No. 782

Evolution of Thoughts: Deductive Game Theories in the Inductive Game Situation
Part II

by

Mamoru Kaneko

June 1998

Evolution of Thoughts: Deductive Game Theories in the Inductive Game Situation Part II

Mamoru Kaneko
Institute of Policy and Planning Sciences, University of Tsukuba,
Ibaraki 305, Japan (kaneko@shako.sk.tsukuba.ac.jp)

June, 1998

Abstract

This is a sequel of our investigation of the evolution of deductive thoughts in the inductive game situation. Here we consider the case where the individual player finds that his payoff function allows no dominant strategies and believes the same for the other player. The interpersonal feature of decision making is expressed by interaction structure $\mathcal{J}^4 = (\{1,2\},\{1,2\})$, which makes the individual thought on decision making reciprocal. Tracing this reciprocal thought, the player goes to transitory phase $k = 1, 2, ...$ We will show that a lot of difficulties appear in these phases. For example, the player is incapable to exploit fully the final decision axioms: this incapability allows only negative recommendations to eliminate noncandidates for decisions. Then we assume that he jumps to the Limit Phase ω. There the axioms are fully exploited as far as a game has a Nash equilibrium and satisfies the interchangeability condition. However, the false subjective thinking may still remain, and even if such difficulties are removed, other problems are waiting for the player, which he may not deductively notice.

1. Introduction

This is a sequel of our investigation of the evolution of deductive thoughts in the inductive game situation. In Part I of this paper, we described the basic inductive situation, and started our discussions with Phases 1 and 2. In Phase 1, the individual player thinks as if he would be a one-person decision maker, and this deductive reasoning could be compatible with his inductive knowledge if his source of knowledge on the game is restricted purely to his active and passive experiences $\mathcal{E}(i \mid a^*)$ in the past. He constructs a true belief on his payoff function over the experienced domain, but it may be false over the unexperienced domain. In Phase 2, a player regards the other player as a one-person decision maker of the type of Phase 1. In both phases, each player may get stuck in a pitfall of induction, but is capable of noticing this pitfall by deductive reasoning. In this Part II, we will leave these phases assuming that they have got more knowledge on his and the other player's payoff functions through communications.

A player, i, would notice that he should leave Phase 1 or 2 when his belief/knowledge on his payoff function allows no dominant strategies and when his belief on the other's does neither. In this case, he starts thinking about (his theory and) the other player's thinking about (his theory and) his thinking, and so on. This situation is expressed with interaction structure $\mathcal{J}^4 = (\{1,2\},\{1,2\})$. As defined in Section 3 of Part I, the belief set generated by $\mathcal{J}^4$, say, $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$, has an infinite number of formulae. This infinity can be interpreted either as representing transitory phases open to further phases, or as representing an infinite set as a total. From the viewpoint of the evolutions of thoughts, these two interpretations should be considered separately.

First, we consider the belief set, $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$, from the viewpoint of transitory phases. Here the repetitive beliefs on each player's decision making appear up to any depths. For example, we have the following: for any $m \geq 0$,

$$\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)) \vdash I_{ii}(a_i) \bigwedge I_{ij}(a_j) \supset (\mathrm{B}_i\mathrm{B}_j)^m \left(\mathrm{Nash}^*(a) \bigwedge \mathrm{B}_i(\mathrm{Nash}^*(a))\right), \qquad (1.1)$$

where $\mathrm{Nash}^*(a)$ is the formula $\mathrm{B}_1(\mathrm{Nash}_1(a_1 \mid a_2)) \bigwedge \mathrm{B}_2(\mathrm{Nash}_2(a_2 \mid a_1))$, which will be discussed in Section 2.[1] It is a point here that the assertion for each m in (1.1) needs only a finite subset of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$. It is shown in Section 3 that to have a conclusion for m, we need a finite subset Γ of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ so that Γ involves the $2m+2$ nesting occurrences of B_1 and B_2. This result means that the players should know each other well (up to some appropriate depth) *or at least this is believed by player i*. As the players have got acquired more and more, they go to further phases and can use a finite assumption set Γ of larger depths. Nevertheless, each phase requires only a finite subset of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$.

As far as the players are in transitory phases, they are with some difficulties but they cannot notice them. Player i is incapable of capturing the totality of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$, which corresponds to our use of a finite subset of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ in (1.1). This incapability makes him to derive only necessary conditions from $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$. A conclusion from this partial exploitation is only the capability of negative thinking to eliminate some strategies from his decision making. This will be discussed in Section 2.

Let us look at the above difficulties from the outsider's point of view. The conclusion from $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ in (1.1) differs from each other for different m. This means that the contents of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ are only partially revealed in each assertion of (1.1). This partial revelation corresponds to the difficulty that the premise of Axiom WD_i cannot be formulated in KD4[2]. However, the difficulty is actually deeper than it appears: there

[1] In fact, $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}2))$ suffices for (1.1). See Theorem 2.A.

are no formulae which satisfy the requirements described by $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$, actually, no formulae satisfy $\mathrm{I2}_1 \bigwedge \mathrm{I2}_2$. Hence there is no hope to have the full exploitation of $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ in epistemic logic $\mathrm{KD4}^2$. This is argued in Section 3. It is important to note that we (outsiders) deductively notice these difficulties using meta-mathematical arguments but not the players (insiders).

The difficulties mentioned above are resolved in the Limit Phase ω, though the players cannot deductively notice the need to go to phase ω. This jump needs a player to take also some big *inductive* (heuristic) step. Suppose that a player reaches the Limit Phase ω. This case is extensively discussed in Kaneko [4] with the Veridicality Axiom (T_i): $\mathrm{B}_i(A) \supset A$. With this axiom, the final decision axiom determines completely the individual decision to be the common knowledge of a Nash strategy. However, since we drop Axiom T_i, the result is the *individual belief* of the common knowledge of a Nash strategy. It may be the case that a player believes that it is common knowledge but is objectively false. Actually, it is logically consistent that each player believes the common knowledge of different game structure. This cannot be the case if we add Axiom T_i. This will be presented as the Konnyaku Mondô Theorem in Section 5. Hence even in the Limit Phase ω, we would have a result similar to the inductive pitfall theorem (Theorem 6.D) of Part I.

Formally, the Limit Phase ω is considered by extending epistemic logic $\mathrm{KD4}^2$. One possible extension is to enrich the language so that infinite conjunctions are allowed to capture the totality of $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$. The other is to introduce common knowledge operator into $\mathrm{KD4}^2$ with appropriate logical axioms and inference rule. The former is taken by Kaneko-Nagashima [11] and [12]. The latter is the approach taken by some authors (e.g., Halpern-Moses [1], Lismont-Mongin [15] and Meyer-van der Hoek [16])[2] In this paper, we adopt the former approach, and the extension of $\mathrm{KD4}^2$ is called *game logic* GL_ω, and give another characterization theorem in Section 5.

We discuss the evolutionary phases $1, 2, ..., \omega$ in epistemic logic $\mathrm{KD4}^2$ and game logic GL_ω. In each phase $k < \omega$, we use a relatively small fragment of the language $\mathcal{P}$, say, Phase 1 involves formulae with no interactive occurrences of belief operators B_1,B_2, and Phase 2 needs formulae with their nesting occurrences up to depth 2. In fact, we can restrict the language to some fragment of $\mathcal{P}$ sufficient for each phase, *a fortiori*, the epistemic axioms as well as inference rules, such as Necessitation, are restricted to such a fragment. Therefore phases evolve with required logics themselves. We will discuss this evolution of required logics and related game theoretical problems in Section 3.

Before going to the main body to the paper, we mention one condition on a game. In this paper, we restrict our attentions to games satisfying the condition:

(Int): if (a_1, a_2) and (b_1, b_2) are Nash equilibria of $g = (g_1, g_2)$, so is (a_1, b_2).

[2] Kaneko [3] showed that the later approach is faithfully embedded into the former with an appropriate translation. Hence the results of the present paper are also translated into the latter approach.

This condition, introduced by Nash [17], is an extension of uniqueness. Therefore it is satisfied by the games of Table 1.1 – 1.3 of Part I, but not the game of the following table (Battle of the Sexes):

	s_{21}	s_{22}
s_{11}	$(2,1)^*$	$(0,0)$
s_{12}	$(0,0)$	$(1,2)^*$

Table 1.1

Since our very basic problem is the individual decision making, an individual player needs to make a decision by himself in either case with and without communications. For the Battle of the Sexes, he needs some additional information, which is discussed in Kaneko [4] with Axiom T_i. Here our purpose is not to investigate these games, but to continue our discussions of the evolution of thoughts to the Limit Phase ω. Therefore we assume Condition Int always in this paper. Under this condition, the following notion is meaningful: a_i is said to be a *Nash strategy* iff $(a_i; a_j)$ is a Nash equilibrium for some a_j.

2. Reciprocal Thoughts with Interaction Structure $\mathcal{J}^4 = (\{1,2\},\{1,2\})$: Transitory Phases

Suppose that the game $g = (g_1, g_2)$ allows neither player to have dominant strategies. Suppose also that the players have communicated with each other and that each player i knows the other player j's payoff function, as well as his own payoff function g_i, to the extent that his belief $\hat{g}_j$ allows no dominant strategies for j. Then he may realize that the situation is reciprocal and that he needs to adopt interaction structure $\mathcal{J}^4 = (\{1,2\},\{1,2\})$. In the following sections, we consider the case where player i adopts this interaction structure.

The case of $\mathcal{J}^4 = (\{1,2\},\{1,2\})$ differs from those of the other interaction structures in that the former generates an infinite flow of reciprocal thoughts, while the flow of thoughts stops at depth 1 or 2 in the other cases. In this sense, $\mathcal{J}^4 = (\{1,2\},\{1,2\})$ requires a lot, which may be regarded as approximating a situation where both players get acquired to each other well. We argue that this situation creates some serious difficulties.

For $\mathcal{J} = (\{1,2\},\{1,2\})$, Axioms $I1_i$ – $I3_i$ ($\{i,j\} = \{1,2\}$) are written as

$$I1_i : \bigwedge_{x} (I_{ii}(x_i) \wedge I_{ij}(x_j) \supset B_i(\mathrm{Nash}_i(x_i \mid x_j)));$$

$$I2_i : \bigwedge_{k=1}^{2} \bigwedge_{l=1}^{2} \bigwedge_{x_l} (I_{il}(x_l) \supset B_i(I_{kl}(x_l)));$$

$$\mathrm{I3}_i : \left(\bigvee_{x_i} I_{ii}(x_i) \supset \bigvee_{x_j} I_{ij}(x_j) \right) \bigwedge \left(\bigvee_{x_j} I_{ij}(x_j) \supset \bigvee_{x_i} I_{ii}(x_i) \right).$$

As mentioned above, the belief set $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ is infinite. This infinity causes the impossibility of finding full solutions for the above axioms in epistemic logic $\mathrm{KD4}^2$. However, the point is not merely this impossibility but that neither player can deductively notice it, which can be proved by us (outsiders).

The above axioms are formulated so as to capture the distinction between belief and knowledge. Accordingly, we should modify the formula $\mathrm{Nash}(a_1, a_2)$ to take epistemic elements into account:

$$\mathrm{B}_1(\mathrm{Nash}_1(a_1 \mid a_2)) \bigwedge \mathrm{B}_2(\mathrm{Nash}_2(a_2 \mid a_1)),$$

which we denote by $\mathrm{Nash}^*(a_1, a_2)$. This states that 1 believes that a_1 is a best response to a_2, and that 2 believes that a_2 is a best response to a_1. This is not the concept we ultimately target, but is a base concept on which we construct a super structure.

In epistemic logic $\mathrm{KD4}^2$, we can only exploit the belief set $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ in the following form, which will be proved in the end of this section.

Theorem 2.A (Partial Characterization). Let m be a nonnegative integer. Then

(1): $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}2)) \vdash I_{ii}(a_i) \bigwedge I_{ij}(a_j) \supset (\mathrm{B}_i\mathrm{B}_j)^m\, (\mathrm{Nash}^*(a) \bigwedge \mathrm{B}_i(\mathrm{Nash}^*(a)))$;

(2): $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)) \vdash I_{ii}(a_i) \supset \bigvee_{x_j} (\mathrm{B}_i\mathrm{B}_j)^m\, (\mathrm{Nash}^*(a_i; x_j) \bigwedge \mathrm{B}_i(\mathrm{Nash}^*(a_i; x_j)))$;

(3): $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)) \vdash I_{ij}(a_j) \supset \bigvee_{x_i} (\mathrm{B}_i\mathrm{B}_j)^m\, (\mathrm{Nash}^*(x_i; a_j) \bigwedge \mathrm{B}_i(\mathrm{Nash}^*(x_i; a_j)))$.

The first states that if $I_{ii}(a_i) \bigwedge I_{ij}(a_j)$, then $(a_i; a_j)$ has the property of Nash^*, player i believes this property, i believes j believes it, and so on. The second and third assert that a_i and a_j are Nash strategies with the additional epistemic structures.

The assertion (1) require only Axioms I1 and I2 in $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}2))$, while (2) and (3) need I3 as well as I1 and I2 in $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$. The main difference between (1) and (2), (3) is to separate $I_{ii}(a_i)$ from $I_{ij}(a_j)$. Even if the players communicate, the ultimate decision made by player i is to choose his own strategy a_i and player i predicts that the ultimate decision by j is to chooses a_j. The ultimate decisions are independent though they are restricted to some extent by their communications. Therefore the assertion (1) should be regarded as an intermediate step to (2) and (3).

The above theorem was stated in the subjective manner. Since, in fact, the objective version is easier to read, we restate Theorem 2.A.(1) in epistemic logic $\mathrm{S4}^2$. Recall that $\mathrm{S4}^2$ is obtained from $\mathrm{KD4}^2$ by adding Axiom $(\mathrm{T}_i) : \mathrm{B}_i(A) \supset A$. If follows from Theorem 2.A.(1) that for any nonnegative integer m,

$$\mathrm{B}^i(\mathcal{J}^4, \mathrm{I}(1\text{–}2)) \vdash_{S4^2} I_{ii}(a_i) \bigwedge I_{ij}(a_j) \supset \mathrm{B}_{i_1} ... \mathrm{B}_{j_m}(\mathrm{Nash}(a)), \qquad (2.1)$$

where $B_{i_1}...B_{i_m}$ is a sequence of B_1,B_2, $i_t \neq i_{t+1}$ for any $t = 1, ..., m-1$. This means that it is common knowledge that a strategy pair $a = (a_1, a_2)$ is a Nash equilibrium, *though "common knowledge" is not formulated as a formula, yet.* It is an significant difference between Theorem 2.A.(1) and (2.1) that the former is only about the subjective belief of player i on the common knowledge of a Nash equilibrium, while the latter states that being a Nash equilibrium is actually common knowledge between the players. Therefore the former allows false belief, but the latter cannot. This will be discussed again in Section 5.

A difficulty arising in the infinity of the belief set $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ is, as stated in Section 4 of Part I, that the axiom scheme WD_i cannot be formulated in $\mathrm{KD4}^2$. It is simply out of the scope of $\mathrm{KD4}^2$ to take the conjunction of an infinite set of formulae. Hence we cannot use Axiom WD_i in $\mathrm{KD4}^2$. This impossibility is deeper than it looks. Although taking the conjunction of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ is not allowed in the assertions of Theorem 2.A, their essential parts are expressed, but only the totality of $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ is not captured in $\mathrm{KD4}^2$. However, the totality is essential for Axiom WD_i. This will be proved in Section 4. In fact, this partial exploitation brings about a serious problem to the players.

For a clear-cut statement, we use the true payoff functions $g = (g_1, g_2)$ for the statements instead of the believed ones, $g^i = (g_i; \hat{g}_j)$, and state the assertions only on $I_{ii}(\cdot)$. The proof of the following theorem will be given in the end of this section.

Theorem 2.B (Negative Recommendation):

(1): If a_i is a not Nash strategy for g, then $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)), \mathsf{B}^i(\mathcal{J}^4, g) \vdash \neg I_{ii}(a_i)$.

(2): For any $a_i \in \Sigma_i$, it is not the case that

$$\bigcup_{k=1}^{2} \mathsf{B}^k(\mathcal{J}^4, \mathrm{I}(1\text{–}3)), \bigcup_{k=1}^{2} \mathsf{B}^k(\mathcal{J}^4, g) \vdash I_{ii}(a_i). \tag{2.2}$$

Note that the first implies $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)), \mathsf{B}^i(\mathcal{J}^4, g) \vdash \mathrm{B}_i(\neg I_{ii}(a_i))$. It asserts that under the belief sets $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ and $\mathsf{B}^i(\mathcal{J}^4, g)$, he can decide a nonequilibrium strategy not to be a candidate for his decision. The second asserts that neither player can decide a decision. In this sense, this theorem states that as far as a player lives in $\mathrm{KD4}^2$, only negative recommendations are available for him.

It is important to emphasize that the unprovability stated in (2) is not in the scope of either player. Although the number of pure strategies is finite, he needs to check an infinite number of proofs to find that (2.2) is not the case. Hence he may get stuck here, and it is an alternative case that after many failures, he may start thinking heuristically and may notice that his language and logic are insufficient. In this case, he may jump to the limit phase ω.

Parallel results hold in the Phases 1 and 2 in the cases of Sections 5 and 6 of Part I if Axiom WD_i is not assumed. In these cases, however, Theorems 5.A and 6.A are full characterizations with Axiom WD_i, and by these, we obtained the decidability and playability results. In the present case, in fact, WD_i cannot be formulated, and, thus, the above theorem is unavoidable as far as we are in $KD4^2$.

We may find some similar fact in the economic literature. We often observe that only necessary conditions for decision making are considered. This partial consideration does not raise any conceptual problem if we (outsider) separate ourselves from the inside players in deriving necessary conditions. However, if we regard the inside players as doing the same, then the partial consideration is problematic. The above theorem states that such considerations provide no positive recommendations for decisions.

Lemma 2.1.(1): $I2_i \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_i(I_{ji}(a_i) \wedge I_{jj}(a_j))$.

(2): $I2_i, B_i(I2_j) \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_iB_j(I_{ii}(a_i) \wedge I_{ij}(a_j))$.

(3): For any $m \geq 0$, $\mathsf{B}^i(\mathcal{J}^4, (I2_i;I2_j)) \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset (B_iB_j)^m(I_{ii}(a_i) \wedge I_{ij}(a_j))$.

Proof. (1): Since $I2_i \vdash I_{ii}(a_i) \supset B_i(I_{ji}(a_i))$ and $I2_i \vdash I_{ij}(a_j) \supset B_i(I_{jj}(a_j))$, we have $I2_i \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_i(I_{ji}(a_i) \wedge I_{jj}(a_j))$.

(2): It follows from (1) that $B_i(I2_j) \vdash B_i(I_{jj}(a_j) \wedge I_{ji}(a_i)) \supset B_iB_j(I_{ii}(a_i) \wedge I_{ji}(a_j))$. Hence $I2_i, B_i(I2_j) \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_iB_j(I_{ii}(a_i) \wedge I_{ij}(a_j))$.

(3): We prove the assertion by induction on m. Since $\mathsf{B}^i(\mathcal{J}^4, (I2_i;I2_j))$ includes $B_i^{+}(I2_i)$, $B_j^{+}B_i(I2_j)$, (2) implies the assertion for $m = 1$. Now, suppose

$$\mathsf{B}^i(\mathcal{J}^4, (I2_i; I2_j)) \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset (B_iB_j)^m(I_{ii}(a_i) \wedge I_{ij}(a_j)). \tag{2.3}$$

Then we have

$$\mathsf{B}^i(\mathcal{J}^4, (I2_i; I2_j)) \vdash B_iB_j(I_{ii}(a_i) \wedge I_{ij}(a_j)) \supset (B_iB_j)^{m+1}(I_{ii}(a_i) \wedge I_{ij}(a_j)).$$

Hence it follows from this and (2.3) that $\mathsf{B}^i(\mathcal{J}^4, (I2_i;I2_j)) \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset (B_iB_j)^{m+1}(I_{ii}(a_i) \wedge I_{ij}(a_j))$. □

Lemma 2.2. $I1_i, I2_i, B_i(I1_j) \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_i(\mathrm{Nash}^*(a))$.

Proof. First, $I1_i \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_i(\mathrm{Nash}_i(a_i \mid a_j))$, which together with Axiom PI_i implies

$$I1_i \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_iB_i(\mathrm{Nash}_i(a_i \mid a_j)). \tag{2.4}$$

Since $I2_i \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_i(I_{jj}(a_j) \wedge I_{ji}(a_i))$ and $B_i(I1_j) \vdash B_i(I_{jj}(a_j) \wedge I_{ji}(a_i)) \supset B_iB_j(\mathrm{Nash}_j(a_j \mid a_i))$, we have

$$I2_i, B_i(I1_j) \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset B_iB_j(\mathrm{Nash}_j(a_j \mid a_i)). \tag{2.5}$$

By (2.4) and (2.5), we have the assertion of the lemma. □

Proof of Theorem 2.A. It follows from Lemmas 2.1.(3) and 2.2 that

$$\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1,2)) \vdash I_{ii}(a_i) \bigwedge I_{ij}(a_j) \supset (\mathrm{B}_i\mathrm{B}_j)^m \mathrm{B}_i(\mathrm{Nash}^*(a_i)).$$

Also, it follows from Lemmas 2.1.(1) and 2.2 that

$$\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1,2)) \vdash I_{ii}(a_i) \bigwedge I_{ij}(a_j) \supset \mathrm{B}_i\mathrm{B}_j(\mathrm{Nash}^*(a)). \tag{2.6}$$

Indeed, since $\mathrm{B}_i(\mathrm{I1}_j \bigwedge \mathrm{I2}_j \bigwedge \mathrm{B}_j(\mathrm{I1}_i)) \vdash \mathrm{B}_i(I_{jj}(a_j) \bigwedge I_{ji}(a_i)) \supset \mathrm{B}_i\mathrm{B}_j(\mathrm{Nash}^*(a))$ by Lemma 2.2 with the permutation of i and j and since $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1,2)) \vdash \mathrm{B}_i(\mathrm{I1}_j \bigwedge \mathrm{I2}_j \bigwedge \mathrm{B}_j(\mathrm{I1}_i))$, we have (2.6) by Lemma 2.1.(1). From (2.6), using Lemma 2.1.(3), we have $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1,2)) \vdash I_{ii}(a_i) \bigwedge I_{ij}(a_j) \supset (\mathrm{B}_i\mathrm{B}_j)^m(\mathrm{Nash}^*(a))$.

Assertions (2) and (3) can be proved in the same manner as in the proof of Lemma 6.2 of Part I. □

Proof of Theorem 2.B.(1): Let a_i be not a Nash strategy. Let a_j be an arbitrary strategy for j. Then

$$g_i \vdash_0 \neg\mathrm{Nash}_i(a_i \mid a_j) \text{ or } g_j \vdash_0 \neg\mathrm{Nash}_j(a_j \mid a_i).$$

Hence $\mathrm{B}_i(g_i) \vdash \mathrm{B}_i(\neg\mathrm{Nash}_i(a_i \mid a_j))$ or $\mathrm{B}_j(g_j) \vdash \mathrm{B}_j(\neg\mathrm{Nash}_j(a_j \mid a_i))$. This implies $\mathrm{B}_i(g_i) \vdash \mathrm{B}_i\mathrm{B}_i(\neg\mathrm{Nash}_i(a_i \mid a_j))$ or $\mathrm{B}_i\mathrm{B}_j(g_j) \vdash \mathrm{B}_i\mathrm{B}_j(\neg\mathrm{Nash}_j(a_j \mid a_i))$. Hence $\mathrm{B}_i(g_i) \vdash \neg\mathrm{B}_i\mathrm{B}_i(\mathrm{Nash}_i(a_i \mid a_j))$ or $\mathrm{B}_i\mathrm{B}_j(g_j) \vdash \neg\mathrm{B}_i\mathrm{B}_j(\mathrm{Nash}_j(a_j \mid a_i))$ by Lemma 3.1.(6) of Part I. Thus $\mathsf{B}^i(\mathcal{J}^4,g) \vdash \neg\mathrm{B}_i\mathrm{B}_i(\mathrm{Nash}_i(a_i \mid a_j))$ or $\mathsf{B}^i(\mathcal{J}^4,g) \vdash \neg\mathrm{B}_i\mathrm{B}_j(\mathrm{Nash}_j(a_j \mid a_i))$. This implies $\mathsf{B}^i(\mathcal{J}^4,g) \vdash \neg\mathrm{B}_i\mathrm{B}_i(\mathrm{Nash}_i(a_i \mid a_j)) \bigvee \neg\mathrm{B}_i\mathrm{B}_i(\mathrm{Nash}_j(a_j \mid a_i)$, which is equivalent to $\mathsf{B}^i(\mathcal{J}^4,g) \vdash \neg\mathrm{B}_i(\mathrm{Nash}^*(a_i;a_j))$. Since this holds for all a_j, we have

$$\mathsf{B}^i(\mathcal{J}^4,g) \vdash \neg \bigvee_{x_j} \mathrm{B}_i(\mathrm{Nash}^*(a_i;x_j)). \tag{2.7}$$

Since $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)) \vdash I_{ii}(a_i) \supset \bigvee_{x_j} \mathrm{B}_i(\mathrm{Nash}^*(a_i;x_j))$ by Theorem 2.A.(2), we have $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)), \mathsf{B}^i(\mathcal{J}^4,g) \vdash \neg \bigvee_{x_j} \mathrm{B}_i(\mathrm{Nash}^*(a_i;x_j)) \supset \neg I_{ii}(a_i)$. This together with (2.7) implies the assertion.

(2): Suppose, on the contrary, that (2.2) holds for some b_i. Since an conjunct of $\mathrm{I2}_k$ is expressed as $I_{kl}(a_l) \supset \mathrm{B}_k(I_{jl}(a_l))$, we have $\{\neg I_{kl}(a_l) : a_l \in \Sigma_l \text{ and } k = 1,2\} \vdash \mathrm{I1}_k \bigwedge \mathrm{I2}_k$. we have

$$\bigcup_k \mathrm{B}_k(\Gamma_k), \Gamma_0, \bigcup_k \{\neg I_{kl}(a_k) : a_l \in \Sigma_l \text{ and } l = 1,2\}, g_1, g_2 \vdash I_{ii}(b_i), \tag{2.8}$$

where $B_k(\Gamma_k)$ is the set of formulae in $B^k(\mathcal{J}^4, I(1–3)) \cup B^k(\mathcal{J}^4, g)$ whose outermost symbols are $B_k(\cdot)$, and $\Gamma_0 = \{I3_k : k = 1, 2\}$. Applying the Separation Theorem (Theorem 5.E of Part I) to (2.8), we have

$$\bigcup_k \{\neg I_{kl}(a_l) : a_l \in \Sigma_l, l = 1, 2\}, \Gamma_0, g_1, g_2 \vdash I_{ii}(b_i),$$
$$B_1(\Gamma_1) \vdash \bot \quad \text{or} \quad B_2(\Gamma_2) \vdash \bot\,. \tag{2.9}$$

We can prove by the belief-elimination operator ε and Soundness for $\vdash_0$ that neither is the case. Here we prove that the first is not the case. Suppose, on the contrary, that the first is the case. Then, by applying the operator ε, we have

$$\bigcup_k \{\neg I_{kl}(a_l) : a_l \in \Sigma_l, l = 1, 2\}, \Gamma_0, g_1, g_2 \vdash_0 I_{ii}(b_i). \tag{2.10}$$

We show that this is not the case by Soundness for $\vdash_0$.

Now we define an assignment σ over the set of atomic formulae by

$$\sigma(I_{kl}(a_l)) = \text{ false} \quad \text{for all } k, l = 1, 2 \text{ and all } a_l \in \Sigma_l;$$

$$\sigma(R_k(a : b)) = \begin{cases} \text{true} & \text{if } g_k(a) \geq g_k(a') \\ \text{false} & \text{otherwise.} \end{cases}$$

In this assignment σ, we have $\models_\sigma A$ for every formula A in $\bigcup_k \{\neg I_{kl}(a_l) : a_l \in \Sigma_l, l = 1, 2\}, \Gamma_0, g_1, g_2$, but $I_{ii}(b_i)$ is false. Hence by Soundness for $\vdash_0$, (2.10) cannot be the case. □

3. Depths of Thoughts in Transitory Phases

We have discussed the evolutionary phases 1, 2, ... in epistemic logic $KD4^2$. In each phase, we use a relatively small fragment of the language $\mathcal{P}$, say, Phase 1 involves formulae with no interactive occurrences of belief operators B_1,B_2, and Phase 2 needs formulae with their nesting occurrences up to depth 2. In fact, we can restrict the language to some fragment of $\mathcal{P}$ sufficient for each phase, *a fortiori*, the epistemic axioms as well as inference rules, such as Necessitation, are restricted to such a fragment. Therefore phases evolve with required logics themselves.

3.1. Evolution of Logics Required for Transitory Phases

The *epistemic depth* $\delta(A)$ of a formula A is crucial for our considerations in this section. The depth $\delta_i(A)$ of a formula A $(i = 1,2)$ is defined by induction on the structure of formula A:

(0): $\delta_i(A) = 0$ if A is atomic;

(1): $\delta_i(\neg A) = \delta_i(A)$;

(2): $\delta_i(A \supset B) = \max(\delta_i(A), \delta_i(B))$;

(3): $\delta_i(\bigwedge \Phi) = \delta_i(\bigvee \Phi) = \max\{\delta_i(A) : A \in \Phi\}$;

(4): for $j \neq i$, $\delta_i(\mathrm{B}_j(A)) = 0$, and $\delta_i(\mathrm{B}_i(A)) = \max(\delta_i(A), \delta_j(A) + 1)$.

We define $\delta(A) = \max(\delta_1(A), \delta_2(A))$. Step (4) counts the successive occurrences of B_1,B_2, ignoring the repetitive occurrence of the same B_i. For example, if A is nonepistemic, then $\delta(\mathrm{B}_2\mathrm{B}_1(A \supset \mathrm{B}_1(A))) = \delta_2(\mathrm{B}_2\mathrm{B}_1(A \supset \mathrm{B}_1(A))) = \max(\delta_2(\mathrm{B}_1(A \supset \mathrm{B}_1(A))),$ $\delta_1(\mathrm{B}_1(A \supset \mathrm{B}_1(A))) + 1) = 2$.

We define the language $\mathcal{P}(m)$ of formulae of epistemic depth up to m by

$$\mathcal{P}(m) = \{A \in \mathcal{P} : \delta(A) \leq m\} \text{ for } m = 0, 1, ... \tag{3.1}$$

For example, the language $\mathcal{P}(0)$ is the set of nonepistemic formulae. Of course, $\mathcal{P}(m) \subsetneq \mathcal{P}(m+1)$ for all m. Now we restrict the language $\mathcal{P}$ to $\mathcal{P}(m)$ in epistemic logic KD4^2, and consider the pair ($\mathcal{P}(m)$,KD4^2). In ($\mathcal{P}(m)$,KD4^2), any formula occurring in a proof is required to be from $\mathcal{P}(m)$. We emphasize that in ($\mathcal{P}(m)$,KD4^2), Necessitation rule is allowed to be applied to A only if $\mathrm{B}_i(A)$ (or $\mathrm{B}_j(A)$) belongs to $\mathcal{P}(m)$. For example, neither Necessitation rule nor any formulae including belief operators $\mathrm{B}_i(\cdot)$ is allowed in ($\mathcal{P}(0)$,KD4^2), and thus it is classical propositional logic.

First, we mention some facts on these logics form the viewpoint of logic. The relation between two logics in the sequence ($\mathcal{P}(0)$,KD4^2), ($\mathcal{P}(1)$,KD4^2), $\cdots$ is as follows: for any k, m with $k > m$,

$$(\mathcal{P}(k), \text{KD4}^2) \text{ is a conservative extension of } (\mathcal{P}(m), \text{KD4}^2), \tag{3.2}$$

that is, (1): if $A \in \mathcal{P}(m)$ is provable in ($\mathcal{P}(m)$,KD4^2), then so is in ($\mathcal{P}(k)$,KD4^2); and (2): if $A \in \mathcal{P}(m)$ is provable in ($\mathcal{P}(k)$,KD4^2), then so is in ($\mathcal{P}(m)$,KD4^2). The second part is called *conservativeness.* This conservativeness is almost a direct consequence of the *cut-elimination theorem* for KD4^2.[3] Also, a model theory for ($\mathcal{P}(m)$,KD4^2) is developed in Kaneko-Suzuki [14], which gives a *sound-completeness theorem* for each ($\mathcal{P}(k)$,KD4^2) with respect a modified Kripke models with the corresponding depths.

[3]The cut-elimination theorem for game logic GL_ω in Kaneko-Nagashima [12] can be simplified for KD4^2.

Let us return to our discussion on the evolutions on game theoretical thoughts. Up to now, we did not give a clear-cut definition of Phase k. Here we identify each Phase k ($k \geq 1$) with the thought processes to be described in logic ($\mathcal{P}(k)$,KD4^2). Now we should argue that this identification capture the original intention of the introduction of phases.

The entire arguments for $\mathcal{J}$ with $J_i = \{i\}$ (Section 5 of Part I) can be done in ($\mathcal{P}(1)$,KD4^2). We remark that WD$_i$ is an axiom schema and includes any formulae potentially, but that it works actually with the restriction of formulae to $\mathcal{P}(1)$. Similarly, we can verify that the entire arguments for $\mathcal{J}$ with $J_i = (\{i,j\},\{j\})$ (Section 6 of Part I) can be stated in ($\mathcal{P}(2)$,KD4^2). Hence these arguments are in Phase 2.

Now consider the assertions of Theorem 2.A. If we look carefully at the proofs of these assertions, we would find that only a finite subset Γ of $\mathrm{B}^i(\mathcal{J}^4$,I(1–3)) having depth $2m+2$ is used in the proofs. Hence Theorem 2.A belongs to Phase $2m+2$. Here the conservativeness of (3.2) has the implication that when a player goes from phase k to phase $k+1$, i.e., from ($\mathcal{P}(k)$,KD4^2) to ($\mathcal{P}(k+1)$, KD4^2), his previous thoughts in phase k are faithfully preserved in phase $k+1$.

Our identification of Phase k by ($\mathcal{P}(k)$,KD4^2) has the intention that if beliefs up to depth k are assumed, the meaningful conclusions should be in ($\mathcal{P}(k)$,KD4^2). If this was not the case, i.e., some meaningful conclusions outside $\mathcal{P}(k)$ were derived, the restriction to $\mathcal{P}(k)$ could be rather artificial: only the results in the above particular cases we have considered happened to be in the corresponding $\mathcal{P}(k)$. Actually, this is not the case, which is shown by the following theorem. The theorem is due to Kaneko-Nagashima [13][4].

Theorem 3.A (Depth Lemma). Suppose $\vdash A \supset \mathrm{B}_{i_1}...\mathrm{B}_{i_k}(B)$, $\delta(A) < k$ and $i_t \neq i_{t+1}$ for $t = 1, ..., k-1$. Then $\vdash \neg A$ or $\vdash B$.

That is, if $\mathrm{B}_{i_1}...\mathrm{B}_{i_k}(B)$ is derived from A with $\delta(A) < k$, then at least one of A and B is trivial in the sense of $\vdash \neg A$ or $\vdash B$. This theorem implies that if beliefs up to depth k are assumed, then all meaningful conclusions from the assumptions are in $\mathcal{P}(k)$. Hence this warrants our identification of Phase k to be ($\mathcal{P}(k)$,KD4^2).

Let us apply Theorem 3.A specifically to Theorem 2.A.(1).

Theorem 3.B (Necessary Depth). Let m be a nonnegative integer, and let Γ be a finite subset of $\mathrm{B}^i(\mathcal{J}^4$,I(1–2)). If $\Gamma \vdash I_{ii}(a_i) \wedge I_{ij}(a_j) \supset (\mathrm{B}_i\mathrm{B}_j)^m\,(\mathrm{Nash}^*(a) \wedge \mathrm{B}_i(\mathrm{Nash}^*(a)))$, then $\max\{\delta(A) : A \in \Gamma\} \geq 2m+2$.

The faithful reading of this theorem is that if the Nash* property is known to the players in the sense of $(\mathrm{B}_i\mathrm{B}_j)^m$ and $(\mathrm{B}_i\mathrm{B}_j)^m\mathrm{B}_i$, the assumption set Γ should include

[4]This theorem is proved for S4^2 in Kaneko-Nagashima [13] using the cut-elimination theorem for S4^2 in the Gentzen style sequent calculus due to Ohnishi-Matsumoto [18]. Since KD4^2 permits cut-elimination, we can prove the theorem in the same (actually simpler) way.

the depth $2m + 2$, where the additional 2 comes form B_i in the second conjunct and B_j in $\text{Nash}^*(a)$. As discussed above, we could read it as meaning that if Γ has at most depth $2m + 2$, we can derive the outer B_iB_j only up to m times. Thus, the depth of the conclusion is getting deeper as player i's thinking (i.e., the assumption set) or the degree of intersubjectivity is getting deeper. Nevertheless, the potential depth of the assumption set, $\mathsf{B}^i(\mathcal{J}^4,\text{I}(1\text{–}2))$, is infinite, and only some finite part of it is used as far as the players are in epistemic logic KD4^2, i.e., it does not allow the players to capture the totality of the belief set. Thus, the evolutionary process of thoughts are getting deeper without reaching infinity.

Proof of Theorem 3.B. Suppose $\Gamma \vdash I_{ii}(a_i) \bigwedge I_{ij}(a_j) \supset (B_iB_j)^m (\text{Nash}^*(a) \bigwedge B_i(\text{Nash}^*(a)))$. This is equivalent to $\vdash (\bigwedge \Gamma) \bigwedge (I_{ii}(a_i) \bigwedge I_{ij}(a_j)) \supset (B_iB_j)^m (\text{Nash}^*(a) \bigwedge B_i(\text{Nash}^*(a)))$. This further implies

$$\vdash (\bigwedge \Gamma) \bigwedge (I_{ii}(a_i) \bigwedge I_{ij}(a_j)) \supset (B_iB_j)^m B_iB_j(\text{Nash}_j(a_j \mid a_i)). \tag{3.3}$$

If neither $\vdash \neg ((\bigwedge \Gamma) \bigwedge (I_{ii}(a_i) \bigwedge I_{ij}(a_j)))$ nor $\vdash \text{Nash}_j(a_j \mid a_i)$, then the application of the depth lemma (Theorem 3.A) to (3.3) implies $\max\{\delta(A) : A \in \Gamma\} \geq 2m + 2$. We can prove $\vdash \neg ((\bigwedge \Gamma) \bigwedge (I_{ii}(a_i) \bigwedge I_{ij}(a_j)))$ nor $\vdash \text{Nash}_j(a_j \mid a_i)$, by using the belief-elimination operator ε and the soundness for $\vdash_0$. □

3.2. No Nontrivial Formulae satisfying the Axioms in KD4^2

As remarked in Section 2, a difficulty is caused for the reason that it is not allowed to take the conjunction of $\mathsf{B}^i(\mathcal{J}^4,\text{I}(1\text{–}3)[\mathcal{A}])$ since it is an infinite set. The reader might wonder whether it would be possible to capture its essential part, as in Theorem 2.A, by considering an arbitrary finite subsets of $\mathsf{B}^i(\mathcal{J}^4,\text{I}(1\text{–}3)[\mathcal{A}])$. However, the difficulty involved is more intrinsic: the language does not allow any formulae to have the property $\text{I2}_1 \bigwedge \text{I2}_2$. In the above axiomatization, Axiom $\text{I2}_1 \bigwedge \text{I2}_2$ is solely responsible to the difficulty. This fact is expressed in the following form, which is proved by using the depth lemma.

Theorem 3.C (Indefinability in KD4^2). If a family of formulae $\{A_{kl} : k, l = 1, 2\}$ satisfies $\vdash \bigwedge_{k=1}^{2} \bigwedge_{l=1}^{2} (A_{il} \supset B_i(A_{kl}))$ for $i = 1, 2$, then $\vdash \neg A_{kl}$ or $\vdash A_{kl}$ for each pair $k, l = 1, 2$.

Thus, the theorem allows only the trivial cases. Since no families of nontrivial formulae satisfy Axioms $\text{I2}_1 \bigwedge \text{I2}_2$, it is impossible to express $I_{ij}(a_j)$'s as equivalent formulae in $\mathcal{P}$. It is an implication that KD4^2 is incapable of full exploitation of the above axioms. Therefore we need to extend KD4^2 in order to exploit fully the final decision axioms.

Proof of Theorem 3.C. Consider a formula A_{1l}. Since $\vdash A_{1l} \supset \mathrm{B}_1(A_{2l})$ and $\vdash A_{2l} \supset \mathrm{B}_2(A_{1l})$, which implies $\vdash \mathrm{B}_1(A_{2l}) \supset \mathrm{B}_1\mathrm{B}_2(A_{1l})$, we have

$$\vdash A_{1l} \supset \mathrm{B}_1\mathrm{B}_2(A_{1l}). \tag{3.4}$$

Now we assume that $\vdash A_{1l} \supset (\mathrm{B}_1\mathrm{B}_2)^m(A_{1l})$. Then $\vdash (\mathrm{B}_1\mathrm{B}_2)A_{1l} \supset (\mathrm{B}_1\mathrm{B}_2)^{m+1}(A_{1l})$. Hence by (3.4), we have $\vdash A_{1l} \supset (\mathrm{B}_1\mathrm{B}_2)^{m+1}(A_{1l})$. Thus we proved

$$\vdash A_{1l} \supset (\mathrm{B}_1\mathrm{B}_2)^m(A_{1l}) \text{ for all } m \geq 0. \tag{3.5}$$

We take m greater than $\delta(A_{1l})$. Then we apply the depth lemma (Theorem 3.A) for $\mathrm{KD4}^2$ to (3.5) with this m, and then have $\vdash \neg A_{1l}$ or $\vdash A_{1l}$. □

4. Game Logic GL_ω

To exploit fully the belief set $\mathrm{B}^i(\mathcal{J},\mathrm{I}(1\text{–}3))$ of the final decision axioms in the case of $\mathcal{J} = \mathcal{J}^4 = (\{1,2\},\{1,2\})$, we need to extend the language $\mathcal{P}$ so that we can represent the contents of certain infinite sets such as $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3))$ by object formulae. As mentioned in Section 1, there are, so far, two possible approaches available: one is to extend $\mathcal{P}$ to an infinitary language to have infinitary conjunctions (the approach taken by Kaneko-Nagashima [11] and [12]), and the second is to introduce new operator symbols and to add certain axioms and inference rule to describe the required properties to the original finitary logic (cf., Halpern-Moses [1], Lismont-Mongin [15] and Meyer-van der Hoek [16]). Here we take the former approach: the extended logic is called *game logic* GL_ω. It is proved in Kaneko [3] that the latter can be faithfully embedded into the former. This implies that the essential part of our argument can be translated into the latter approach. Nevertheless, we will mention briefly the latter approach, too, since it gives some hints for the understanding of the evolution of logics.

First, we extend the language $\mathcal{P}$ to $\mathcal{P}^1, \mathcal{P}^2, \ldots$. by the following induction. We call the formulae in $\mathcal{P}^0 = \mathcal{P}$ the *0-formulae*. Suppose that the set $\mathcal{P}^{k-1}$ of $(k-1)$-formulae is already defined $(k = 1, \ldots)$. Then $\mathcal{P}^k$ is the set of *k-formulae* defined by the following induction:

(k-i): any expression in $\mathcal{P}^{k-1} \cup \{(\bigwedge \Phi), (\bigvee \Phi) : \Phi$ is a nonempty countable subset of $\mathcal{P}^{k-1}\}$ is a k-formula;

(k-ii): if A and B are k-formulae, so are $(\neg A), (A \supset B)$ and $\mathrm{B}_i(A)$ for $i = 1, 2$.

We denote $\bigcup_{k<\omega} \mathcal{P}^k$ by $\mathcal{P}^\omega$. We call an expression in $\mathcal{P}^\omega$ simply a *formula*, and also Φ an *allowable set* iff Φ is a nonempty countable subset of $\mathcal{P}^k$ for some $k < \omega$.

The primary reason for our infinitary language is to express the concept of common knowledge explicitly as a conjunctive formula. The common knowledge of a formula A is defined as follows: For any $m \geq 0$, we denote the set $\{\mathrm{B}_{i_1}\mathrm{B}_{i_2}\ldots\mathrm{B}_{i_m} : i_t \neq i_{t+1}$ for

$t = 1, ..., m-1\}$ by $\mathcal{B}(m)$, where $\mathcal{B}(0)$ is stipulated to consist of the null symbol e, i.e., eA is A itself. We define the *common knowledge* of A as

$$\bigwedge\{B^k(A) : B^k \in \bigcup_{0 \leq m < \omega} \mathcal{B}(m)\}, \tag{4.1}$$

which we denote by $C(A)$. Hence $C(A)$ is common *knowledge* in the sense that it includes A, instead of common *belief*, where the common belief of A is defined by deleting A in (4.1). The common belief of A is expressed as $C(B_1(A) \bigwedge B_2(A))$ using $C(\cdot)$. To capture the relativization of "knowledge" to "belief", it will be also shown that the necessary modification of common knowledge is not common belief but the individual belief of common knowledge, instead.

The space $\mathcal{P}^\omega$ is closed with respect to the operation $C(\cdot)$. Indeed, if A is in $\mathcal{P}^{\ell-1}$, the set $\{B^k(A) : B^k \in \bigcup_{m<\omega} \mathcal{B}(m)\}$ is a countable subset of $\mathcal{P}^{\ell-1}$, and its conjunction, $C(A)$, belongs to $\mathcal{P}^\ell$ by (ℓ-i).

To state one more axiom, we define the notion of a cc-formula. We say that A in $\mathcal{P}^\omega$ is a *cc-formula* iff (i) it contains no infinitary disjunctions and (ii) if it contains an infinitary conjunction, then it is written as $C(B)$ for some $B \in \mathcal{P}^\omega$. Any subformula of a cc-formula is also a cc-formula.

Game logic GL_ω is obtained from $KD4^2$ by substituting the infinitary language $\mathcal{P}^\omega$ for the finitary $\mathcal{P}$ together with the following additional axiom (schema):

(C-Barcan): $\bigwedge\{B_iB^k(A) : B^k \in \bigcup_{0 \leq m < \omega} \mathcal{B}(m)\} \supset B_iC(A)$,

where A is assumed to be a cc-formula.[5]

The definition of a proof in GL_ω is extended slightly from that in $KD4^2$. A *proof* P in GL_ω is defined as: (i) it is a countable tree with no infinite path from the root; (ii) a formula is associated with each node in P, and the formula associated with each leaf is an instance of the logical axioms; and (iii) adjoining nodes together with the associated formulae form an instance of the inference rules. We write $\vdash_\omega A$ iff there is a proof P in GL_ω such that A is associated with the root of P. For any set Γ of formulae, we write $\Gamma \vdash_\omega A$ iff $\vdash_\omega \bigwedge \Phi \supset A$ for some allowable subset Φ of Γ.

In the finitary $KD4^2$, $\vdash B_i(\bigwedge \Phi) \equiv \bigwedge B_i(\Phi)$ for any nonempty finite set of formulae in $\mathcal{P}$, where $B_i(\Phi) = \{B_i(A) : A \in \Phi\}$. Even if Φ is an infinite set, the one direction,

[5]In the game logic approach of Kaneko-Nagashima [11] and [12], the restriction on A to be a cc-formula is not assumed. Without this restriction, we have not succeeded in obtaining the faithful embedding result of the common knowledge logic to the game logic. Since our purpose of the introduction of infinitary conjunctions and disjunctions is to treat common knowledge as object formulae, the restriction to cc-formulae is not an obstrucle in our game theoretical applications. On the other hands, a reader may find that in our game theoretical considerations, the restriction is not used at all. Only with it, we can obtain the embedding result (Kaneko [3]), which associates model theoretic counterparts (Halpern-Moses [1] and Lismont-Mongin [15]) with our considerations.

$B_i(\bigwedge \Phi) \supset \bigwedge B_i(\Phi)$, is easily extended into GL_ω without using the C-Barcan axiom, but the converse cannot be provable in a direct infinitary extension of $KD4^2$. By the C-Barcan axiom, we allow this converse to the extent that it holds for the infinitary conjunctions of the common knowledge type and a cc-formula A. Without adding the C-Barcan axiom, Theorem 2.C would hold in the direct infinitary extension of $KD4^2$. Hence, without the C-Barcan axiom, the Limit Phase would not give a remedy to the problems in the transitory phases.[6]

Game logic GL_ω is an extension of finitary $KD4^2$ and, in fact, it is a conservative extension: for any $A \in \mathcal{P}$,

$$\vdash A \text{ if and only if } \vdash_\omega A, \tag{4.2}$$

where the *if* part is conservativeness. This conservativeness is proved in Kaneko [8]. We use the *only-if* part without referring to (4.2). After all, the logics required are evolving in the form:

$$(\mathcal{P}(0), KD4^2) \rightarrow (\mathcal{P}(1), KD4^2) \rightarrow \cdots \rightarrow (\mathcal{P}(k), KD4^2) \rightarrow \cdots \ GL_\omega$$

Each logic is a conservative extension of its predecessors. Game logic GL_ω is also a conservative extension of all predecessors. As discussed in Section 3, each ($\mathcal{P}(k)$,$KD4^2$) is needed for the consideration of Phase k, but the totality of $B^i(\mathcal{J}^4, I(1\text{–}3))$ is captured by none of these logics. Game logic GL_ω allows each player to capture the totality and gives a solution to his decision making problem.

First, we state the following simple facts on common knowledge formulae (Kaneko-Nagashima [11]).

Lemma 4.1. Let Φ be a finite set of formulae, and $A_1, A_2 \in \mathcal{P}^\omega$. Then

(1): $\vdash_\omega C(\bigwedge \Phi) \equiv \bigwedge C(\Phi)$;

(2): $\vdash_\omega \bigvee C(\Phi) \supset C(\bigvee \Phi)$;

(3): $\vdash_\omega C(A_1 \supset A_2) \supset (C(A_1) \supset C(A_2))$;

(4): $C(\Gamma) \vdash_\omega A_1$ implies $C(\Gamma) \vdash_\omega C(A_1)$.

Now we express the belief set $B^i(\mathcal{J}^4, (A_1, A_2))$ in terms of the common knowledge operator $C(\cdot)$, which becomes much easier to read.

Lemma 4.2. Let A_1, A_2 be cc-formulae in $\mathcal{P}^\omega$. Then $\vdash_\omega \bigwedge B^i(\mathcal{J}^4, (A_1, A_2)) \equiv B_i C(A_1 \wedge A_2)$.

Thus, $B^i(\mathcal{J}^4, (A_1, A_2))$ means that player i believes that $A_1 \wedge A_2$ is common knowledge. In the S4-type game logic, i.e., we add Axiom (T_i) : $B_i(A) \supset A$ to GL_ω,

[6] This difficulty has a counterpart in logic. The fragment of all cc-formulae in GL_ω (with the C-Barcan axiom) is proved to be sound and complete with respect to Kripke semantics. This is a consequence of the faithful embedding theorem of Kaneko [3] on the common knowledge logic to game logic GL_ω. Without the C-Barcan axiom, it is proved that this fragment is Kripke incomplete (Kaneko [3]). This is a counterpart in logic.

$B_iC(A_1 \bigwedge A_2)$ is equivalent to $C(A_1 \bigwedge A_2)$. Hence the relevant modification of common knowledge in our context is not the common belief, but the individual belief of common knowledge.

Proof of Lemma 4.2. Recall that the set $\mathsf{B}^i(\mathcal{J}^4, (A_1, A_2))$ is written as

$$\{B_i^{\dagger}(A_i), (B_iB_j)^1B_i^{\dagger}(A_i), (B_iB_j)^2B_i^{\dagger}(A_i), ...\}\cup$$

$$\{B_iB_j^{\dagger}(A_j), (B_iB_j)^1B_iB_j^{\dagger}(A_j), (B_iB_j)^2B_iB_j^{\dagger}(A_j), ...\}.$$

We denote the first and second sets by Γ_1 and Γ_2. Then we can prove that for all $m \geq 0$, $\Gamma_i \vdash_\omega B_i(B_iB_j)^m B_i(A_i)$, $\Gamma_i \vdash_\omega B_iB_j(B_iB_j)^m B_i(A_i)$, and $\Gamma_i \vdash_\omega B_i(B_iB_j)^m(A_i)$, $\Gamma_i \vdash_\omega B_iB_j(B_iB_j)^m(A_i)$. Hence $\vdash_\omega \bigwedge \Gamma_i \supset \bigwedge\{B_iB^k(A_i) : B^k \in \bigcup_{m<\omega} \mathcal{B}(m)\}$. The converse is straightforward. Hence $\vdash_\omega \bigwedge \Gamma_i \equiv \bigwedge\{B_iB^k(A_i) : B^k \in \bigcup_{m<\omega} \mathcal{B}(m)\}$. By the C-Barcan axiom, we have

$$\vdash_\omega \bigwedge \Gamma_i \equiv B_iC(A_i).$$

In the same manner, we have

$$\vdash_\omega \bigwedge \Gamma_j \equiv B_iC(A_j).$$

Hence we have $\vdash_\omega \bigwedge \mathsf{B}^i(\mathcal{J}^4, (A_1, A_2)) \equiv B_iC(A_1 \bigwedge A_2)$ by Lemma 4.1.(1). □

Also, the conclusion of Theorem 2.A.(1) is represented as $B_iC(\text{Nash}^*(a))$, which means that player i believes the common knowledge of $\text{Nash}^*(a)$. Note that $\text{Nash}^*(a) = B_1(\text{Nash}_1(a_1 \mid a_2)) \bigwedge B_2(\text{Nash}_2(a_2 \mid a_1))$ itself includes beliefs. In the S4-type game logic, $B_iC(\text{Nash}^*(a)))$ is shown to be equivalent to $C(\text{Nash}(a))$.

Lemma 4.3. $\vdash_\omega \bigwedge_{m<\omega} (B_iB_j)^m (\text{Nash}^*(a) \bigwedge B_i(\text{Nash}^*(a))) \equiv B_iC(\text{Nash}^*(a))$.

Proof. First, we prove that the latter implies the former. It holds that $\vdash_\omega B_iC(\text{Nash}^*(a)) \supset (B_iB_j)^m(\text{Nash}^*(a))$ and $\vdash_\omega B_iC(\text{Nash}^*(a)) \supset (B_iB_j)^m B_i(\text{Nash}^*(a))$ for all $m \geq 0$. Hence

$$\vdash_\omega B_iC(\text{Nash}^*(a)) \supset \bigwedge_{m<\omega} (B_iB_j)^m (\text{Nash}^*(a)) \bigwedge B_i(\text{Nash}^*(a))).$$

Consider the converse. The former formula is equivalent to

$$\bigwedge\{B_iB^m\text{Nash}^*(a) : B^m \in \bigcup_{m<\omega} \mathcal{B}(m)\},$$

Here we note that Axiom PI_i is used. By the C-Barcan axiom, we have

$$\vdash_\omega \bigwedge\{B_iB^m\text{Nash}^*(a) : B^m \in \bigcup_{m<\omega} \mathcal{B}(m)\} \supset B_i \left(\bigwedge\{B^m\text{Nash}^*(a) : B^m \in \bigcup_{m<\omega} \mathcal{B}(m)\}\right).$$

This is the converse itself. □

Before going to our game theoretical considerations in Section 5, we mention briefly the common knowledge logic (cf., Halpern-Moses [1], Lismont-Mongin [15] and Meyer-van der Hoek [16]). First, observe that the common knowledge operator has the following properties (cf., Kaneko [3]).

Lemma 4.4. Let $A, B \in \mathcal{P}^{\omega}$. Then

(1): $\vdash_{\omega} C(A) \supset A \wedge \mathrm{B}_1 C(A) \wedge \mathrm{B}_2 C(A)$, where A is a cc-formulae;

(2): if $\vdash_{\omega} B \supset A \wedge \mathrm{B}_1(B) \wedge \mathrm{B}_2(B)$, then $\vdash_{\omega} B \supset C(A)$.

The common knowledge logic CKL is formulated by focussing these properties and by requiring these as axioms for new symbols $C_0(\cdot)$ in the *finitary* extension of our language $\mathcal{P}$ including this unary operator symbol $C_0(\cdot)$. Then one axiom and one inference rule corresponding to (1) and (2) are assumed on $C_0(\cdot)$. Then it is shown by proving the sound and completeness theorems with respect to Kripke semantics that this extension of the finitary KD4^2 determines $C_0(\cdot)$ to be common knowledge. This common knowledge logic CKL is embedded into GL_{ω} by translating $C_0(A)$ into $C(A^*)$, where A^* is also obtained by the same principle, and GL_{ω} is a conservative extension of the embedded fragment.

The following lemma will be used in Section 5.

Lemma 4.5. Let $A_1, A_2, C_1, C_2 \in \mathcal{P}^{\omega}$. Then

(1): if $\mathsf{B}^i(\mathcal{J}^4, (A_1, A_2)) \vdash_{\omega} C_i$ and $\mathsf{B}^i(\mathcal{J}^4, (A_1, A_2)) \vdash_{\omega} \mathrm{B}_i \mathrm{B}_j^{+}(C_j)$, then $\mathsf{B}^i(\mathcal{J}^4, (A_1, A_2)) \vdash_{\omega} \bigwedge \mathsf{B}^i(\mathcal{J}^4, (C_1, C_2))$;

(2): if $\vdash_{\omega} C_i$ and $\vdash_{\omega} \mathrm{B}_i \mathrm{B}_j^{+}(C_j)$, then $\vdash_{\omega} \bigwedge \mathsf{B}^i(\mathcal{J}^4, (C_1, C_2))$.

Proof. The assertion (2) can be regarded as a special case of (1). For (1), it suffices to show that for all $m \geq 0$,

$$\begin{gathered} \mathsf{B}^i(\mathcal{J}^4, (A_1, A_2)) \vdash_{\omega} (\mathrm{B}_i \mathrm{B}_j)^m \mathrm{B}_i^{+}(C_i) \\ \mathsf{B}^i(\mathcal{J}^4, (A_1, A_2)) \vdash_{\omega} (\mathrm{B}_i \mathrm{B}_j)^m \mathrm{B}_i \mathrm{B}_j^{+}(C_j). \end{gathered} \tag{4.3}$$

Indeed, since the set $\{(\mathrm{B}_i\mathrm{B}_j)^m \mathrm{B}_i^{+}(C_i) : m \geq 0\} \cup \{(\mathrm{B}_i\mathrm{B}_j)^m \mathrm{B}_i \mathrm{B}_j^{+}(C_j) : m \geq 0\}$ is exactly $\mathsf{B}^i(\mathcal{J}^4, (C_1, C_2))$, we take the conjunction of this set, which is the desired result. Let us prove (4.3). For $m = 0$, (4.3) holds. Indeed, the second assertion is the assumption. Consider the first assertion for $m = 0$. By the assumption and Lemma 4.3, we have $\mathrm{B}_i C(A_1 \wedge A_2) \vdash_{\omega} C_i$. This together with Nec and MP_i implies $\mathrm{B}_i \mathrm{B}_i C(A_1 \wedge A_2) \vdash_{\omega} \mathrm{B}_i(C_i)$. Since $\vdash_{\omega} \mathrm{B}_i C(A_1 \wedge A_2) \supset \mathrm{B}_i \mathrm{B}_i C(A_1 \wedge A_2)$, we have $\mathrm{B}_i C(A_1 \wedge A_2) \vdash_{\omega} \mathrm{B}_i^{+}(C_i)$. By Lemma 4.3, this is equivalent to the desired one.

Now suppose that (4.3) holds for m. Then it follows from Lemma 4.2

$$\mathrm{B}_i C(A_1 \wedge A_2) \vdash_\omega (\mathrm{B}_i \mathrm{B}_j)^m \mathrm{B}_i^+(C_i).$$

Hence by Nec and MP_i,

$$\mathrm{B}_i \mathrm{B}_j \mathrm{B}_i C(A_1 \wedge A_2) \vdash_\omega (\mathrm{B}_i \mathrm{B}_j)^{m+1} \mathrm{B}_i^+(C_i).$$

Since $\vdash_\omega C(A_1 \wedge A_2) \supset \mathrm{B}_j C(A_1 \wedge A_2)$ by Lemma 4.4, we have $\vdash_\omega \mathrm{B}_i C(A_1 \wedge A_2) \supset \mathrm{B}_i \mathrm{B}_j C(A_1 \wedge A_2)$ and $\vdash_\omega \mathrm{B}_i \mathrm{B}_j C(A_1 \wedge A_2) \supset \mathrm{B}_i \mathrm{B}_j \mathrm{B}_i C(A_1 \wedge A_2)$ by Nec and MP_i. Thus we have

$$\mathrm{B}_i C(A_1 \wedge A_2) \vdash_\omega (\mathrm{B}_i \mathrm{B}_j)^{m+1} \mathrm{B}_i^+(C_i).$$

Hence we have the first one of (4.3). In the same manner, we have the second. □

5. Reciprocal Thoughts with Interaction Structure $\mathcal{J}^4 = (\{1,2\},\{1,2\})$: the Limit Phase ω

The reciprocal thoughts in the Limit Phase ω can be fully captured in game logic GL_ω. The final decisions are determined to be the individual beliefs of the common knowledge of a Nash strategy. It is still a *subjective* belief and may have a discrepancy from the objective reality. This possibility will be well reflected in the result we call the Konnyaku Mondô Theorem to be given in Subsection 5.2.

5.1. Characterization, Decidability and Playability

In the following, we denote, by $\hat{I}_{ii}(a_i)$ and $\hat{I}_{ij}(a_j)$, respectively,

$$\bigvee_{x_j} \mathrm{B}_i C(\mathrm{Nash}^*(a_i; x_j)) \text{ and } \bigvee_{x_i} \mathrm{B}_i C(\mathrm{Nash}^*(x_i; a_j)). \tag{5.1}$$

The following is the characterization result, which will be proved in the end of this subsection.

Theorem 5.A. (Characterization III): Let $g^i = (g_i; \hat{g}_j)$ satisfy Condition Int. Then, for $k = 1, 2$,

$$\mathrm{B}^i(\mathcal{J}^4, \mathrm{I}(1\text{–}3)), \mathrm{B}^i(\mathcal{J}^4, \mathrm{WD}), \mathrm{B}^i(\mathcal{J}^4, g^i) \vdash_\omega \bigwedge_{x_k} \left(I_{ik}(x_k) \equiv \hat{I}_{ik}(x_k) \right). \tag{5.2}$$

We emphasize that in Theorem 5.A, $\hat{I}_{ik}(x_k)$ has the outer B_i and means that player i believes the common knowledge of a Nash strategy. Also, player i believes that it is common knowledge that j has and knows his own payoff function $\hat{g}_j$. These beliefs $\hat{g}_1$ and

$\hat{g}_2$ may different. Hence each player may believe the common knowledge of a different situation. That is, the assumption sets $\bigcup_i \mathsf{B}^i(\mathcal{J}^4, \mathrm{I}(1\text{-}3)), \bigcup_i \mathsf{B}^i(\mathcal{J}^4, \mathrm{WD}), \bigcup_i \mathsf{B}^i(\mathcal{J}^4, g^i)$ are consistent in GL_ω even if $\hat{g}_1$ and $\hat{g}_2$ are different from g_1 and g_2, respectively. This possibility will be discussed in Subsection 5.2.

In the S4-type game logic, $I_{ii}(a_i)$ and $I_{ij}(a_j)$ are determined to be

$$\bigvee_{x_j} C(\mathrm{Nash}(a_i; x_j)) \text{ and } \bigvee_{x_i} C(\mathrm{Nash}(x_i; a_j)). \tag{5.3}$$

This result is what is considered in Kaneko-Nagashima [11] and Kaneko [4]. These papers evaluated this version of the theorem. In this case, it is impossible that each player believes the common knowledge of a different situation. That is, $\bigcup_i \mathsf{B}^i(\mathcal{J}^4, g^i)$ is inconsistent in the S4-type game logic if $\hat{g}_i$ is different from g_i for at least one of $i = 1, 2$.

Without Condition Int on $g^i = (g_i; \hat{g}_j)$, the above theorem need to be modified so that Nash equilibrium is replaced by the subsolution concept given by Nash [17]. In the S4-type game logic, Kaneko [4], Section 6 obtained a result corresponding to Theorem 5.A under some additional assumption that they have some communication to guarantee the common knowledge of the choice of a subsolution, and also discussed new difficulties arising from such games. We will discuss fully the cases without Conditions Int and Conc_i in a separate paper.

Let us return to game logic GL_ω and mention the decidability and playability. In the following, we denote $\mathsf{B}^i(\mathcal{J}^4, \mathrm{I}(1\text{-}3)), \mathsf{B}^i(\mathcal{J}^4, \mathrm{WD}), \mathsf{B}^i(\mathcal{J}^4, g^i)$ by $\Pi_i(g^i)$. We will give the proof of Theorem 5.B.(3-1) in the end of this subsection.

Theorem 5.B. (Decidability III): Let $g^i = (g_i; \hat{g}_j)$ satisfy Condition Int. Then, for $k = 1, 2$,

(3-1): a_k is a Nash strategy for g^i if and only if $\Pi_i(g^i) \vdash_\omega \mathrm{B}_i^+(I_{ik}(a_k))$;

(3-2): a_k is not a Nash strategy for g^i if and only if $\Pi_i(g^i) \vdash_\omega \mathrm{B}_i^+(\neg I_{ik}(a_k))$.

Theorem 5.C. (Playability III): Let $g^i = (g_i; \hat{g}_j)$ satisfy Condition Int. Then, for $k = 1, 2$,

(3-3): g^i has a Nash equilibrium if and only if $\Pi_i(g^i) \vdash_\omega \mathrm{B}_i^+(\bigvee_{x_k} I_{ik}(x_k))$;

(3-4): g^i has no Nash equilibrium if and only if $\Pi_i(g^i) \vdash_\omega \mathrm{B}_i^+(\neg \bigvee_{x_k} I_{ik}(x_k))$.

Thus, a player who has reached the Limit Phase ω can decide his final decision relative to his belief on the game $g^i = (g_i; \hat{g}_j)$. Here player i has no longer a difficulty mentioned by Theorem 2.B.(2). Statement (3-2) means that he finds that a_i is not a solution. If (3-4) is the case, he finds neither a Nash equilibrium nor a Nash strategy in pure strategies. One possible way out is to consider mixed strategies, which case will be remarked in Section 6.

Let us prove Theorems 5.A and 5.B.(3-1). First, the following lemma is an almost direct conclusion from Theorem 2.A.(1) and Lemma 4.3.

Lemma 5.1. $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}2)) \vdash_\omega I_{ii}(a_i) \wedge I_{ij}(a_j) \supset \mathrm{B}_i C(\mathrm{Nash}^*(a))$.

The following lemma can be proved from Lemma 5.1 in the same manner as the proof of Lemma 6.2 of Part I.

Lemma 5.2. $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)) \vdash_\omega I_{ik}(a_k) \supset \bigvee_{x_l} \mathrm{B}_i C(\mathrm{Nash}^*(a_k;x_l))$ for $k = 1,2$.

Lemma 5.3.($\mathrm{I}\hat{1}$): Let $g^i = (g_i;\hat{g}_j)$ satisfy Condition Int. Then

$$\mathsf{B}^i(\mathcal{J}^4, g^i) \vdash_\omega \hat{I}_{ii}(a_i) \wedge \hat{I}_{ij}(a_j) \supset \mathrm{B}_i(\mathrm{Nash}_i(a_i \mid a_j));$$

$$\mathsf{B}^i(\mathcal{J}^4, g^i) \vdash_\omega \mathrm{B}_i\mathrm{B}_j^+ \left(\hat{I}_{jj}(a_j) \wedge \hat{I}_{ji}(a_i) \supset \mathrm{B}_j(\mathrm{Nash}_j(a_j \mid a_i))\right).$$

($\mathrm{I}\hat{2}$): $\vdash_\omega \hat{I}_{il}(a_l) \supset \mathrm{B}_i(\hat{I}_{kl}(a_l))$ for $i,k,l = 1,2$, and $a_l \in \Sigma_l$.

($\mathrm{I}\hat{3}$): $\vdash_\omega \bigvee_{x_l} \hat{I}_{il}(x_l) \supset \bigvee_{x_k} \hat{I}_{lk}(x_k))$ for $i,k,l = 1,2$.

Once this lemma is proved, $\mathsf{B}^i(\mathcal{J}^4, g^i) \vdash_\omega \mathrm{I}_i(1\text{–}3)[\mathcal{I}]$ and $\mathsf{B}^i(\mathcal{J}^4, g^i) \vdash_\omega \mathrm{B}_i\mathrm{B}_j^+(\mathrm{I}_j(1\text{–}3)[\mathcal{I}])$, where $\mathcal{I} = \{\hat{I}_{kl}(a_l)\}_{k,l,a_l}$. Hence by Lemma 4.5, $\mathsf{B}^i(\mathcal{J}^4, g^i) \vdash_\omega \mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)[\mathcal{I}])$. Hence using WD_i, we have $\mathsf{B}^i(\mathcal{J}^4, g^i), \mathsf{B}^i(\mathcal{J}^4,\mathrm{WD}) \vdash_\omega \hat{I}_{ik}(a_k) \supset I_{ik}(a_k)$. This together with Lemma 5.1 implies that $\mathsf{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)), \mathsf{B}^i(\mathcal{J}^4, g^i), \mathsf{B}^i(\mathcal{J}^4,\mathrm{WD}) \vdash_\omega \hat{I}_{ik}(a_k) \equiv I_{ik}(a_k)$. This complete the proof of Theorem 5.A.

Proof of Lemma 5.3. Assertions ($\mathrm{I}\hat{2}$) and ($\mathrm{I}\hat{3}$) are straightforward. Now we prove only the first assertion of ($\mathrm{I}\hat{1}$). Before going into details, we mention one fact:

$$\mathrm{B}_i^+(g_i), \mathrm{B}_j^+(\hat{g}_j) \vdash_\omega \mathrm{Nash}^*(a_i;a_j) \supset \mathrm{Nash}(a_i;a_j). \tag{5.4}$$

Since $\vdash_\omega \mathrm{Nash}^*(a_i;a_j) \supset \mathrm{B}_i(\mathrm{Nash}_i(a_i;a_j))$ and $\mathrm{B}_i^+(g_i) \vdash_\omega \mathrm{B}_i(\mathrm{Nash}(a_i \mid a_j)) \supset \mathrm{Nash}_i(a_i \mid a_j)$ by Lemma 3.2.(3) of Part I, we have $\mathrm{B}_i^+(g_i) \vdash_\omega \mathrm{Nash}^*(a_i;a_j) \supset \mathrm{Nash}_i(a_i \mid a_j)$. In the same manner, $\mathrm{B}_j^+(\hat{g}_j) \vdash_\omega \mathrm{Nash}^*(a_i;a_j) \supset \mathrm{Nash}_j(a_j \mid a_i)$. These imply (5.4).

By (5.1),

$$\vdash_\omega \hat{I}_{ii}(a_i) \wedge \hat{I}_{ij}(a_j) \supset \left(\bigvee_{x_j} \mathrm{B}_i(\mathrm{Nash}^*(a_i;x_j))\right) \wedge \left(\bigvee_{x_i} \mathrm{B}_i(\mathrm{Nash}^*(x_i;a_j))\right).$$

Since $\vdash_\omega \bigvee B_i(\Phi) \supset B_i(\bigvee \Phi)$ for any allowable set Φ, we have

$$\vdash_\omega \hat{I}_{ii}(a_i) \wedge \hat{I}_{ij}(a_j) \supset B_i \left((\bigvee_{x_j} \text{Nash}^*(a_i;x_j)) \wedge (\bigvee_{x_i} \text{Nash}^*(x_i;a_j)) \right). \quad (5.5)$$

It follows from (5.4) that

$$\begin{aligned} B_i^+(g_i), B_j^+(\hat{g}_j) \vdash_\omega & (\bigvee_{x_j} \text{Nash}^*(a_i;x_j)) \wedge (\bigvee_{x_i} \text{Nash}^*(x_i;a_j)) \\ & \supset (\bigvee_{x_j} \text{Nash}(a_i;x_j)) \wedge (\bigvee_{x_i} \text{Nash}(x_i;a_j)) \end{aligned}$$

Since g^i satisfies Int, i.e., $g^i \vdash_0 (\bigvee \text{Nash}(a_i;x_j)) \wedge (\bigvee_{x_i} \text{Nash}(x_i;a_j)) \supset \text{Nash}(a_i;a_j)$, we have

$$B_i^+(g_i), B_j^+(\hat{g}_j) \vdash_\omega (\bigvee_{x_j} \text{Nash}^*(a_i;x_j)) \wedge (\bigvee_{x_i} \text{Nash}^*(x_i;a_j)) \supset \text{Nash}(a_i;a_j).$$

Hence

$$\begin{aligned} B_iB_i^+(g_i), B_iB_j^+(\hat{g}_j) \vdash_\omega & B_i(\bigvee \text{Nash}^*(a_i;x_j)) \wedge (\bigvee_{x_i} \text{Nash}^*(x_i;a_j)) \\ & \supset B_i(\text{Nash}(a_i;a_j)). \end{aligned} \quad (5.6)$$

Thus it follows from (5.5) and (5.6), noting $\vdash_\omega B_i^+(g_i) \supset B_iB_i^+(g_i)$, that

$$B_i^+(g_i), B_iB_j^+(\hat{g}_j) \vdash \hat{I}_{ii}(a_i) \wedge \hat{I}_{ij}(a_j) \supset B_i(\text{Nash}(a_i;a_j)).$$

This implies $B^i(\mathcal{J}^4, g^i) \vdash_\omega \hat{I}_{ii}(a_i) \wedge \hat{I}_{ij}(a_j) \supset B_i(\text{Nash}_i(a_i \mid a_j))$. □

Proof of Theorem 5.B.(3-1): Suppose that a_k is a Nash strategy for g^i. Then there is a strategy a_l for player l such that $(a_k;a_l)$ is a Nash equilibrium. Then $B^i(\mathcal{J}^4, g^i) \vdash_\omega B_iC(\text{Nash}(a_k;a_l))$. Thus $B^i(\mathcal{J}^4, g^i) \vdash_\omega \bigvee_{x_l} B_iC(\text{Nash}(a_k;x_l))$. By Theorem 5.A, we have $\Pi_i(g^i) \vdash_\omega I_{ik}(a_k)$. Then $\Pi_i(g^i) \vdash_\omega B_i^+(I_{ik}(a_k))$.

The converse can be proved in the manner parallel to Theorem 5.B of Part I. □

5.2. Konnyaku Mondô – – Devil's Tongue Dialogue

We emphasized after Theorem 5.A that the final decision was an *individual belief* of the common knowledge of a Nash strategy. The Limit Phase ω is interpreted as meaning that the players have been talking, face to face, about their game situation as well as their thought making. The belief sets $B^i(\mathcal{J}^4$,I(1–3)), $B^i(\mathcal{J}^4$,WD), $B^i(\mathcal{J}^4, g^i)$ mean that player i *believes* that the situation and their ways of thought making are common knowledge between the players. Nevertheless, we often observe in our life that one regards something as common knowledge, though it is not really common knowledge. The following theorem is a specific version of such situations. Recall that $\Pi(g^i)$ denotes $\mathrm{B}^i(\mathcal{J}^4,\mathrm{I}(1\text{–}3)), \mathrm{B}^i(\mathcal{J}^4,\mathrm{WD}), \mathrm{B}^i(\mathcal{J}^4, g^i)$.

Theorem 5.D (Konnyaku Mondô): Suppose that each g_i allows no dominant strategies, and that (a_1^*, a_2^*) is an inductively stable stationary state. Suppose that (a_1^*, a_2^*) is a strict Nash equilibrium, i.e., for $i = 1, 2$,

$$g_i(a_i^*; a_j^*) > g_i(a_i; a_j^*) \text{ for all } a_i \neq a_i^*.$$

Then there are payoff functions $\hat{g}_1$ and $\hat{g}_2$ such that

(1): each $\hat{g}_i$ allows no dominant strategies;

(2): each believed game $g^i = (g_i; \hat{g}_j)$ satisfies Condition Int;

(3): each player i has interaction structure $\mathcal{J}^4 = (\{1,2\},\{1,2\})$;

(4): $\Pi_i(g^i) \vdash_\omega I_{ik}(a_k^*)$ and $\Pi_i(g^i) \vdash_\omega \neg I_{ik}(a_k)$ for all $a_k \neq a_k^*$ and $i, k = 1, 2$;

(5): $\Pi_1(g^1) \cup \Pi_2(g^2)$ is consistent in GL_ω.

Recall that Theorem 5.A that each $I_{ik}(a_k)$ is determined to be $\bigvee_{x_j} \mathrm{B}_i C(\mathrm{Nash}(a_k; x_j))$.

Each player i believes that I(1–3),WD and g^i are common knowledge (Lemma 4.2), but $g^1 = (g_1, \hat{g}_2)$ and $g^2 = (\hat{g}_1, g_2)$ may be objectively false. For example, consider the following game $g = (g_1, g_2)$ given by Table 5.1.

	s_{21}	s_{22}	s_{23}
s_{11}	$(5,5)^*$	$(0,3)$	$(5,5)^*$
s_{12}	$(3,0)$	$(3,3)^*$	$(0,1)$
s_{13}	$(0,2)$	$(2,0)$	$(2,2)$

Table 5.1

	s_{21}	s_{22}	s_{23}
s_{11}	$(5,\hat{2})$	$(0,3)$	$(5,\hat{2})$
s_{12}	$(3,0)$	$(3,3)^*$	$(0,1)$
s_{13}	$(0,2)$	$(2,0)$	$(2,2)$

Table 5.2: $g^1 = (g_1, \hat{g}_2)$

Suppose that (s_{12}, s_{22}) is the inductively stable state. We modify g_1 and g_2 so that each player mistakes $\hat{2}$ for 5. Table 5.2 describes $g^1 = (g_1, \hat{g}_2)$. Each believed game $g^i = (g_i; \hat{g}_j)$ has a unique Nash equilibrium, *a fortiori*, it satisfies Condition Int. This may be caused by their misunderstandings of payoff $\hat{2}$ for 5 through their communication. The point here is the possibility that each player develops a false and different belief of the common knowledge of the situation.

Stories of this kind can be found in literature and folklore. A Japanese traditional rakugo (comic story), "Konnyaku mondô" (Devil's Tongue dialogue), describes exactly such a situation: A (devil's tongue) jelly maker lived in a Buddhist temple pretending to be a monk. A real Buddhist monk came to visit this temple to have a dialogue on Buddhism thoughts. The jelly maker first refused but eventually agreed to have a dialogue. Since the jelly maker did not know how he could communicate with the Monk on Buddhism, he answered the questions of the Monk in gestures. The Monk took this as a style of dialogue, and also responded in gestures. After several exchanges of gestures, both thought that the jelly maker defeated the Monk. After the dialogue, a witness asked the Monk about the dialogue. The Monk said that the jelly maker had a great Buddhism thought shown by his gestures and should be respected. Another witness asked the jelly maker about it, and the jelly maker answered: the Monk communicated poorly about jelly products and the jelly maker defeated the Monk with his gestures.[7] Thus each of them believed that they had perfectly meaningful dialogue and that it was common knowledge that the jelly maker defeated the monk in the dialogue. The Monk believed that they had a Buddhism dialogue, but the jelly maker believed that they had discussed jelly products (pp.61–70 in [20]).[8]

The above reciprocal form is interesting, but a one-person version of this theorem, i.e., one believes that something is common knowledge but the other does not, may be more often observed. For example, consider the following game $g = (g_1, g_2)$ given by Table 5.3:

	s_{21}	s_{22}	s_{23}
s_{11}	$(3,2)$	$(0,3)$	$(6,1)$
s_{12}	$(6,0)$	$(3,3)^*$	$(3,0)$

Table 5.3

	s_{21}	s_{22}	s_{23}
s_{11}	$(3,\hat{6})$	$(0,3)$	$(6,\hat{5})$
s_{12}	$(6,0)$	$(3,3)^*$	$(3,0)$

Table 5.4: $g^1 = (g_1, \hat{g}_2)$

In this game, player 2 has a dominant strategy s_{22}, but 1 has no dominant strategies. If $g = (g_1, g_2)$ is truly believed by player 1, then interaction structure $\mathcal{J}^3 = (\{1,2\},\{2\})$ suffices and the argument of Section 6 of Part I is applied. Suppose, however, that player 1 mistakes payoff $\hat{5}$ and $\hat{6}$ for 1 and 2 for player 2's payoffs through their communication. The resulting matrix is Table 5.4. For this matrix, player 1 need interaction structure $\mathcal{J}^4 = (\{1,2\},\{1,2\})$ and may believe the common knowledge of Nash equilibrium (s_{12}, s_{22}). Nevertheless, player 2 himself may truly believe that s_{22} is a unique dominant strategy, and that he does not care about the decision making of player 1 (2 may be kind enough to talk to 1, though he is indifferent).

[7]This part is slightly modified from the original (pp.61–70 in [20]) to have a shorter consistent story, but the essential part is not changed.

[8]This analogy was suggested by T. Nagashima. Kurosawa's movie, "Rashomon", (based on a short story, "Yabunonaka", by R. Akutagawa) can be regarded as having a similar point.

Here we emphasize that the evolutions of thoughts for the players are interactive but do not necessarily progress together. The above example suggests that player 2 stays in Phase 1, while player 1 goes to Phase ω.

Sketch of the Proof of Theorem 5.D. This proof is parallel to that of Theorem 5.D of Part I. First, we construct $\hat{g}_1$ and $\hat{g}_2$ so that each allows no dominant strategies and (a_1^*, a_2^*) is a unique Nash equilibrium for each of $g^1 = (g_1, \hat{g}_2)$ and $g^2 = (\hat{g}_1, g_2)$. Hence each satisfies the assertions (1) and (2). The assertion (3) is an assumption, and (4) follows from Theorem 5.B. The main step is to prove that $\Pi_1(g^1) \cup \Pi_2(g^2)$ is consistent in GL_ω. For this, we first prove the lemma parallel to Lemma 5.1 of Part I. Then we refer to the separation theorem for GL_ω of Kaneko-Nagashima [12] (p.281, Theorem 3.3), which is a stronger version of Theorem 5.E of Part I. The remaining is essentially the same as the proof of Theorem 5.D of Part I. □

6. Conclusions

6.1. Evolution of Thoughts

In Parts I and II, we have discussed an evolution of intersubjective thoughts on a game (payoff functions) and decision making. Here we summarize the process and its further continuation. As in Parts I and II, we describe one phase to next phase, but this phase-to-phase presentation is rather for simplicity. It may happen that some phases may go simultaneously in the mind of a player.

Phase 0: In this phase, a player does not construct a theory: he maximizes his payoff function based on his memory of experiences. In Section 2 of Part I, we postulate that he has made trials over all possible actions, and that he has a capacity good enough to recall information, received in the past, on the events of unilateral deviations from the stationary state. Proposition 2.1 of Part I states that these memories on experiences are sufficient for his behavior.

It is often more convenient to summarize his experiences and to construct a "theory" (or "model") of the game situation from them. In our setting, the players are isolated and playing solely the game g, but this is a simplicity assumption for our consideration of modeling some social situation. If they are perfectly isolated and can concentrate on playing the game g, they might not need to construct a theory. The people we would like to target are not isolated but live in social situations facing other problems. The recurrent situation of the game constitutes a rather small part of the social world of each player. In such a situation, it would be useful for the individual player to construct a summarizing theory of the problem in question than always to recall experiences in his memory. This is our basic postulate for the present research program.

Now an individual player leaves Phase 0, and as stated by Postulate 7 of Part I, he starts constructing a theory to explain the situation based on his experiences. If he thinks only about his own payoff function, then he is in Phase 1.

Phase 1: Here the player thinks only about his own payoff function, and chooses a payoff maximizing strategy – – a dominant strategy. Theorem 5.B of Part I states that he can decide whether or not a given strategy is a dominant strategy. If he finds a dominant strategy, he would stay in Phase 1. On the other hand, if he finds no dominant strategies, then he should think about the other player's decision. Theorem 5.D of Part I states that if he can know his payoff function up to the experienced domain $\mathcal{E}(i \mid a^*)$ – – Postulate 5 of Part I – –, then it would be always possible for him to construct a believed payoff function so that his stationary behavior is a dominant strategy. If the player is cautious enough to find that there are many other possibilities for his payoff function to allow this explanation, he may become more cautious about the less frequent events, and may find that his payoff function allows no dominant strategies. If this is the case, he would think about the other player's behavior. One possibility is to go to Phase 2.

Phase 2: In this phase, the player starts thinking about the other player's behavior but regards the other player as a one-person decision maker of the type of Phase 1. Without communication, the player has only the knowledge on the other player's choices. Hence it may be easier for the player to construct a belief on the other's payoff function in the sense that his thinking on the other player's payoff function is less restrictive. Here he could find also the great possibilities for his explanation, but would have no source to narrow down such possibilities without having communications to the other player. Once he starts talking to the other player, he may get some information about the other player's payoff function.

If he finds still that the other player has a dominant strategy, he would stay in Phase 2. If he finds that his belief on the player's payoff function allows no dominant strategies, he can nó longer regard the other player as a one-person decision maker. Now he leaves Phase 2.

Transitory Phases k: First, suppose that the player happens to believe that neither he nor the other has a dominant strategy. Here he needs to think about the other player's thoughts on decision making. However, without communications, he has no hint for the other player's thinking. If this is so, he does not actually reach the transitory phase. To think about the other player's decision making in the way of $\mathcal{J}^4 = (\{1,2\},\{1,2\})$, he needs to have some communication with the other.

Having some communications with the other player, the individual player starts making some beliefs on the other player's thoughts on decision making. In general, communication goes deeper, the belief is getting deeper and he goes to a further phase.

Here we emphasize that the face-to-face communication is important. If their communication is not face-to-face but takes a step for one direction, such as computer message communication, then the reciprocal depth does not go deep, and is constrained by the number of messages. However, the face-to-face communication has the special feature that it allows visual verifications and the players' mutual understanding goes almost instantaneously to any finite depth.[9] Strictly speaking, however, to achieve the common knowledge, each player needs to have an inductive decision from any finite depths to the limit. It is the point here is that this inductive decision is rather a good approximation. Subject to possible misunderstanding, the individual player may believe that something becomes common knowledge. This jump also includes some decision making of his thought.

Here we should realize that this step from transitory phases to the Limit Phase ω need also a big jump of the language as well as a logic. If the player sticks to living in the finitary logic $KD4^2$, then he would meet serious difficulties but would not realize them as far as he stay in the same language and logic. This is the subject discussed in Sections 2 and 3 of Part II. To go to the Limit Phase ω, the individual player should make a heuristic jump. This jump is also regarded as an induction, since the player can experience many failures and decides the need of the jump.

The Limit Phase ω: In this phase, if the game has a Nash equilibrium and satisfies Condition Int, the player reaches some decision, though they might be committed to the beliefs of the common knowledge of false facts, as stated in Subsection 5.2.

We need an additional consideration on decision making if the game does not satisfy Condition Int. In this case, the players need to talk to make some coordination on strategies. Kaneko [4] has discussed this difficulty in the S4-type game logic. This consideration is carried over to our GL_ω with some modifications.

A more serious difficulty would appear when a game does not have a Nash equilibrium in pure strategies, *a fortiori*, no dominant strategies for each player. When $g^i = (g_i; \hat{g}_j)$ has no Nash equilibria, player i can notice, by the results of Subsection 5.1, that he has no decisions. In this case, it is one possibility for him to think about a Nash equilibrium in mixed strategies. Then the inductive game theory needs to be modified so that each player may notice probabilistic behavior.

Then there are two problems newly emerging, both of which are related to the probabilistic behavior. One is related to the deductive game theory and the second is inductive. Both problems are simultaneously emerging, but again, we discuss each separately.

First, if an individual player considers his probabilistic behavior in his theory, his theory should include some real number theory. In the two-person case, in fact, the

[9] Plato [19], Book VI started *the analogy of the Sun* with writing "Then have you noticed,' I asked, 'how extremely lavish the designer of our senses was when he gave us the faculty of sight and make objects visible ?' Then Socates (Plato) continues arguing that vision is helped by light (the Sun), while the other senses do not have a help of a third element. Now we know that the speed of light is almost infinite relative to our individual visual scope. The common knowledge may be regarded as an (ideal) approximation of the situation where people are looking at each other.

ordered field theory suffices (cf., Kaneko [5]). In this case, the problem is not very difficult. However, if the game includes at least three players, then we need more complicated equilibria including potentially any real algebraic numbers as equilibrium strategy probabilities. In this case, the choice of language matters, and a natural choice of a language and a logic leads to a serious undecidability, which implies that the game is deductively not playable (Kaneko-Nagashima [11]). Kaneko [5] gave full discussions on this problem.

From the inductive point of view, we meet the problem of how or whether the player can notice probabilistic behavior from his experiences. This is related to the definition of a frequentist probability (cf., von Mises [22] and Wald [21]). According to this frequentist probability theory, if the player can recall his experiences in the past up to any length (to the infinite past), he could identify the probabilistic behavior of the other players. However, this assumption is already far beyond our intention. Another possible approach can be taken by using the Kolmogorov complexity measure, which defines an approximate randomness of a given finite sequence. Probably, this is more appropriate here than the direct von Mises-Wald approach. However, this is not studied.

6.2. Games with More Players

For a game with more than two players, the basic evolution story is essentially the same, but there are much more possible interactive patterns. Even if we assume no differences in the players' subjectivities, there are *very* many possibilities: if we take subjectivities into account, the multitude of possibilities is almost explosive. Here we consider a few simple 3-person games without differentiating subjectivities.

	s_{21}	s_{22}
s_{11}	(5,5)	(1,2)
s_{12}	(6,1)	(3,3)

Table 6.1

	s_{31}	s_{32}
s_{21}	1	0
s_{22}	0	1

Table 6.2

	s_{31}	s_{32}
s_{11}	1	0
s_{12}	0	1

Table 6.3

Consider the 3-person game defined by Tables 6.1 and 6.2. Table 6.1 gives the payoffs to players 1, 2 which depend upon their choices, and Table 6.2 gives the payoffs to player 3 which depend upon the choices of 2 and 3. In this game, player 1 can choose a payoff maximizing strategy s_{11} by thinking about his own payoffs, 2 should think about 1's decision making, and 3 should think about 1's and as well as 2's. The interactions are described as Diagram 6.1. Expression $i \leftarrow j$ means that j needs to think about i's decision making. Hence the beliefs of depth 3 are involved in this game.

1 ← 2 ← 3

Diagram 6.1

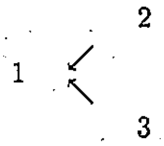

Diagram 6.2

If the payoffs for player 3 are changed into Table 6.3, i.e., 3's payoffs are determined by 1's and 3's choices. Here the interactions are described by Diagram 6.2. The beliefs of depth 2 are required.

Consider the following game:

	s_{21}	s_{22}	s_{23}
s_{11}	(5,5)	(1,2)	(4,3)
s_{11}	(6,1)	$(3,3)^*$	(3,1)

Table 6.4

	s_{31}	s_{32}
s_{11}	1	0
s_{21}	0	1

Table 6.5: 3's payoffs

In this game, the common knowledge of the decision making postulates between 1 and 2 is required, and 3 needs to know this common knowledge. In this game, players 1 and 2 need not care about 3's decision at all. Then the interactions are described as Diagram 6.3.

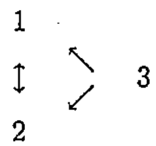

Diagram 6.3

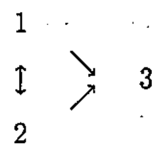

Diagram 6.4

The following is a 3-person game, where player 3's choice determines his payoff but affects the payoffs of 1 and 2. The roles of s_{11} and s_{12} (s_{21} and s_{22}) are permuted by 3's choice. In this game, players 1 and 2 need the common knowledge of 3's decision making. The interactive structure is described by Diagram Diagram 6.4.

s_{31}	1
s_{32}	0

Table 6.6

	s_{21}	s_{22}	s_{23}
s_{11}	(5,5)	(1,2)	(4,3)
s_{11}	(6,1)	$(3,3)^*$	(3,1)

Table 6.7: s_{31}

	s_{21}	s_{22}	s_{23}
s_{11}	$(3,3)^*$	(6,1)	(3,1)
s_{11}	(1,2)	(5,5)	(4,3)

Table 6.8: s_{32}

When we have more players, the possible interactive patterns are very many. The evolution of each player's thought may stop in some pattern.

Kaneko [6] gave a systematic (game theoretical) analysis of these possible patterns, and [7] considered these problems from the viewpoint of game logic.

References

[1] Halpern, J. H. and Y. Moses, (1992), A Guide to Completeness and Complexity for Modal Logics of Knowledge and Beliefs, *Artificial Intelligence* 54, 319-379.

[2] Honderich, T. (1995), Ed. *The Oxford Companion to Philosophy*, Oxford University Press. Oxford.

[3] Kaneko, M., (1996), Common Knowledge Logic and Game Logic, ISEP. DP. 694. To appear in *Journal of Symbolic Logic.*

[4] Kaneko, M., (1997a), Epistemic Considerations of Decision Making in Games, IPPS. DP. 724. To appear in *Mathematical Social Sciences.*

[5] Kaneko, M., (1997b), Mere and Specific Knowledge of the Existence of a Nash Equilibrium, IPPS. DP. 741.

[6] Kaneko, M., (1997c), Decision Making in Partially Interactive Games I: Game Theoretical Development, IPPS. DP. 743.

[7] Kaneko, M., (1998), Decision Making in Partially Interactive Games II: Game Logic Development. To be completed.

[8] Kaneko, M., (1998b), Depth of Knowledge and the Barcan Inferences in Game Logic. To be completed.

[9] Kaneko, M. and A. Matsui, (1997), Inductive Game Theory: Discrimination and Prejudices, IPPS. DP. 711, University of Tsukuba.

[10] Kaneko, M. and T. Nagashima, (1991), Final Decisions, Nash Equilibrium and Solvability in Games with the Common Knowledge of Logical Abilities, *Mathematical Social Sciences* 22, 229-255.

[11] Kaneko, M., and T. Nagashima, (1996), Game Logic and its Applications I. *Studia Logica* 57, 325-354.

[12] Kaneko, M. and T. Nagashima, (1997a), Game Logic and its Applications II. *Studia Logica* 58, 273–303.

[13] Kaneko, M., and T. Nagashima, (1997b), Axiomatic Indefinability of Common Knowledge in Finitary Logics, *Epistemic Logic and the Theory of Games and Decision,* eds. M. Bacharach, L. A. Gerard-Varet, P. Mongin and H. Shin, Kluwer Academic Press, 69–93.

[14] Kaneko, M. and N-Y. Suzuki, (1998), Game Semantics I: Completeness for Formulae of Finite Depths, to be completed.

[15] Lismont, L. and P. Mongin, (1994), On the Logic of Common Belief and Common Knowledge, *Theory and Decision 37,* 75–106.

[16] Meyer, J.-J. Ch., and W. van der Hoek, *Epistemic Logic for AI and Computer Science*, Cambridge University Press, (1995).

[17] Nash, J. F., (1951), Noncooperative Games, *Annals of Mathematics* 54, 286-295.

[18] Ohnishi, M., and K. Matsumoto, (1957), Gentzen Method in Modal Calculi I, *Osaka Math. J.* 9, 113-130.

[19] Plato, (1955), *The Republic*, translated by G. D. P. Lee, Penguin Books Inc, Baltimore.

[20] Terutoshi, Y. Okitsu K. and S. Enomoto (eds.), (1980), Collections of Meiji-Taisho comic stories (in Japanese), Vol. 3. Kodansha, Tokyo.

[21] Wald, A., (1937), Die Widerspruchsfreiheit des Kollektivbegriffes der Wahrscheinlichkeitsrechnung, *Ergebnisse eines mathematischen Kolloquiums* 8, 38–72.

[22] von Mises, R., (1931), Wahrscheinlichkeitsrechnuug, Vorlesungen aus dem Gebiet der angewandten Mathematik, Band I, Leipzig and Wien.