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by

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Abstract

This paper presents a discrete-time optimal stopping problem with a finite planning horizon. For a search cost s, a random offer, $w \sim F(w)$, will be found for each time. Offers appearing subsequently are allowed to be not only accepted or passed up but also "reserved" for recall later. To reserve an offer with value w, a reserving cost $r(w)$ is incurred with each reserved offer expiring k periods after. The objective is to maximize the expected discounted net profit with the provision that an offer must be accepted. The major finding is that an offer reserved during the search process must not be accepted prior to its maturity of reservation, however, it may be accepted on the maturity.

1 Introduction

Suppose that a piece of land must be disposed of by a certain date in the future (the deadline). It costs a certain amount of money, say advertising costs, to find a buyer, and offers made by buyers are considered to be random samples from a known distribution. Now, suppose that the asset holder has aimed at a selling price so high that all buyers appearing prior to the deadline have been rejected and the deadline has come. At this point, the land owner will have no choice but to sell to that buyer who appears on the deadline however low the price he offers. This situation is quite dire. However, if a too low selling price is accepted early in the time frame because of the fear of the risk, there is every chance that better buyers who may have appeared later would be missed. To avoid these two extremes, the owner of the land should have a rule to guide to the best action on a daily basis.

Such a problem can be well explained by using the so-called optimal stopping problem. This is a stochastic decision problem of determining an offer to accept among offers appearing sequentially and randomly. In terms of future availabilities of offers once passed up, almost all optimal stopping problems presented so far can be classified into three models: Models with recall (Björn [1] and Kang [4]), without recall (Björn [1] and Sakaguchi [9]), and with uncertain recall (Ikuta [2], and Karni and Schwartz [5]). In the first model, an offer once passed up is assumed to be available forever, in the second, it is instantly lost and unavailable forever, and in the third, it may become unavailable with a certain probability. In all of the models, actions the decision maker can take for each offer appearing are limited to accepting or rejecting it. As a result, whether an offer will be available in the future or not is determined to be out of the decision maker's reach.

However, what if the decision maker can keep offers available by reserving them for certain periods in exchange for paying some money? Rose [6] [7] and Saito [8] tried to answer this question. In Rose [7], each offer is allowed to be held for k periods in return for a cost bk where b is a given nonnegative number, where only one offer can be held at a time and it is prohibited to renew the reservation of an offer at the time of its maturity. Rose [6] deals with a similar model in which $k = 1$ and renewals are permitted. In these two papers, offers are estimated not by absolute values but by relative ranks, and the objectives are to maximize the probabilities to accept the best offer. In Saito [8], a reserving cost depends on the offer value and more than one reserved offer can be held at a time, where once an offer is reserved, it is available forever, that is, the reserving period is infinite.

In the current paper, although the reserving cost depends on the offer value, the reserving period is restricted to be a given finite number k, independent of the offer value. In the land selling problem, for example, the owner is assumed to be allowed to make agreements with each buyer to sell the land to him in the following k days at the price he offers. The major finding in this paper is that an offer reserved during the search process must not be accepted prior to its maturity of reservation, however, it may be accepted on the maturity.

The precise description of the model is provided in Section 2. In Section 3, three basic assumptions of the model are described. In Section 4, the model is formulated mathematically. Section 5 is only devoted to the analysis for clarifying the optimal decision rule, which is summarized in Section 6. In Section 7, some numerical examples are given. Section 8 treats the case without the third assumption stated in Section 3. Section 9 deals with the case that the planning horizon is infinite.

2 Model

Suppose a person periodically searches for offers and must accept one of them up to the deadline. He can find an offer at each time if a *search cost* $s > 0$ was paid at the previous time, however, he does not know in advance what offer will come out. The only information available is the distribution function $F(w)$ of offer value w where values of subsequent offers w, w', $\cdots$, are stochastically independent. For the most recently found offer, the so-called *current offer*, he is allowed not only to accept or pass up but also to reserve. Reserving an offer with value w, simply referred to as *offer* w later on, gives him a right to recall and accept it in the future, but this right is effective for only k periods and a *reserving cost* $r(w)$ is incurred. On the other hand, he can never return to any of offers passed up or expired.

His problem here is to find a rule to guide him to which action should be taken for each offer appearing so as to maximize the total expected discounted net profit obtainable in the process ahead, that is, the expectation of the present discounted value of an accepted offer minus that of the amount of search costs and reserving costs paid over the periods from the present point in time to the termination of the search by accepting an offer.

3 Assumptions and Preliminaries

We make the following three assumptions in this paper.

Assumption 1: The offer value distribution function F satisfies $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \le w < b$, and $F(w) = 1$ for $b \le w$ where a and b are such that $0 \le a < b < \infty$, and μ denotes the mean of $F(w)$.

Assumption 2: The reserving cost $r(w)$ is nondecreasing and continuous with $0 < r(w) < \infty$.

Assumption 3: $a < \alpha$ where $\alpha = \beta\mu - s$ and β is a per-period discount factor with $0 < \beta \le 1$. The case $\alpha \le a$ will be discussed in Section 8.

The following two functions and their properties are often used throughout the paper.

$$S(x) = \int_a^b \max\{w, x\} dF(w), \tag{3.1}$$

$$K(x) \;=\; \beta \int_a^b \max\{w, x\} dF(w) - x - s \;=\; \beta S(x) - x - s. \tag{3.2}$$

Let θ denote the root of equation $K(x) = 0$, whose existence is verified in the lemma below.

Lemma 3.1

(a) *$S(x)$ is continuous, convex, and nondecreasing in x and strictly increasing for $a \le x$.*
(b) *$S(x) = \mu$ for $x \le a$, $x < S(x)$ for $x < b$, and $S(x) = x$ for $b \le x$.*
(c) *$K(x)$ is continuous, convex, and nonincreasing in x and strictly decreasing for $x \le b$.*
(d) *θ exists uniquely with $\alpha \le \theta < b$. And, $\alpha < \theta$ if and only if $a < \alpha$.*
(e) *θ is continuous and strictly increasing in β.*

PROOF. See Ikuta [3]. ∎

In general, let $\hat{p}$ denote the maximum element of a vector $\boldsymbol{p}$, and $\boldsymbol{p}_i$ denote the vector defined by removing the i-th element p_i from $\boldsymbol{p}$, that is, if $\boldsymbol{p} = (p_1, p_2, \cdots, p_k) \in R^k$, then $\boldsymbol{p}_i = (p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_k) \in R^{k-1}$.

4 Optimal Equation

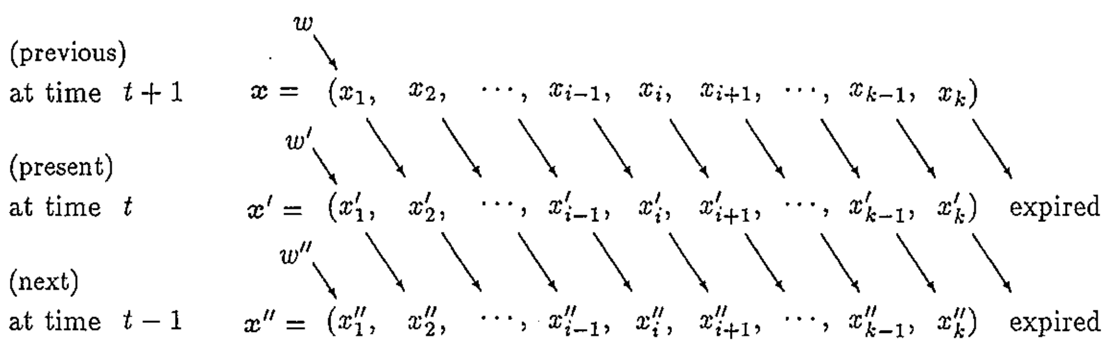
Let the time points t be taken equally spaced and numbered backward from the deadline $t = 0$, hence t also represents the number of periods remaining. Suppose that we are at time t and w_i is an offer found i periods ago, or at time $t + i$, and let

$$x_i = \begin{cases} w_i, & \text{if the offer } w_i \text{ was reserved,} \\ 0, & \text{if it was not reserved.} \end{cases} \tag{4.1}$$

Since each reserved offer is available for only k periods, the vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_k) \in R^k$ represents all available reserved offers where $x_i = 0$ can be regarded as a fictitious reserved offer and the best offer of $\boldsymbol{x}$ is $\hat{x}$, that is,

$$\hat{x} = \max\{x_1, x_2, \cdots, x_k\}. \tag{4.2}$$

We call the vector $\boldsymbol{x}$ the *reserved offer vector* and the offer $\hat{x}$ the *leading offer*. The relation among reserved offer vectors at the previous, present, and next time point can be depicted as in



(w, w', and w'' are offers found at times $t + 2$, $t + 1$, and t, respectively)

Figure 4.1: Reserved offer vector $\boldsymbol{x}$

Figure 4.1.

For convenience in later discussions, we define

$$y = x_k = (x_1, x_2, \cdots, x_{k-1}), \tag{4.3}$$

and use expressions $(w, y) = (w, x_1, \cdots, x_{k-1})$ and $(0, y) = (0, x_1, \cdots, x_{k-1})$.

For each time except for the deadline, we have four choices: Accepting the current offer w and stopping the search (AS), reserving w and continuing the search (RC), passing up w and stopping the search by accepting the leading offer $\hat{x}$ (PS), and passing up w and continuing the search (PC). Of course, at the deadline, only decisions AS or PS can be taken.

Then, letting $v_t(x)$ denote the maximum expected net profit attainable by starting from time t with a reserved offer vector x, we have

$$v_0(x) = \int_a^b \max\left\{\begin{array}{lcl} AS & : & w, \\ PS & : & \hat{x} \end{array}\right\} dF(w) = S(\hat{x}), \tag{4.4}$$

$$v_t(x) = \int_a^b \max\left\{\begin{array}{lcl} AS & : & w, \\ RC & : & -r(w) - s + \beta v_{t-1}(w, y), \\ PS & : & \hat{x}, \\ PC & : & -s + \beta v_{t-1}(0, y) \end{array}\right\} dF(w), \quad t \geq 1. \tag{4.5}$$

Let us define the two functions

$$z_t^o(x) = \max\{\hat{x}, -s + \beta v_{t-1}(0, y)\}, \qquad t \geq 1, \tag{4.6}$$

$$z_t^r(w, y) = \max\{w, -r(w) - s + \beta v_{t-1}(w, y)\}, \quad t \geq 1, \tag{4.7}$$

$z_0^o(x) = \hat{x}$, and $z_0^r(w, y) = w$. From Eqs. (4.4) and (4.5), we know that $z_t^o(x)$ and $z_t^r(w, y)$ stand for the expected net profits from deciding, respectively, to pass up the current offer w and not to pass it up at time t, provided that an optimal decision rule is applied after that. Then, $v_t(x)$, $t \geq 0$, can be rewritten as

$$v_t(x) = \int_a^b \max\{z_t^r(w, y), z_t^o(x)\} dF(w), \quad t \geq 0. \tag{4.8}$$

Let us denote the set of current offers not to be passed up, or either accepted or reserved, by

$$W_t(x) = \{w \mid z_t^o(x) \leq z_t^r(w, y)\}, \quad t \geq 0. \tag{4.9}$$

Additionally, we define

$$g_t^i(x_i | x_i) = -s + \beta v_{t-1}(0, y) - x_i, \qquad t \geq 1, \quad i \leq k, \tag{4.10}$$

$$f_t(w | y) = -r(w) - s + \beta v_{t-1}(w, y) - w, \qquad t \geq 1, \tag{4.11}$$

and let $\theta_t^i(x_i)$ and $\lambda_t(y)$ be the respective roots of $g_t^i(x_i | x_i) = 0$ and $f_t(w | y) = 0$, if any. Then, $\theta_t^i(x_i)$ represents the indifferent point in terms of x_i between accepting it and continuing the search under a given $x_i \in R^{k-1}$, and $\lambda_t(y)$ the indifferent point in terms of w between accepting it and reserving it under a given $y \in R^{k-1}$.

Therefore, the optimal decision rule is characterized by $\theta_t^i(x_i)$, $\lambda_t(y)$, and $W_t(x)$.

5 Analysis

In light of the context, without loss of generality, we can consider w and x_i to be in $(-\infty, b]$. Throughout the paper, we shall merely write w and x_i in the sense of $w \le b$ and $x_i \le b$, respectively. Hence, $\hat{x} \le b$ and $\hat{y} \le b$ everywhere.

Lemma 5.1

(a) $v_t(x)$, *thus* $z_t^o(x)$ *and* $z_t^r(w,y)$ *are continuous, convex, and nondecreasing in* x *and nondecreasing in* t.

(b) $\mu \le v_t(x)$ *for any* x, $\hat{x} < v_t(x)$ *for* $\hat{x} < b$, *and* $v_t(x) = b$ *for* $\hat{x} = b$.

(c) *Let* x^a *be such that* $x_i^a = x_i$ *for* $a \le x_i$ *and* $x_i^a = a$ *for* $x_i < a$. *Then* $v_t(x) = v_t(x^a)$.

(d) *If* $\mathbf{0} \le \mathbf{q}' \le \mathbf{q}$ *and* $v_t(\mathbf{p},\mathbf{q}) = v_t(\mathbf{p},\mathbf{0})$, *then* $v_t(\mathbf{p},\mathbf{q}) = v_t(\mathbf{p},\mathbf{q}')$.

(e) *For any given* n, *if* x^1 *and* x^2 *satisfy* $\hat{x}^1 = x_n^1 = x_n^2 = \hat{x}^2$ *and* $x_i^1 = x_i^2$ *for* $i < n$, *then* $v_t(x^1) = v_t(x^2)$.

(f) *Suppose* $t < k$ *and let* x^1 *and* x^2 *be such that* $\max\{x_i^1 \mid i \le k-t\} = \max\{x_i^2 \mid i \le k-t\}$ *and that* $x_i^1 = x_i^2$ *for* $k-t < i$. *Then* $v_t(x^1) = v_t(x^2)$.

PROOF. (a) Easily shown by induction starting with Eq. (4.4) and Lemma 3.1(a) as to x and with $v_1(x) \ge \int_a^b \max\{w, \hat{x}\} dF(w) = v_0(x)$ as to t.

(b) First, clearly $v_t(x) \ge \int_a^b w dF(w) = \mu$ for all t. Secondly, if $\hat{x} < b$, then $\hat{x} < S(\hat{x}) = v_0(x) \le v_1(x) \le \cdots$ from Lemma 3.1(b), Eq. (4.4), and assertion (a).

Finally, if $\hat{x} = b$, then $v_0(x) = b$ by Eq. (4.4) and Lemma 3.1(b). Assume the assertion to be true for $t-1$ and let $\mathbf{b} = (b, \cdots, b) \in R^{k-1}$. If $\hat{x} = b$, then $y \le \mathbf{b}$, thus $v_{t-1}(w, y) \le v_{t-1}(w, \mathbf{b}) = b$ for any $w \ge 0$ from (a) and the assumption. Hence, if $\hat{x} = b$, then $\max\{\cdots\}$ in Eq. (4.5) becomes equal to b, thereby, $v_t(x) = b$.

(c) Clearly $\hat{x} \le \hat{x}^a$ for any x. If $a \le x_i$ for at least one i, then $\hat{x} = \hat{x}^a$. Hence, by contraposition, if $\hat{x} < \hat{x}^a$, then $x_i < a$ for all i, thus $\hat{x}^a = a$.

For $t = 0$, the assertion is clear from Eq. (4.4) and Lemma 3.1(b). Assume the assertion to be true for $t-1$. Then, for any $w \ge a$, since $(w, y)^a = (w, y^a)$, we have $v_{t-1}(w, y) = v_{t-1}(w, y^a)$ by the assumption, thus $z_t^r(w, y) = z_t^r(w, y^a)$ for any $w \ge a$ by Eq. (4.7). Since $v_{t-1}(0, y) \le v_{t-1}(0, y^a) \le v_{t-1}(a, y^a)$ due to (a), and $v_{t-1}(0, y) = v_{t-1}(a, y^a)$ by the assumption, we get $v_{t-1}(0, y) = v_{t-1}(0, y^a)$. Hence, if $\hat{x} = \hat{x}^a$, then $z_t^o(x) = z_t^o(x^a)$ by Eq. (4.6). Even if $\hat{x} < \hat{x}^a$, we also get $z_t^o(x) = z_t^o(x^a)$ since $\hat{x} < \hat{x}^a = a < \alpha = -s + \beta\mu \le -s + \beta v_{t-1}(0, y) = -s + \beta v_{t-1}(0, y^a)$ by Assumption 3 and (b). Consequently, from Eq. (4.8), we get $v_t(x) = v_t(x^a)$.

(d) If $\mathbf{0} \le \mathbf{q}' \le \mathbf{q}$, then $v_t(\mathbf{p}, \mathbf{0}) \le v_t(\mathbf{p}, \mathbf{q}') \le v_t(\mathbf{p}, \mathbf{q})$ due to (a). Hence, if $v_t(\mathbf{p}, \mathbf{q}) = v_t(\mathbf{p}, \mathbf{0})$, then $v_t(\mathbf{p}, \mathbf{q}) = v_t(\mathbf{p}, \mathbf{q}')$.

(e) Since $\hat{x}^1 = \hat{x}^2$, the assertion proves true for $t = 0$ from Eq. (4.4).

Assume the assertion to be true for $t-1$. Choose x^1 and x^2 satisfying the condition of the assertion with $n < k$ and let $p^1 = (w, y^1)$ and $p^2 = (w, y^2)$ with any $w \ge 0$. Then, $p_1^1 = w = p_1^2$ and $p_i^1 = x_{i-1}^1 = x_{i-1}^2 = p_i^2$ for $2 \le i \le n+1$. Hence, if $w \le x_n^1 = x_n^2$, then $\hat{p}^1 = x_n^1 = x_n^2 = \hat{p}^2$, or else $\hat{p}^1 = w = \hat{p}^2$. Accordingly, since p^1 and p^2 satisfy the condition with $n+1$ or 1, we get $v_{t-1}(w, y^1) = v_{t-1}(w, y^2)$ for any $w \ge 0$ by the assumption. Hence, since $\hat{x}^1 = \hat{x}^2$, we get $v_t(x^1) = v_t(x^2)$ by Eq. (4.5). If $n = k$, since $x^1 = x^2$, the assertion holds true.

(f) Easily shown by noting $\hat{x}^1 = \hat{x}^2$ and applying an argument similar to that in the proof of (e). ∎

Lemma 5.2

(a) $g_t^i(x_i|x_i)$ *is strictly decreasing in* x_i *for each* i *and any* x_i, *and* $f_t(w|y)$ *is strictly decreasing in* w *for any* y.

(b) $\theta_t^i(x_i)$ *exists uniquely with* $\theta_t^i(x_i) \in [\alpha, b)$ *for each* i *and any* x_i, *and so also does* $\lambda_t(y)$ *with* $\lambda_t(y) \in [\alpha - r(b), b)$ *for any* y.

(c) $W_t(x) \neq \phi$ *and* $W_t(x)^c \neq \phi$ *for any* x.

PROOF. (a) We first show the assertion as to $g_t^i(x_i|x_i)$.

If $i = k$, since y is independent of x_k, so also is $v_{t-1}(0, y)$, thus the assertion proves true.

Fix any $i < k$. If $\hat{y}_i = b$, then $\hat{y} = b$ for any x_i, thus $v_{t-1}(0, y) = b$ for any x_i because of Lemma 5.1(b), hence the assertion holds. For $\hat{y}_i < b$, choose x^1, x^2, and x^b with $x_i^1 < x_i^2 < x_i^b = b$ and $x_j^1 = x_j^2 = x_j^b < b$ for $j \neq i, k$. Then, $\hat{y}^1 \leq \hat{y}^2 < \hat{y}^b = b$, and by Lemma 5.1(a,b), $v_{t-1}(0, y)$ is convex in x_i, $v_{t-1}(0, y^b) = b$, and $x_i^1 \leq \hat{y}^1 < v_{t-1}(0, y^1)$. Hence,

$$\beta \frac{v_{t-1}(0, y^2) - v_{t-1}(0, y^1)}{x_i^2 - x_i^1} \leq \beta \frac{v_{t-1}(0, y^b) - v_{t-1}(0, y^1)}{b - x_i^1} < \beta \frac{b - x_i^1}{b - x_i^1} \leq 1. \tag{5.1}$$

Consequently, $\beta v_{t-1}(0, y^1) - x_i^1 > \beta v_{t-1}(0, y^2) - x_i^2$, which also leads us to the assertion.

By noting Assumption 2, the assertion as to $f_t(w|y)$ can be verified in a like manner.

(b) Since $g_t^i(\alpha|x_i) = \beta\,(v_{t-1}(0, y) - \mu) \geq 0$ and $g_t^i(b|x_i) \leq (\beta - 1)b - s < 0$ by Lemma 5.1(b), we conclude that $g_t^i(x_i|x_i) = 0$ has a unique root $\theta_t^i(x_i) \in [\alpha, b)$ due to (a).

By noting Assumption 2, we obtain the assertion as to $\lambda_t(y)$ in the same way.

(c) Since $v_{t-1}(0, y) \leq v_{t-1}(b, y) = b$ due to Lemma 5.1(a,b), we get $z_t^o(x) \leq \max\{b, -s + \beta b\} = b = \max\{b, -r(b) - s + \beta b\} = z_t^r(b, y)$, or $b \in W_t(x)$. Hence, $W_t(x) \neq \phi$.

By Lemma 5.1(a,c), we get $v_{t-1}(0, y) \leq v_{t-1}(a, y) \leq v_{t-1}(a, y^a)$ and $v_{t-1}(0, y) = v_{t-1}(a, y^a)$, thus $v_{t-1}(a, y) = v_{t-1}(0, y)$, implying $-r(a) - s + \beta v_{t-1}(a, y) < -s + \beta v_{t-1}(0, y)$. Hence, since $a < -s + \beta\mu \leq -s + \beta v_{t-1}(0, y)$ by Assumption 3 and Lemma 5.1(b), we get $z_t^r(a, y) < -s + \beta v_{t-1}(0, y) \leq z_t^o(x)$, or $a \notin W_t(x)$. Therefore, $W_t(x)^c \neq \phi$. ∎

Remark 5.3 We have $a < \alpha \leq \theta_t^i(x_i)$ for every t from Assumption 3 and Lemma 5.2(b), while $\lambda_t(y) < a$ occurs since, if $r(a) > \beta b - s - a$, then $f_t(a|y) \leq -r(a) - s + \beta b - a < 0$.

Corollary 5.4 *For each* i *and any* x_i, *thus for any* y,

$$\text{(a)}\quad x_i \begin{cases} < -s + \beta v_{t-1}(0, y), & \textit{if } x_i < \theta_t^i(x_i), \\ = -s + \beta v_{t-1}(0, y), & \textit{if } x_i = \theta_t^i(x_i), \\ > -s + \beta v_{t-1}(0, y), & \textit{if } \theta_t^i(x_i) < x_i. \end{cases}$$

$$\text{(b)}\quad w \begin{cases} < -r(w) - s + \beta v_{t-1}(w, y), & \textit{if } w < \lambda_t(y), \\ = -r(w) - s + \beta v_{t-1}(w, y), & \textit{if } w = \lambda_t(y), \\ > -r(w) - s + \beta v_{t-1}(w, y), & \textit{if } \lambda_t(y) < w. \end{cases}$$

PROOF. Clear from Lemma 5.2(a). ∎

Theorem 5.5

(a) *For each* $i < k$, *if* x_i *is such that* $\hat{y}_i \leq \theta$, *then* $\theta_t^i(x_i) = \theta$.

(b) *If* y *is such that* $\hat{y} \leq \theta$, *then* $\hat{y} \leq \theta_t^k(y) = -s + \beta v_{t-1}(0, y) \leq \theta$.

(c) *If y is such that $\hat{y} \leq \theta$, then $\lambda_t(y) < \theta$.*
(d) *$\theta_t^k(y)$ and $\lambda_t(y)$ are continuous and nondecreasing in y and nondecreasing in t.*

PROOF. Assertions (a-c) are shown together by induction. First, from Eq. (4.4) we have $v_0(x) = S(\theta)$ for any x with $\hat{x} = \theta$. Assume $v_{t-1}(x) = S(\theta)$ for any x with $\hat{x} = \theta$.

(a) Choose any $i < k$ and any x_i with $\hat{y}_i \leq \theta$. If $x_i = \theta$, then $\hat{y} = \theta$, so $v_{t-1}(0, y) = S(\theta)$ by the assumption, thus $g_t^i(\theta|x_i) = K(\theta) = 0$. Hence, $\theta_t^i(x_i) = \theta$ by the uniqueness of $\theta_t^i(x_i)$.

(b) Since $v_{t-1}(0, y)$ is independent of x_k, we get $\theta_t^k(y) = -s + \beta v_{t-1}(0, y)$. Hence, if $\hat{y} \leq \theta$, it follows that $\theta_t^k(y) \geq -s + \beta v_0(0, y) = K(\hat{y}) + \hat{y} \geq \hat{y}$ by Lemmas 5.1(a) and 3.1(c), furthermore, $v_{t-1}(0, y) \leq v_{t-1}(0, \theta, \cdots, \theta) = S(\theta)$ by Lemma 5.1(a) and the assumption, from which $\theta_t^k(y) \leq -s + \beta S(\theta) = K(\theta) + \theta = \theta$.

(c) Since $v_{t-1}(\theta, y) = S(\theta)$ for any y with $\hat{y} \leq \theta$ by the assumption, we get $f_t(\theta|y) = -r(\theta) + K(\theta) = -r(\theta) < 0$. Hence, from Lemma 5.2(a) we claim $\lambda_t(y) < \theta$.

(a-c) To complete the proofs we show $v_t(x) = S(\theta)$ for any x with $\hat{x} = \theta$.

Since $\hat{y} \leq \theta$, by (b) we get $-s + \beta v_{t-1}(0, y) \leq \theta$, thus $z_t^o(x) = \hat{x} = \theta$.

Since $\hat{y} \leq \theta$, if $w < \theta$, then $v_{t-1}(w, y) \leq v_{t-1}(\theta, y) = S(\theta)$ by Lemma 5.1(a) and the assumption, so $-r(w) - s + \beta v_{t-1}(w, y) < -s + \beta S(\theta) = K(\theta) + \theta = \theta$, thus $z_t^r(w, y) < \max\{\theta, \theta\} = \theta$. If $\theta \leq w$, then $\lambda_t(y) < \theta \leq w$ by (c), thus $z_t^r(w, y) = w$ by Corollary 5.4(b). Thereby, from Eq. (4.8), we conclude that, if $\hat{x} = \theta$,

$$v_t(x) = \int_a^b \max\{z_t^r(w, y), \theta\} dF(w) = \int_a^\theta \theta dF(w) + \int_\theta^b w dF(w) = S(\theta). \tag{5.2}$$

(d) By Lemma 5.1(a), both $g_t^k(x_k|y)$ and $f_t(w|y)$ are continuous and nondecreasing in y. From this and Lemma 5.2(a), the assertion as to y holds. Similarly, we get the assertion as to t. ∎

Lemma 5.6

(a) *If $\theta \leq \hat{x}$, then $z_t^o(x) = \hat{x}$, $z_t^r(w, y) < \hat{x}$ for $w < \hat{x}$, $z_t^r(w, y) = w$ for $\hat{x} \leq w$, and $v_t(x) = S(\hat{x})$.*
(b) *If $\theta < \hat{x}$, then $v_t(x') < v_t(x)$ for any x' such that $\hat{x}' < \hat{x}$.*

PROOF. (a) All of them are obvious for $t = 0$ by their definitions. Assume $v_{t-1}(x) = S(\hat{x})$ for any x with $\theta \leq \hat{x}$.

Suppose $\hat{y} \leq \theta$. If $\theta \leq \hat{x}$, then $z_t^o(x) = \max\{\hat{x}, \theta_t^k(y)\} = \hat{x}$ by Theorem 5.5(b). In the proof of Theorem 5.5(a-c) we have $z_t^r(w, y) < \theta$ for $w < \theta$ and $z_t^r(w, y) = w$ for $\theta \leq w$, from which, if $\theta \leq \hat{x}$, then $z_t^r(w, y) < \hat{x}$ for $w < \hat{x}$ and $z_t^r(w, y) = w$ for $\hat{x} \leq w$ by considering three cases: $w < \theta$, $\theta \leq w < \hat{x}$, and $\hat{x} \leq w$.

Suppose $\theta < \hat{y}$. Then, for any w, we have $\theta < \max\{w, \hat{y}\}$, thus $v_{t-1}(w, y) = S(\max\{w, \hat{y}\})$ by the assumption, from which and Lemma 3.1(c),

$$-s + \beta v_{t-1}(w, y) - \max\{w, \hat{y}\} = K(\max\{w, \hat{y}\}) < 0. \tag{5.3}$$

Setting $w = 0$ in Eq. (5.3), we get $-s + \beta v_{t-1}(0, y) < \hat{y} \leq \hat{x}$, thus $z_t^o(x) = \hat{x}$. Eq. (5.3) yields $-r(w) - s + \beta v_{t-1}(w, y) < \max\{w, \hat{y}\} \leq \max\{w, \hat{x}\}$, hence $z_t^r(w, y) < \max\{\hat{x}, \hat{x}\} = \hat{x}$ for $w < \hat{x}$, and $z_t^r(w, y) = w$ for $\hat{x} \leq w$.

Finally, we complete the proof by showing, for any x with $\theta \leq \hat{x}$,

$$v_t(x) = \int_a^b \max\{z_t^r(w,y), \hat{x}\}dF(w) = \int_a^{\hat{x}} \hat{x}dF(w) + \int_{\hat{x}}^b wdF(w) = S(\hat{x}). \tag{5.4}$$

(b) Suppose $\theta < \hat{x}$. Let x' be such that $\hat{x}' < \hat{x}$ and choose x'' such that $\max\{\theta, \hat{x}'\} < \hat{x}'' < \hat{x}$. Since $a < \alpha \leq \theta$ by Assumption 3 and Lemma 3.1(d), we get $a < \theta < \hat{x}'' < \hat{x}$, thus $v_t(x'') = S(\hat{x}'') < S(\hat{x}) = v_t(x)$ by (a) and Lemma 3.1(a). Hence, if $v_t(x) \leq v_t(x')$, then $v_t(x'') < v_t(x')$, contradicting Lemma 5.1(a). So, the assertion proves true. ∎

Corollary 5.7 *Let $\theta \leq \hat{x}$. If $w < \hat{x}$, then* $\max\{\cdots\}$ *in* Eq. (4.8) *becomes $\hat{x}$, or else w.*

PROOF. From Lemma 5.6(a), if $w < \hat{x}$, then $z_t^r(w,y) < z_t^o(x) = \hat{x}$, or else $z_t^o(x) \leq z_t^r(w,y) = w$. ∎

Here, define

$$\begin{aligned} x_i^L &= (x_1, \cdots, x_i, 0, \cdots, 0) \in R^k, \\ x_i^R &= (0, \cdots, 0, x_{i+1}, \cdots, x_k) \in R^k, \\ X_t^i(x_i) &= \{x_i \mid v_t(x) = v_t(x_i^R)\} \end{aligned}$$

where clearly $\hat{x} = \max\{\hat{x}_i^L,\ \hat{x}_i^R\}$ for $i \leq k$, and $\hat{y} = \max\{\hat{y}_i^L,\ \hat{y}_i^R\}$ for $i < k$.

Lemma 5.8 *For any given i, if $x_i \notin X_t^i(x_i)$, then $v_t(x) = v_t(x_i^L)$.*

PROOF. Suppose $\hat{x} > \theta$. If $\hat{x} = \hat{x}_i^R$, then $v_t(x) = S(\hat{x}) = S(\hat{x}_i^R) = v_t(x_i^R)$ by Lemma 5.6(a). If $\hat{x} > \hat{x}_i^R$, then $v_t(x) > v_t(x_i^R)$ by Lemma 5.6(b). Therefore, since $v_t(x) = v_t(x_i^R)$ if and only if $\hat{x} = \hat{x}_i^R$, we get $X_t^i(x_i) = \{x_i \mid \hat{x} = \hat{x}_i^R\}$. Hence, if $x_i \notin X_t^i(x_i)$, then $\hat{x} > \hat{x}_i^R$, so $\hat{x} = \hat{x}_i^L > \theta$, thus $v_t(x) = S(\hat{x}) = S(\hat{x}_i^L) = v_t(x_i^L)$.

Let $\theta \geq \hat{x}$ below. If $\hat{x} = \hat{x}_i^R$, then $v_0(x) = v_0(x_i^R)$. Suppose $\hat{x} > \hat{x}_i^R$. If $a \geq \hat{x}$, then $v_0(x) = v_0(x_i^R)$ $(= \mu)$ by Lemma 3.1(b), or else $v_0(x) > v_0(x_i^R)$ by Lemma 3.1(a). Hence, $X_0^i(x_i) = \{x_i \mid \hat{x} = \hat{x}_i^R \text{ or } a \geq \hat{x}\}$. If $x_i \notin X_0^i(x_i)$, then $\hat{x} > \hat{x}_i^R$, thus $\hat{x} = \hat{x}_i^L$, from which $v_0(x) = v_0(x_i^L)$. Consequently, the assertion holds true for $t = 0$.

Assume the assertion to be true for $t-1$. Note that the assumption with $i = 1$ is that, if $x_1 \notin X_{t-1}^1(x_1) = \{x_1 \mid v_{t-1}(x_1, x_1) = v_{t-1}(0, x_1)\}$, then $v_{t-1}(x_1, x_1) = v_{t-1}(x_1, 0)$, equivalently,

$$w \notin X_{t-1}^1(y) = \{w \mid v_{t-1}(w,y) = v_{t-1}(0,y)\} \implies v_{t-1}(w,y) = v_{t-1}(w,0). \tag{5.5}$$

To begin with, we let x and x' be such that $x \geq x'$, so $y \geq y'$, and show

$$v_t(x) = v_t(x') \iff z_t^o(x) = z_t^o(x'). \tag{5.6}$$

If $\hat{y} \leq w$, then $v_{t-1}(w,y) = v_{t-1}(w,y')$ by Lemma 5.1(e) with $n = 1$. If $w \notin X_{t-1}^1(y)$, then $v_{t-1}(w,y) = v_{t-1}(w,0)$ by Eq. (5.5), from which and Lemma 5.1(d) we also get $v_{t-1}(w,y) = v_{t-1}(w,y')$. Hence, if $\hat{y} \leq w$ or $w \notin X_{t-1}^1(y)$, then $z_t^r(w,y) = z_t^r(w,y')$. Conversely, if $w < \hat{y}$ and $w \in X_{t-1}^1(y)$, then $w < \hat{x}$ and $v_{t-1}(w,y) = v_{t-1}(0,y)$ by Eq. (5.5), thus $z_t^r(w,y) < \max\{\hat{x}, -s + \beta v_{t-1}(0,y)\} = z_t^o(x)$. Thereby, any w satisfies $z_t^r(w,y) = z_t^r(w,y')$ or $z_t^r(w,y) < z_t^o(x)$. From this and Lemma 5.1(a),

$$z_t^r(w,y) \neq z_t^r(w,y') \implies z_t^r(w,y') < z_t^r(w,y) < z_t^o(x). \tag{5.7}$$

Suppose $z_t^o(x) = z_t^o(x')$. Then, $\max\{z_t^r(w,y), z_t^o(x)\} = \max\{z_t^r(w,y'), z_t^o(x')\}$ for any w from Eq. (5.7), thus $v_t(x) = v_t(x')$ by Eq. (4.8). Conversely, suppose $z_t^o(x) > z_t^o(x')$. Since any $w \notin W_t(x)$ satisfies $z_t^o(x) > z_t^r(w,y) \geq z_t^r(w,y')$ by Eq. (4.9) and Lemma 5.1(a), we get $\max\{z_t^r(w,y), z_t^o(x)\} = z_t^o(x) > \max\{z_t^r(w,y'), z_t^o(x')\}$ for any $w \notin W_t(x)$, thus $v_t(x) > v_t(x')$ by Eq. (4.8). Therefore, we have confirmed Eq. (5.6).

Next, we shall prove the case $i < k$ by (i) showing

$$X_t^i(x_i) = \{x_i \mid \theta_t^k(y) \leq x_k \text{ or } x_i \in X_{t-1}^{i+1}(0, y_i)\}, \quad i < k, \tag{5.8}$$

and then (ii) verifying $v_t(x) = v_t(x_i^L)$ for $x_i \notin X_t^i(x_i)$.

(i) Since having supposed $\hat{x} \leq \theta$, so $\hat{y} = \max\{\hat{y}_i^L, \hat{y}_i^R\} \leq \theta$, from Theorem 5.5(b),

$$z_t^o(x) = \max\{\hat{x},\ \theta_t^k(y)\} = \max\{\hat{y},\ x_k,\ \theta_t^k(y)\} = \max\{x_k,\ \theta_t^k(y)\}, \tag{5.9}$$

$$z_t^o(x_i^L) = \max\{\hat{x}_i^L,\ \theta_t^k(y_i^L)\} = \max\{\hat{y}_i^L,\ 0,\ \theta_t^k(y_i^L)\} = \theta_t^k(y_i^L), \tag{5.10}$$

$$z_t^o(x_i^R) = \max\{\hat{x}_i^R,\ \theta_t^k(y_i^R)\} = \max\{\hat{y}_i^R,\ x_k,\ \theta_t^k(y_i^R)\} = \max\{x_k,\ \theta_t^k(y_i^R)\}. \tag{5.11}$$

It follows from the assumption that, for each $i < k$,

$$x_i \notin X_{t-1}^{i+1}(0, y_i) = \{x_i \mid v_{t-1}(0, y) = v_{t-1}(0, y_i^R)\} \implies v_{t-1}(0, y) = v_{t-1}(0, y_i^L). \tag{5.12}$$

If $\theta_t^k(y) \leq x_k$, then $\theta_t^k(y_i^R) \leq \theta_t^k(y) \leq x_k$ from Theorem 5.5(d). If $x_i \in X_{t-1}^{i+1}(0, y_i)$, then $v_{t-1}(0,y) = v_{t-1}(0, y_i^R)$ by Eq. (5.12), thus $\theta_t^k(y) = \theta_t^k(y_i^R)$ by Theorem 5.5(b). Hence, if $\theta_t^k(y) \leq x_k$ or $x_i \in X_{t-1}^{i+1}(0, y_i)$, then $z_t^o(x) = z_t^o(x_i^R)$ by Eqs. (5.9) and (5.11).

If $x_i \notin X_{t-1}^{i+1}(0, y_i)$, then $v_{t-1}(0,y) > v_{t-1}(0, y_i^R)$ by Eq. (5.12), thus $\theta_t^k(y) > \theta_t^k(y_i^R)$. Hence, if $\theta_t^k(y) > x_k$ and $x_i \notin X_{t-1}^{i+1}(0, y_i)$, then $z_t^o(x) = \theta_t^k(y) > z_t^o(x_i^R)$ by Eqs. (5.9) and (5.11).

Hence, $z_t^o(x) = z_t^o(x_i^R)$ if and only if $\theta^k(y) \leq x_k$ or $x_i \in X_{t-1}^{i+1}(0, y_i)$. Due to this fact and Eq. (5.6), we have confirmed Eq. (5.8).

(ii) If $x_i \notin X_{t-1}^{i+1}(0, y_i)$, then $v_{t-1}(0,y) = v_{t-1}(0, y_i^L)$ by Eq. (5.12), thus $\theta_t^k(y) = \theta_t^k(y_i^L)$. Hence, if $x_i \notin X_t^i(x_i)$, that is, if $x_k < \theta_t^k(y)$ and $x_i \notin X_{t-1}^{i+1}(0, y_i)$, then $z_t^o(x) = z_t^o(x_i^L)$ by Eqs. (5.9) and (5.10). This and Eq. (5.6) complete the proof for $i < k$.

Finally, if $i = k$, since $x = x_k^L$, we claim the assertion without considering $X_t^k(y)$. ∎

Theorem 5.9

(a) *If $x^1 \leq x^2$, then $W_t(x^1) \supseteq W_t(x^2)$.*

(b) *If $t < k$ and $w < \max\{x_i \mid i \leq k - t\}$, then $w \notin W_t(x)$.*

Proof. (a) Choose w such that $w \in W_t(x^2)$ and $w < \lambda_t(y^2)$. Then, by Eqs. (4.6) and (4.9) and Corollary 5.4(b), we get $-s + \beta v_{t-1}(0, y^2) \leq z_t^o(x^2) \leq z_t^r(w, y^2) = -r(w) - s + \beta v_{t-1}(w, y^2)$, from which $0 < r(w) \leq \beta\left(v_{t-1}(w, y^2) - v_{t-1}(0, y^2)\right)$, implying $v_{t-1}(w, y^2) > v_{t-1}(0, y^2)$. So, by Eq. (5.5), we know $w \notin X_{t-1}^1(y^2)$, thus $v_{t-1}(w, y^2) = v_{t-1}(w, 0)$, from which $v_{t-1}(w, y^2) = v_{t-1}(w, y^1)$ by Lemma 5.1(d). Hence, $z_t^r(w, y^2) = z_t^r(w, y^1)$.

Whether $w \in W_t(x^2)$ or not, if $\lambda_t(y^2) \leq w$, since $\lambda_t(y^1) \leq \lambda_t(y^2)$ due to Theorem 5.5(d), we get $z_t^r(w, y^1) = z_t^r(w, y^2)$ $(= w)$ from Corollary 5.4(b).

From Lemma 5.1(a), Eq. (4.9), and the above facts, if $w \in W_t(x^2)$, then $z_t^o(x^1) \leq z_t^o(x^2) \leq z_t^r(w, y^2) = z_t^r(w, y^1)$, yielding $w \in W_t(x^1)$. This proves the assertion.

(b) If $t = 0$, then $\max\{x_i \mid i \leq k - 0\} = \hat{x}$ and $W_0(x) = \{w \mid \hat{x} \leq w\}$ by Eq. (4.9). Hence, if $w < \max\{x_i \mid i \leq k - 0\}$, then $w \notin W_0(x)$. So, the assertion holds for $t = 0$.

Suppose $0 < t < k$ and $w < \max\{x_i \mid i \leq k-t\}$, thus $w < \hat{x}$, and let $p^1 = (w, y)$ and $p^2 = (0, y)$. Then,

$$\begin{aligned}\max\{p_i^1 \mid i \leq k-(t-1)\} &= \max\{w, x_1, \cdots, x_{k-t}\} \\ &= \max\{0, x_1, \cdots, x_{k-t}\} = \max\{p_i^2 \mid i \leq k-(t-1)\}\end{aligned}$$

and $p_i^1 = x_{i-1} = p_i^2$ for $k-(t-1) < i$. So, p^1 and p^2 satisfy the condition of Lemma 5.1(f) with $t-1$, thus $v_{t-1}(w, y) = v_{t-1}(0, y)$. Hence, $z_t^r(w, y) < \max\{\hat{x}, -r(w) - s + \beta v_{t-1}(0, y)\} \leq z_t^o(x)$, or $w \notin W_t(x)$. ∎

Lemma 5.10

(a) *If $\hat{y} \leq \lambda_t(0)$, then $\lambda_t(y) = \lambda_t(0)$, and if $\lambda_t(0) < \hat{y}$, then $\lambda_t(y) \notin W_t(x)$.*

(b) *If $w \in W_t(x)$, then either $w \leq \lambda_t(0)$ or $\lambda_t(y) < w$.*

Proof. (a) If $\hat{y} \leq \lambda_t(0)$, it follows from Lemma 5.1(e) with $n = 1$ that $v_{t-1}(\lambda_t(0), y) = v_{t-1}(\lambda_t(0), 0)$, implying $f_t(\lambda_t(0)|y) = f_t(\lambda_t(0)|0) = 0$, thus $\lambda_t(y) = \lambda_t(0)$. The latter part is proven by contraposition. Suppose $\lambda_t(y) \in W_t(x)$. Then, $\hat{y} \leq \hat{x} \leq z_t^o(x) \leq z_t^r(\lambda_t(y), y) = \lambda_t(y)$, yielding $\lambda_t(y) = \lambda_t(0)$ in exactly the same way as above. Hence, if $\lambda_t(y) \in W_t(x)$, then $\hat{y} \leq \lambda_t(y) = \lambda_t(0)$.

(b) Since either $w \leq \lambda_t(0)$ or $\lambda_t(0) < w$ for any w, the assertion holds for $\hat{y} \leq \lambda_t(0)$ from (a). The proof for $\lambda_t(0) < \hat{y}$ is by contradiction. Choose x^2 with $\lambda_t(0) < \hat{y}^2$ and suppose that a certain $w \in W_t(x^2)$ satisfies $\lambda_t(0) < w \leq \lambda_t(y^2)$. Then, by Theorem 5.5(d) and the intermediate value theorem, there is an x^1 such that $x^1 \leq x^2$ and $w = \lambda_t(y^1)$. Since $\lambda_t(0) < w = \lambda_t(y^1)$, we get $\lambda_t(0) < \hat{y}^1$ by the contraposition of the former part of (a), thus $w = \lambda_t(y^1) \notin W_t(x^1)$ by the latter part of (a). Hence, if there is a $w \in W_t(x^2)$ such that $\lambda_t(0) < w \leq \lambda_t(y^2)$, then $w \notin W_t(x^1)$ for a certain x^1 with $x^1 \leq x^2$, which contradicts Theorem 5.9(a). This complete the proof. ∎

Corollary 5.11 *For any $w \in W_t(x)$, we have $w \leq \lambda_t(y)$ if and only if $w \leq \lambda_t(0)$.*

Proof. Any $w \in W_t(x)$ such that $w \leq \lambda_t(y)$ satisfies $w \leq \lambda_t(0)$ since $\lambda_t(0) < w \leq \lambda_t(y)$ is impossible for $w \in W_t(x)$ from Lemma 5.10(b), and any $w \in W_t(x)$ such that $\lambda_t(y) < w$ satisfies $\lambda_t(0) < w$ since $\lambda_t(0) \leq \lambda_t(y)$ by Theorem 5.5(d). ∎

6 Optimal Decision Rule

From Corollary 5.4, the optimal decision rule can be generally prescribed as follows:

Optimal Decision Rule: Suppose you are at time t with a reserved offer vector x and have just drawn an offer w. The choices are:

(a) If $w \in W_t(x)$ and $\lambda_t(y) < w$, accept the current offer w (AS).

(b) If $w \in W_t(x)$ and $w \leq \lambda_t(y)$, reserve the current offer w (RC).

(c) If $w \notin W_t(x)$, pass up the current offer w, and then:

- (i) If $\theta_t^i(x_i) < x_i = \hat{x}$ for a certain i, accept the leading offer x_i (PS).
- (ii) If $x_i \leq \theta_t^i(x_i)$ for all i, continue the search (PC).

In this section, we shall reveal properties of the above optimal decision rule.

1. *For each time t with any x, an offer $w \in W_t(x)$ must be reserved if and only if $w \leq \lambda_t(0)$.*

The result is immediate from Corollary 5.11. Note that, whatever x we have, we only have to compare an offer w with $\lambda_t(0)$ to decide whether to accept or reserve it.

Besides, since every offer w satisfies $a \leq w$, the case $\lambda_t(0) < a$ stated in Remark 5.3 only implies that no offers should be reserved.

2. *If you have a reserved offer vector x such that $\theta \leq \hat{x}$ and have just drawn an offer w, accept the more lucrative between the leading offer $\hat{x}$ and the current offer w.*

The result is the restatement of Corollary 5.7.

As seen in the next result, however, no reserved offer vector x ever satisfies $\theta \leq \hat{x}$ except for the case that such x is given as an initial proposal before entering the search process.

3. *An offer reserved during the search process must not be accepted prior to its maturity of reservation, however, it may be accepted on the maturity.*

From Assumption 1, 3, and Lemma 3.1(d), we have $0 \leq a < \alpha \leq \theta$, thus $\hat{0} < \theta$, yielding $\lambda_t(0) < \theta$ for every t by Theorem 5.5(c). Consequently from Result 1, all offers to be reserved throughout the search process have less value than θ. Hence, if the search starts with a reserved offer vector x such that $\hat{x} < \theta$, the inequality holds forever, or $x_i < \theta$ for all i for every t. So, $\hat{y} < \theta$ and $\hat{y}_i < \theta$ for all $i < k$ for every t. In this case, $\theta_t^i(x_i) = \theta$ for $i < k$ by Theorem 5.5(a), thus $\theta_t^i(x_i) < x_i$ never happens for $i < k$. However, $\theta_t^k(y) < x_k$ is possible. In fact, for example, in the case that $k = 2$ (so, $y = (x_1)$ and $x_k = x_2$), $s = 1/10$, $\beta = 9/10$, $r(w) = w/1000$, and $F(w) = 0$ for $w < 1/4$, $1/4$ for $1/4 \leq w < 2/4$, $2/4$ for $2/4 \leq w < 3/4$, $3/4$ for $3/4 \leq w < 1$, and 1 for $1 \leq w$, we can have, without any computational error, $(x_1, x_2) = (0, 1/2)$ at $t = 1$ and then $\theta_1^2(x_1) = \theta_1^2(0) = 74/160 < 1/2 = x_2$.

The above facts suggest that no reserved offer x_i must be accepted if it is still available at the next time, or $i < k$, and that only the offer x_k, which is at maturity, has the chance to be accepted. This can be interpreted as follows: Since an offer once reserved is assumed not to deteriorate in its value over the reserving period, it seems a waste to accept an offer while some reserving periods still remain.

This is one of points different from the results obtained in Saito [8] which dissuades us from accepting any reserved offer prior to the deadline. However, since the reserving period in Saito [8] is assumed to be infinite, any reserved offer available at a certain time t is still available at the next time $t - 1$, so no offer reaches the maturity. For the reason, the result in Saito [8] does not contradict Result 3 in the current paper.

What is to be emphasized here is that, although in the case of an infinite reserving period, offers are reserved only to prevent the risk at the deadline as stated in Section 1, in the case of a finite reserving period, we reserve offers so as not only to prevent that risk but also to facilitate stopping the search when we see no reason to pursue it further.

4. *If x is better, the range of offers to be passed up should be wider.*

This is clear since Theorem 5.9(a) is equivalent to $W_t(x^1)^c \subseteq W_t(x^2)^c$ for $x^1 \leq x^2$.

5. *Although every offer should be passed up if it is inferior to any of the reserved offers which will be still available at the deadline, it may prove wise to reserve even an offer inferior to some of the reserved offers which will expire prior to the deadline.*

Theorem 5.9(b) indicates the result. In fact we have a case that, with a reserved offer x_i, an offer w is to be reserved despite $w < x_i$ (see Table 1 in Section 7).

In Saito [8], an offer to be reserved must be superior to the leading offer at each time. This is different from our Result 5 but can be taken as consistent with it from the viewpoint that each reserved offer is assumed to be available at the deadline in Saito [8].

We should notice that, although it seems better to accept a reserved offer x_i than to reserve an offer w with $w < x_i$, it can be optimal to reserve such an offer w if it is close to x_i. Undoubtedly such a w will not be recalled and accepted while the x_i is available, but it is to be reserved as further insurance against any unfortunate situation after the expiration of the x_i.

Now, from Results 1 to 5, the optimal decision rule can be rewritten as follows:

Optimal Decision Rule: Suppose you are at time t with a reserved offer vector x and have just drawn an offer w. The choices are:

(A) In the case of $\hat{x} < \theta$:

- (a) If $w \in W_t(x)$ and $\lambda_t(0) < w$, accept the current offer w (AS).
- (b) If $w \in W_t(x)$ and $w \leq \lambda_t(0)$, reserve the current offer w (RC).
- (c) If $w \notin W_t(x)$, pass up the current offer w, and then:
  - (i) If $\theta_t^k(y) < x_k = \hat{x}$, accept the leading offer x_k (PS).
  - (ii) If $x_k \leq \theta_t^k(y)$, continue the search (PC).

(B) In the case of $\theta \leq \hat{x}$:

- (a) If $\hat{x} \leq w$, accept the current offer w (AS).
- (b) If $w < \hat{x}$, accept the leading offer $\hat{x}$ (PS).

7 Numerical Examples

We here depict the optimal decision rule by using some numerical examples.

Figure 7.1 illustrates the optimal decision rule for $t = 1$, calculated on the condition that $F(w)$ is the uniform distribution on $[1, 2]$, $\beta = 0.95$, $k = 2$, $s = 0.005$, and $r(w) = 0.002w$.

If $k = 2$, all offers to be considered for each time are two reserved offers x_1 and x_2, and a current offer w. Hence, optimal decision rules with $k = 2$ can be schematized in 3-dimensional diagrams like Figure 7.1.

The left of Figure 7.2 is the cross section of Figure 7.1 with $x_2 = 1.3$ and the right is the one with $x_1 = 1.4$, where the areas above the bold lines represent $W_1(x_1, 1.3)$ and $W_1(1.4, x_2)$, respectively. Either of the diagrams indicates the optimal decision rule for $t = 1$ with $(x_1, x_2) = (1.4, 1.3)$: If a current offer w is such that $w < 1.410$, pass it up, if $1.410 \leq w \leq 1.570$, reserve it, and if $1.570 < w$, accept it.

Here, it can be shown that the set of offers to be reserved is given by a union of disjoint sets and so also is the one to be passed up. This phenomenon depends on the shape of $r(w)$.

Figure 7.3 illustrates the optimal decision rule for $t = 1$ with $(x_1, x_2) = (0, 0)$, calculated on the same condition as that used in Figure 7.1 except for $r(w) = 0.002$ for $w < 1.15$, $0.4w - 0.458$ for $1.15 \leq w < 1.20$, and 0.022 for $1.20 \leq w$. Since $z_t^o(0,0) = z_t^o(a,a)$ and $z_t^r(w,0) = z_t^r(w,a)$, the rule with $(x_1, x_2) = (0,0)$ is equivalent to the one with $(x_1, x_2) = (a, a)$. Hence, the diagrams tell that, if $w < 1.070$, pass up w, if $1.070 \leq w \leq 1.185$, reserve w, if $1.185 < w < 1.220$, pass up w again, and if $1.220 \leq w \leq 1.535$, reserve w again. Let us call such a property the *Multiple Critical Value Property*. According to the many numerical calculations the author made, the phenomenon tends to happen when $r(w)$ is flat or increases slightly until a certain w and then rises rapidly. It was theoretically proven that, if $r(w)$ is concave, the phenomenon never happens. The result is also described in Saito [8].

Finally, we shall show a scenario of a search. Table 1 is calculated on the same condition as that used in Figure 7.1. If $(x_1, x_2) = (0, 0)$ at $t = 4$ and an offer $w = 1.55$ appears, the offer should be reserved, so the next search ($t = 3$) starts with $(x_1, x_2) = (1.55, 0)$. If an offer $w = 1.54$ is found at $t = 3$, it should be reserved in spite of $w < x_1$ (Result 5). Since decision

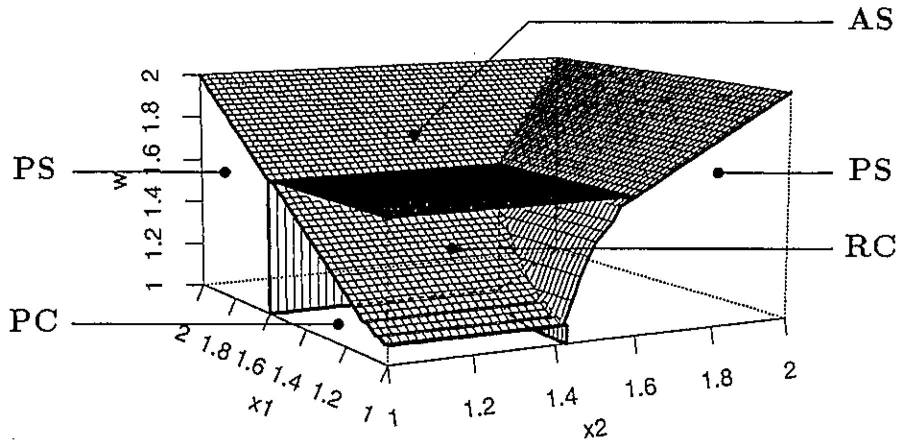


Figure 7.1: Optimal Decision Rule for $t = 1$ with $k = 2$

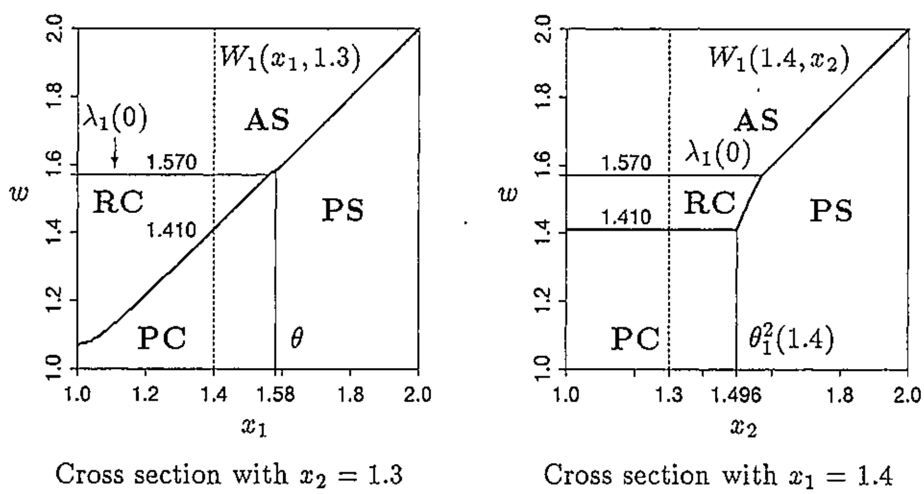
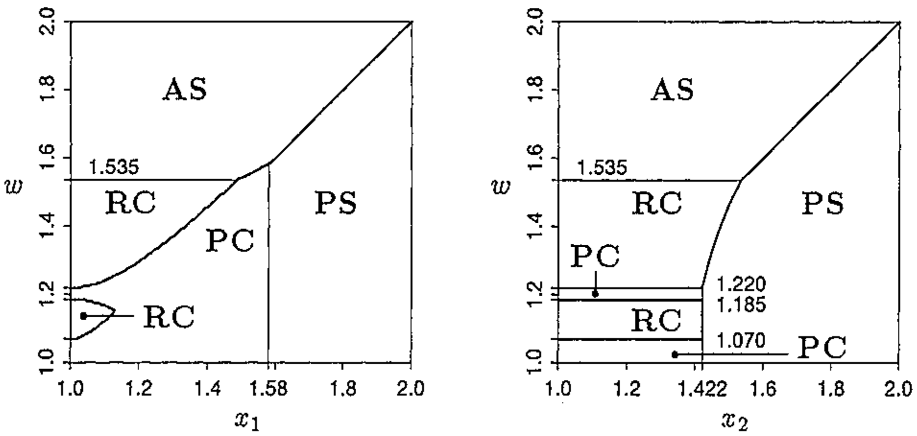


Figure 7.2: Cross sections of Figure 7.1

PC should be taken at $t = 2$ if $w = 1.50$, we reach $t = 1$ with $(x_1, x_2) = (0, 1.54)$. If $w = 1.50$ at $t = 1$, the reserved offer $x_2 = 1.54$ should be accepted (Result 3).

Table 1: A scenario of a search process

t	(x_1, x_2)	θ	$\theta_t^2(x_1)$	$W_t(x_1, x_2)$	$\lambda_t(0)$	w	Decision
4	(0 ,0)	1.580	1.569	$1.545 \leq w$	1.575	1.55	RC (Reserve $w = 1.55$)
3	(1.55 ,0)	1.580	1.566	$1.535 \leq w$	1.575	1.54	RC (Reserve $w = 1.54$)
2	(1.54 ,1.55)	1.580	1.560	$1.515 \leq w$	1.575	1.50	PC
1	(0 ,1.54)	1.580	1.422	$1.510 \leq w$	1.570	1.50	PS (Accept $x_2 = 1.54$)



Cross section with $x_2 = 0$ Cross section with $x_1 = 0$

Figure 7.3: Cross sections of the other example

8 The Case Void of Assumption 3

We here assume $\alpha \leq a$, the converse of Assumption 3. Note that the assumption is not used at all in Corollary 5.4(b), Theorem 5.5(b,c), and Corollary 5.7.

Suppose $\hat{x} < \theta$, so $\hat{y} < \theta$. Since $-s + \beta v_{t-1}(0, y) \leq \theta$ by Theorem 5.5(b), we get $z_t^o(x) \leq \max\{\theta, \theta\} = \theta$ by Eq. (4.5). Since $\alpha \leq a$ implies $\theta = \alpha \leq a$ due to Lemma 3.1(d), we have $\lambda_t(y) < \theta \leq a$ by Theorem 5.5(c). Hence, if $a \leq w$, then $a \leq z_t^r(w, y) = w$ by Eq. (4.7) and Corollary 5.4(b). Thereby, for any $w \geq a$, we get $\hat{x} < \theta \leq a \leq w$ and $z_t^o(x) \leq \theta \leq a \leq z_t^r(w, y) = w$. That is, if $\hat{x} < \theta$, any offer w satisfies $\hat{x} < w$ and should be accepted immediately.

Combining this fact and Corollary 5.7 for the case $\theta \leq \hat{x}$, we conclude that, if Assumption 3 is invalid, or if $\alpha \leq a$, it is optimal to quit the search immediately by accepting the more lucrative between the leading offer $\hat{x}$ and the current offer w.

9 Infinite Planning Horizon

Theorem 9.1 *As $t \to \infty$, we have $v_t(x) \to v(x) = \max\{(\theta + s)/\beta, S(\hat{x})\}$, $W_t(x) \to W(x) = \{w | \max\{\theta, \hat{x}\} \leq w\}$, and $\lambda_t(y) \to \lambda(y) < \max\{\theta, \hat{y}\} \leq \max\{\theta, \hat{x}\}$.*

Proof. Use an argument similar to the proof of Theorem 7.1 in Saito [8]. ∎

Theorem 9.1 presents the optimal decision rule with infinite planning horizon: Accept an offer w if $\max\{\theta, \hat{x}\} \leq w$, or else continue the search. This is the same as the result in Saito [8].

The point to notice is that no offer needs to be reserved if the planning horizon is infinite.

Now, if $\theta \leq \hat{x}$, the optimal decision rule (B) as stated in Section 6 can be applied. If $\hat{x} < \theta$, then $\max\{\theta, \hat{x}\} = \theta$. Thereby, we can arrange the above rule as follows: Accept the more lucrative between the leading offer $\hat{x}$ and a current offer w if $\theta \leq \max\{\hat{x}, w\}$, or else continue. This is substantially the same as the rules for the case of infinite planning horizon in models with recall, without recall, and with uncertain recall (Björn [1] and Ikuta [2]).

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