

$$x_i \succ y_i \quad \text{and} \quad x_i\lambda x_{i+1} \succ y_i\lambda y_{i+1} \quad \text{whenever } 1 \le i \le n \text{ is odd},$$
$$y_i \succ x_i \quad \text{and} \quad y_i\lambda y_{i+1} \succ x_i\lambda x_{i+1} \quad \text{whenever } 1 \le i \le n \text{ is even},$$

then not$(y_{n+1} \succ x_{n+1})$ *if* n *is odd, and* not$(x_{n+1} \succ y_{n+1})$ *if* n *is even.*

Observe that A4(1) is equivalent to AA2. We say that, for $x, y, z, w \in \mathcal{M}$, a set $\{(x,y),(z,w)\}$ of pairs (x,y) and (z,w) violates AA2 if either $w \succ z$, $x \succ y$, and $x\lambda z \succ y\lambda w$ for all $\lambda \in I^0$, or $z \succ w$, $y \succ x$, and $y\lambda w \succ x\lambda z$ for all $\lambda \in I^0$. Then axiom A4(n) requires that there is no n-sequence of sets, $\{(x_1,y_1),(x_2,y_2)\}$, $\{(x_2,y_2),(x_3,y_3)\}, \ldots, \{(x_n,y_n),(x_{n+1},y_{n+1})\}$, that violate AA2.

The main theorem of the paper is stated as follows.

Theorem 1 *Let n be a positive integer. Suppose that $\mathcal{M}$ is a mixture set that satisfies axioms* A1–A3, *and* A4(n). *Then $(\mathcal{M}, \succ)$ has an at most n-dimensional quasilinear lexicographic representation. If $\mathcal{M}$ is a mixture space, then axioms* A1–A3 *and* A4(n) *hold if and only if $(\mathcal{M}, \succ)$ has an at most n-dimensional linear lexicographic representation.*

The proof for sufficiency of the axioms will appear in the next section. It is easy to see that axioms A1–A3 are necessary for both representations in the theorem. To show the necessity of A4(n) when $\mathcal{M}$ is a mixture space, suppose that $(\mathcal{M}, \succ)$ has a linear lexicographic representation $(u_1, \ldots, u_n)$. Given $x, y \in \mathcal{M}$ with not$(x \sim y)$, let k_{xy} denote a positive integer such that $u_{k_{xy}}(x) \neq u_{k_{xy}}(y)$ and $u_i(x) = u_i(y)$ for all $i < k_{xy}$. Of course, $1 \le k_{xy} \le n$. If a set $\{(x,y),(z,w)\}$ violates AA2, then it follows from linearity of the representation that $k_{xy} < k_{zw}$.

Assume that the hypotheses of axiom A4(n) hold. Then by the last claim in the preceding paragraph, $1 \le k_{x_1y_1} < \cdots < k_{x_ny_n} \le n$. Therefore, $k_{x_{n+1}y_{n+1}} = k_{x_ny_n}$. Hence it follows again from the last claim of the preceding paragraph that the set $\{(x_n,y_n),(x_{n+1},y_{n+1})\}$ cannot violate AA2, so that not$(y_{n+1} \succ x_{n+1})$ if n is odd, and not$(x_{n+1} \succ y_{n+1})$ if n is even. Thus the conclusions of axiom A4(n) obtain.

The following example illustrates insufficiecy of axiom A4(n) for the quasilinear lexicographic representation. Let $n = 2$, so that, for all $x, y \in \mathcal{M}$,

$$x \succ y \Longleftrightarrow (u_1(x), u_2(x)) >_L (u_1(y), u_2(y)).$$

Fix $x, y, z \in \mathcal{M}$ with $x \succ y \succ z$. Since u_1 is linear and u_2 is quasilinear, we define $u_1(x) = 1$, $u(y) = 0$, $u(z) = -1$, and, for all $\lambda \in I$,

$$\begin{aligned} u_2(x\lambda y) &= \lambda, \\ u_2(z\lambda y) &= -\lambda, \\ u_2(x\lambda z) &= \begin{cases} 0 & \text{if } 0 < \lambda \le \frac{1}{4}, \\ -1 & \text{if } \frac{1}{4} < \lambda \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < \lambda \le 1. \end{cases} \end{aligned}$$

Then the 2-sequence of the sets $\{(x\frac{3}{4}z, x\frac{1}{2}y), (y, x\frac{1}{2}z)\}$ and $\{(y, x\frac{1}{2}z), (x\frac{1}{4}z, y\frac{1}{2}z)\}$ violates axiom A4(2).

3 Sufficiency Proof of the Theorem

Throughout the section, we assume that $\mathcal{M}$ is a mixture set, and axioms A1–A3 and A4(n) hold for a positive integer n. We shall prove the sufficiency of the axioms in three steps by deriving Fishburn's hierarchical axiom A(m) introduced below in Step 3. Fisrt in Step 1, we introduce several auxilliary relations and their properties proved in Fishburn (1971a, 1982). Step 2 proves additional properties of those relations. Finally Step 3 completes the proof.

Step 1. We introduce several auxilliary relations. Four lemmas will be cited from Fishburn (1982, Chapter 4) for the later developments, and their correspondent numbers will be stated as L 4.n. Following Fishburn (1982), we shall construct the hierarchical structure of closed preference intervals $\langle x, y\rangle$, where

$$\langle x, y\rangle = \{z \in \mathcal{M} : x \succeq z \succeq y\}.$$

It follows that $\langle x, y\rangle$ is empty if $y \succ x$, since $\succ$ is a weak order. By $\mathcal{N}$, we shall denote the set of all nonempty closed preference intervals with nonindifferent end points, i.e., $\mathcal{N} = \{\langle x, y\rangle : x \succ y\}$. When $\langle x, y\rangle, \langle z, w\rangle \in \mathcal{N}$, the minimal element in $\mathcal{N}$ that includes both $\langle x, y\rangle$ and $\langle z, w\rangle$ is easily seen to be $\langle x, w\rangle \cup \langle z, y\rangle$.

An immediate implication of A1–A3 is that, for all $\langle x, y\rangle \in \mathcal{N}$, there is a function ϕ_{xy} on $\langle x, y\rangle$ taking value in I such that, for all $z \in \langle x, y\rangle$,

$$\begin{aligned} x\lambda y \succ z \quad & \text{for all } \lambda > \phi_{xy}(z), \\ z \succ x\lambda y \quad & \text{for all } \lambda < \phi_{xy}(z). \end{aligned}$$

Note that either $x\alpha y \sim z$ or $x\alpha y \succ z$ or $z \succ x\alpha y$ holds if $\alpha = \phi_{xy}(z)$. The following lemma gives linearity and order-preserving properties of the ϕ_{xy} function.

Lemma 1 (L4.1) *If $\langle x, y\rangle \in \mathcal{N}$, then ϕ_{xy} is linear, and, for all $z, w \in \langle x, y\rangle$, if $z \succeq w$, then $\phi_{xy}(z) \geq \phi_{xy}(w)$.*

We define two binary relations, $\sqsupseteq$ and $\sqsupset^*$, on $\mathcal{N}$ by

$$\begin{aligned} \langle x, y\rangle \sqsupseteq \langle z, w\rangle \quad & \text{iff} \quad \langle x, y\rangle \supseteq \langle z, w\rangle \text{ and } \phi_{xy}(z) > \phi_{xy}(w), \\ \langle x, y\rangle \sqsupset^* \langle z, w\rangle \quad & \text{iff} \quad \langle x, y\rangle \supseteq \langle z, w\rangle \text{ and } \phi_{xy}(z) = \phi_{xy}(w). \end{aligned}$$

When $\langle x, y\rangle \supseteq \langle z, w\rangle$ and both intervals are in $\mathcal{N}$, exactly one of $\langle x, y\rangle \sqsupseteq \langle z, w\rangle$ and $\langle x, y\rangle \sqsupset^* \langle z, w\rangle$ must hold. It is easy to see from the definitions that $\sqsupseteq$ is reflexive and $\sqsupset^*$ is irreflexive. The failure of $\phi_{xy}(z) = \phi_{xy}(w)$ can happen only if the Archimedean axiom, AA1 or AA2, is false. We may say that $\sqsupseteq$ is *commensurable*, since the ϕ_{xy} function may scale utilities for elements in $\langle z, w\rangle$ along with those in $\langle x, y\rangle$ when $\langle x, y\rangle \sqsupseteq \langle z, w\rangle$, and $\sqsupset^*$ is *noncommensurable*, since otherwise.

The following lemma shows properties of $\sqsupset^*$ and some relations between $\sqsupseteq$ and $\sqsupset^*$.

Lemma 2 (L4.4(a)–(d))
(a) *If* $\langle x,y\rangle \sqsupseteq \langle z,w\rangle$ *and* $\langle z,w\rangle \sqsupset^* \langle r,s\rangle$, *then* $\langle x,y\rangle \sqsupset^* \langle r,s\rangle$; *if* $\langle x,y\rangle \sqsupset^* \langle z,w\rangle$ *and* $\langle z,w\rangle \sqsupseteq \langle r,s\rangle$, *then* $\langle x,y\rangle \sqsupset^* \langle r,s\rangle$.
(b) *If* $\langle x,y\rangle \sqsupset^* \langle r,s\rangle$, $\langle z,w\rangle \sqsupseteq \langle r,s\rangle$, *and* $\langle x,y\rangle \supseteq \langle z,w\rangle$, *then* $\langle x,y\rangle \sqsupset^* \langle z,w\rangle$.
(c) $\sqsupset^*$ *is transitive and irreflexive.*
(d) *If* $\langle x,y\rangle \sqsupset^* \langle z,w\rangle$, $\langle x,y\rangle \sqsupset^* \langle r,s\rangle$, *and* $\langle z,w\rangle \cap \langle r,s\rangle \neq \emptyset$, *then* $\langle x,y\rangle \sqsupset^* \langle z,w\rangle \cup \langle r,s\rangle$.

We now define a key binary relation $=_0$ on $\mathcal{N}$ induced by $\sqsupseteq$ as follows:

$$\langle x,y\rangle =_0 \langle z,w\rangle \quad \text{iff} \quad \langle x,w\rangle \cup \langle z,y\rangle \sqsupseteq \langle x,y\rangle \text{ and } \langle x,w\rangle \cup \langle z,y\rangle \sqsupseteq \langle z,w\rangle.$$

The following relation between $\sqsupseteq$ and $=_0$ will be frequently used in the later proofs: for all $x,y,z,w \in \mathcal{M}$,

$$\langle x,y\rangle \sqsupseteq \langle z,w\rangle \Longrightarrow \langle x,y\rangle =_0 \langle z,w\rangle.$$

To see this, suppose that $\langle x,y\rangle \sqsupseteq \langle z,w\rangle$. Noting that $\langle x,y\rangle = \langle x,w\rangle \cup \langle z,y\rangle$, we obtain that $\langle x,w\rangle \cup \langle z,y\rangle \sqsupseteq \langle z,w\rangle$ and $\langle x,w\rangle \cup \langle z,y\rangle \sqsupseteq \langle x,y\rangle$. Then by definition, $\langle x,y\rangle =_0 \langle z,w\rangle$.

An important property of $=_0$ to formulate the hierarchical structure of mixture subsets is stated by

Lemma 3 (L4.5) $=_0$ *on* $\mathcal{N}$ *is an equivalence relation.*

We partition $\mathcal{N}$ into equivalence class by $=_0$, and let $\mathcal{N}_0 = \mathcal{N}/=_0$. $\mathcal{N}_0$ consists of a single class $\mathcal{N}$ if and only if the Archimedean axiom, AA1 or AA2, holds.

Given an equivalence class $A \in \mathcal{N}_0$, let $\mathcal{M}(A)$ denote the set of all elements in $\mathcal{M}$ that appear in at least one interval in A, i.e., $\mathcal{M}(A) = \cup_A \langle x,y\rangle$. It follows that $\mathcal{M}(A)$ is a mixture subset of $\mathcal{M}$. The next lemma shows the hierarchical relations between the different mixture subsets, $\mathcal{M}(A)$, induced by $=_0$.

Lemma 4 (L4.8) *For any two distinct* $A, B \in \mathcal{N}_0$, *either* $\mathcal{M}(A) \cap \mathcal{M}(B) = \emptyset$ *or* $\mathcal{M}(A) \supset \mathcal{M}(B)$ *or* $\mathcal{M}(B) \supset \mathcal{M}(A)$.

Step 2. This step proves three lemmas which show additional properties of how hierarchical structures of mixture subsets induced by $=_0$ and commensurable and noncommensurable relations can be related.

Lemma 5 *If* $\langle x,y\rangle \sqsupset^* \langle z,w\rangle$, $\langle x,y\rangle \in A$, *and* $\langle z,w\rangle \in B$, *then* $\mathcal{M}(A) \supseteq \mathcal{M}(B)$.

Proof. Suppose that the hypotheses of the lemma hold. Take any r in $\mathcal{M}(B)$. We are to show that $r \in \mathcal{M}(A)$. Let $\langle z',w'\rangle$ be an interval in B that contains r. Also, let $\langle z'',w''\rangle = \langle z,w'\rangle \cup \langle z',w\rangle$, i.e., the smallest interval that includes $\langle z,w\rangle$ and $\langle z',w'\rangle$.

Since $\langle z,w\rangle =_0 \langle z',w'\rangle$, it follows from the definition of $=_0$ that $\langle z'',w''\rangle \sqsupseteq \langle z,w\rangle$, so $\langle z'',w''\rangle =_0 \langle z,w\rangle$. Thus $\langle z'',w''\rangle \in B$, since $\langle z,w\rangle \in B$.

We note that $\langle x,y\rangle \cup \langle z'',w''\rangle$ is in $\mathcal{N}$, since $\langle x,y\rangle \cap \langle z'',w''\rangle \neq \emptyset$. We obtain that $\langle x,y\rangle \cup \langle z'',w''\rangle \supseteq \langle x,y\rangle$, so that

$$\begin{array}{rlcl} \text{either} & \langle x,y\rangle \cup \langle z'',w''\rangle & \sqsupseteq & \langle x,y\rangle, \\ \text{or} & \langle x,y\rangle \cup \langle z'',w''\rangle & \sqsupset^* & \langle x,y\rangle. \end{array}$$

Since $\langle x,y\rangle \sqsupset^* \langle z,w\rangle$, it follows from Lemmas 2(a) and 2(c) that $\langle x,y\rangle \cup \langle z'',w''\rangle \sqsupset^* \langle z,w\rangle$. Therefore, noting $\langle z'',w''\rangle \sqsupseteq \langle z,w\rangle$, Lemma 2(b) implies that $\langle x,y\rangle \cup \langle z'',w''\rangle \sqsupset^* \langle z'',w''\rangle$.

Assume that $\langle x,y\rangle \cup \langle z'',w''\rangle \sqsupset^* \langle x,y\rangle$. Since $\langle x,y\rangle \cap \langle z'',w''\rangle \neq \emptyset$, a contradiction to irreflexivity of $\sqsupset^*$ follows from the last claim in the preceding paragraph and Lemma 2(d). Hence we must have that $\langle x,y\rangle \cup \langle z'',w''\rangle \sqsupseteq \langle x,y\rangle$, so $\langle x,y\rangle \cup \langle z'',w''\rangle =_0 \langle x,y\rangle$. Thus $\langle x,y\rangle \cup \langle z'',w''\rangle \in A$. Since $r \in \langle z'',w''\rangle$, we obtain that $r \in \mathcal{M}(A)$. □

Lemma 6 *If* $\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle x,y\rangle$, $\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle z,w\rangle$, $\langle x,y\rangle \in A$, $\langle z,w\rangle \in B$, *and* $\langle x,y\rangle \cap \langle z,w\rangle = \emptyset$, *then* $\mathcal{M}(A) \cap \mathcal{M}(B) = \emptyset$.

Proof. Suppose that the hypotheses of the lemma hold. Assume that $\mathcal{M}(A) \cap \mathcal{M}(B) \neq \emptyset$. Then by Lemma 4, either $\mathcal{M}(A) \supset \mathcal{M}(B)$ or $\mathcal{M}(B) \supset \mathcal{M}(A)$. Suppose that $\mathcal{M}(A) \supset \mathcal{M}(B)$. Then we are to derive a contradiction. When $\mathcal{M}(B) \supset \mathcal{M}(A)$, a similar proof leads to a contradiction. Hence we must have $\mathcal{M}(A) \cap \mathcal{M}(B) = \emptyset$.

Since $\langle z,w\rangle \in B$ is not empty, let $t \in \langle z,w\rangle$. Since $t \in \mathcal{M}(B)$, we have $t \in \mathcal{M}(A)$, so that we take any $\langle x',y'\rangle \in A$ that contains t. Since A and B are distinct, not$(\langle x',y'\rangle =_0 \langle z,w\rangle)$. Noting that $\langle x',y'\rangle \cap \langle z,w\rangle \neq \emptyset$ and $\langle x',y'\rangle \cup \langle z,w\rangle = \langle x',w\rangle \cup \langle z,y'\rangle$, Lemma 2(d) implies that either one of the following two cases holds:

$$\begin{array}{ll} \text{Case 1:} & \langle x',w\rangle \cup \langle z,y'\rangle \sqsupset^* \langle z,w\rangle, \\ \text{Case 2:} & \langle x',w\rangle \cup \langle z,y'\rangle \sqsupset^* \langle x',y'\rangle. \end{array}$$

Case 1. Suppose that $\langle x',w\rangle \cup \langle z,y'\rangle \sqsupset^* \langle z,w\rangle$. Then $\langle x',w\rangle \cup \langle z,y'\rangle \sqsupseteq \langle x',y'\rangle$. Let $\langle x'',y''\rangle = \langle x',w\rangle \cup \langle z,y'\rangle$, i.e., the smallest interval that includes $\langle x',y'\rangle$ and $\langle z,w\rangle$. Then $\langle x'',y''\rangle =_0 \langle x',y'\rangle$, so that $\langle x'',y''\rangle \in A$. Since $\langle x,y\rangle \in A$, $\langle x,y\rangle =_0 \langle x'',y''\rangle$, so that $\langle x'',y\rangle \cup \langle x,y''\rangle \sqsupseteq \langle x,y\rangle$.

Since $\langle x'',y\rangle \cup \langle x,y''\rangle \supseteq \langle x,w\rangle \cup \langle z,y\rangle$, we have that

$$\begin{array}{rlcl} \text{either} & \langle x'',y\rangle \cup \langle x,y''\rangle & \sqsupset^* & \langle x,w\rangle \cup \langle z,y\rangle, \\ \text{or} & \langle x'',y\rangle \cup \langle x,y''\rangle & \sqsupseteq & \langle x,w\rangle \cup \langle z,y\rangle. \end{array}$$

If the former holds, then it follows from Lemma 2(c) that $\langle x'',y\rangle \cup \langle x,y''\rangle \sqsupset^* \langle x,y\rangle$, since $\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle x,y\rangle$. This contradicts the last claim of the preceding paragraph. If the latter holds, then it similarly follows from Lemma 2(a) that $\langle x'',y\rangle \cup \langle x,y''\rangle \sqsupset^* \langle x,y\rangle$, a similar contradiction. Hence Case 1 fails to hold.

Case 2. Suppose that $\langle x'',y''\rangle \sqsupset^* \langle x',y'\rangle$. Then $\langle x'',y''\rangle \sqsupseteq \langle z,w\rangle$. Thus $\langle x'',y''\rangle =_0 \langle z,w\rangle$, so that $\langle x'',y''\rangle \in B$. It follows from Lemma 5 that not$(\mathcal{M}(A) \supset \mathcal{M}(B))$, a contradiction. Hence Case 2 fails to hold. □

Lemma 7 *If $\langle x,y\rangle \in A$, $\langle z,w\rangle \in B$, and $\mathcal{M}(A) \supset \mathcal{M}(B)$, then*

$$\begin{aligned} \langle x,w\rangle \cup \langle z,y\rangle &\sqsupseteq \langle x,y\rangle, \\ \langle x,w\rangle \cup \langle z,y\rangle &\sqsupset^* \langle z,w\rangle. \end{aligned}$$

Proof. Suppose that the hypotheses of the lemma hold. Then not($\langle x,y\rangle =_0 \langle z,w\rangle$), so that

$$\begin{aligned} &\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle x,y\rangle, \\ \text{or}\quad &\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle z,w\rangle. \end{aligned}$$

Suppose that $\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle x,y\rangle$. Assume first that $\langle x,w\rangle \cup \langle z,y\rangle \sqsupseteq \langle z,w\rangle$. Then $\langle x,w\rangle \cup \langle z,y\rangle =_0 \langle z,w\rangle$. Thus $\langle x,w\rangle \cup \langle z,y\rangle \in B$. Hence by Lemma 5, not($\mathcal{M}(A) \supset \mathcal{M}(B)$), a contradiction. Assume next that $\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle z,w\rangle$. Then $\langle x,y\rangle \cap \langle z,w\rangle = \emptyset$, otherwise contradicting irreflexivity of $\sqsupset^*$. Hence by Lemma 6, $\mathcal{M}(A) \cap \mathcal{M}(B) = \emptyset$, a contradiction.

Therefore, we must have that $\langle x,w\rangle \cup \langle z,y\rangle \sqsupseteq \langle x,y\rangle$, so that $\langle x,w\rangle \cup \langle z,y\rangle \sqsupset^* \langle z,w\rangle$. □

Step 3. This step completes the sufficiency proof by deriving Fishburn's hierarchical structures of mixture subsets induced by $=_0$.

Adjacent mixture subsets induced by $=_0$ are identified by $\supset_1$, so that, for all $A, B \in \mathcal{N}_0$,

$$\begin{aligned} &\mathcal{M}(A) \supset_1 \mathcal{M}(B) \\ &\quad \text{iff}\ \ \mathcal{M}(A) \supset \mathcal{M}(B) \text{ and } \mathcal{M}(A) \supset \mathcal{M}(C) \supset \mathcal{M}(B) \text{ for no } C \in \mathcal{N}_0. \end{aligned}$$

Furthermore, mixture subsets separated by $k-1$ other ordered mixture subsets are identified by $\supset_k$, so that, for $k \geq 2$ and for all $A, B \in \mathcal{N}_0$,

$$\begin{aligned} &\mathcal{M}(A) \supset_k \mathcal{M}(B) \\ &\quad \text{iff}\ \ \mathcal{M}(A) \supset_1 \mathcal{M}(C) \text{ and } \mathcal{M}(C) \supset_{k-1} \mathcal{M}(B) \text{ for some } C \in \mathcal{N}_0. \end{aligned}$$

Now we state Fishburn's hierarchical axiom as follows, understood as applying to all positive integers m.

A(m). *For $m = 1$, there is no $A, B \in \mathcal{N}_0$ such that $\mathcal{M}(A) \supset \mathcal{M}(B)$. For $m > 1$, there are some $A, B \in \mathcal{N}_0$ such that $\mathcal{M}(A) \supset_{m-1} \mathcal{M}(B)$, and for all $A, B \in \mathcal{N}_0$, if $\mathcal{M}(A) \supset \mathcal{M}(B)$ then $\mathcal{M}(A) \supset_k \mathcal{M}(B)$ for some $1 \leq k \leq m-1$.*

The sufficiency proof is completed by showing that axiom A(m) holds for some $m \leq n$. Suppose on the contrary that there are $A_1, \ldots A_{n+1} \in \mathcal{N}_0$ such that $\mathcal{M}(A_1) \supset \cdots \supset \mathcal{M}(A_{n+1})$, i.e., $\mathcal{M}(A_1) \supset_n \mathcal{M}(A_{n+1})$. It suffices to prove the following claim, since Axiom A4(n) is easily seen to be violated.

Claim 1 *If $\langle x,y\rangle \in A$, $\langle z,w\rangle \in B$, and $\mathcal{M}(A) \supset \mathcal{M}(B)$, then $x\lambda w \succ y\lambda z$ for all $\lambda \in I^0$.*

Proof. Let $\langle r,s\rangle = \langle x,w\rangle \cup \langle z,y\rangle$, the smallest interval that includes $\langle x,y\rangle$ and $\langle z,w\rangle$. Then by Lemma 7, $\langle r,s\rangle \sqsupseteq \langle x,y\rangle$ and $\langle r,s\rangle \sqsupset^* \langle z,w\rangle$.

Since $w \in \langle r,s\rangle$ and $\langle r,s\rangle \sqsupseteq \langle x,y\rangle$, $x\lambda w$ and $y\lambda w$ are in $\langle r,s\rangle$. It follows from the definition of $\sqsupseteq$ that $\phi_{rs}(x) > \phi_{rs}(y)$. Thus by Lemma 1, $\phi_{rs}(x\lambda w) > \phi_{rs}(y\lambda w)$. Hence $\langle r,s\rangle \sqsupseteq \langle x\lambda w, y\lambda w\rangle$ for all $\lambda \in I^0$.

Since $y \in \langle r,s\rangle$ and $\langle r,s\rangle \sqsupset^* \langle z,w\rangle$, $z\lambda y$ and $w\lambda y$ are in $\langle r,s\rangle$. It follows from the definition of $\sqsupseteq$ that $\phi_{rs}(z) = \phi_{rs}(w)$. Thus by Lemma 1, $\phi_{rs}(z\lambda y) = \phi_{rs}(w\lambda y)$. Hence $\langle r,s\rangle \sqsupset^* \langle y\lambda z, y\lambda w\rangle$ for all $\lambda \in I^0$.

It follows from preceding two paragraphs, Lemma 1, and the definitions of $\sqsupseteq$ and $\sqsupset^*$ that $\phi_{rs}(x\lambda w) > \phi_{rs}(y\lambda w) = \phi_{rs}(y\lambda z)$. Hence $x\lambda w \succ y\lambda z$ for all $\lambda \in I^0$. □

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