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Threshold Models for Comparative Probability on Finite Sets

by

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Abstract

Let $\succ$ be a comparative probability relation on the set $\mathcal{B}_S$ of all subsets of a finite state space S. This paper presents and discusses necessary and sufficient axioms for several threshold models of $\succ$, whose general representational form yields a probability measure P on $\mathcal{B}_S$ and a bivariate set function $\Omega \geq 0$ on $\mathcal{B}_S \times \mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$, $A \succ B$ if and only if $P(A) > P(B) + \Omega(A, B)$. Several conditions such as skew-monotonicity and additive separability will be imposed on the functional form of Ω.

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1 Introduction

The aim of this paper is to identify and discuss necessary and sufficient axioms for several threshold models of a binary comparative probability relation $\succ$, read as *is more probable than*, on the algebra $\mathcal{B}_S$ of all subsets of a *finite* state space S. Subsets of S are called *events*. The most general representational form that we shall consider yields a probability measure P on $\mathcal{B}_S$ and a nonnegative bivariate set function Ω on $\mathcal{B}_S \times \mathcal{B}_S$ such that, for all $A, B \in \mathcal{B}_S$,

$$A \succ B \iff P(A) > P(B) + \Omega(A, B),$$

where Ω can be interpreted as threshold of probability numbers, i.e., event A is more probable than event B if and only if the probability difference $P(A) - P(B)$ is greater than a nonnegative number $\Omega(A, B)$.

Kraft, Pratt, and Seidenberg (1959) were the first to present necessary and sufficient conditions for the existence of a probability measure P on $\mathcal{B}_S$ that strictly agrees with $\succ$ without threshold, i.e., $\Omega(A, B) = 0$ for all $A, B \in \mathcal{B}_S$. The first attempt to allow for imprecise probability judgments was made by Adams (1965) and Fishburn (1969), who gave necessary and sufficient axiomatizations that result in partial rather than strict agreement as follows: for all $A, B \in \mathcal{B}_S$,

$$A \succ B \Longrightarrow P(A) > P(B).$$

This one-way representation is undesirable, because we cannot infer any comparative probability judgments among events from the probability measure P. In addition to P, we need more information to recover all such judgments. Such an information may be provided by threshold functions, since it will be shown later that the one-way representation is tantamount to our general threshold representation. However, the threshold function Ω is too general for practical reason, since no restriction other than nonnegativity is imposed. Thus it is important to examine axiomatic structures that have more specific functional properties of Ω.

Threshold representations were first explored by Fishburn (1969) and Domotor and Stelzer (1971), who introduced a constant threshold, i.e., $\Omega(A, B) = \epsilon$ for all $A, B \in \mathcal{B}_S$ and some nonnegative number ϵ, and provided necessary and sufficient axioms for that representation. Fishburn (1969) also considered a univariate threshold, i.e., $\Omega(A, B) = \omega(B)$ for all $A, B \in \mathcal{B}_S$ and a nonnegative set function ω on $\mathcal{B}_S$. His axioms are sufficient but not necessary for the representation. Since then, much progress on axiomatization of imprecise probability judgments has not been done except Fishburn (1986), who examined several interval representations of $\succ$ on finite sets, but did not focus on specific structures of the threshold function Ω.

We are concerned with three types of representational forms of Ω such as bivariate (nonseparable) forms, (additively) separable forms, or univariate forms. Since it is most likely that an individual would not regard event A as less probable than event B when A properly includes B, it seems also likely that when event B is a proper subset of event A, event C would be more probable than B when C is more probable than A, or A would be more probable than C when B is more probable than C. We regard such inclusion monotonicity properties fundamental to any reasonable comparative probabilty judgments. The inclusion monotonicity is reflected by skew-monotonicity of Ω, which says that Ω is decreasingly monotonic in its first argument and increasingly monotnoic in its second argument.

The paper is organized as follows. Section 2 discusses axiomatizations of the general threshold model and the skew-monotonic threshold model. Sections 3 and 4 respectively explore additively separable and univariate forms of Ω. Then Section 5 deals with the cases that univariate thresholds are given by additive set functions. All sufficiency proofs of the theorems are deferred to Section 6.

2 Bivariate Thresholds

A univariate set function Q on $\mathcal{B}_S$ is *increasingly* (respectively, *decreasingly*) *monotonic* if $Q(A) \geq Q(B)$ (respectively, $Q(A) \leq Q(B)$) whenever $A \supseteq B$, and *additive* if, for all $A, B \in \mathcal{B}_S$ for which $A \cap B = \emptyset$,

$$Q(A \cup B) + Q(\emptyset) = Q(A) + Q(B).$$

It is not generally assumed that $Q(\emptyset) = 0$. A *probability measure* P on $\mathcal{B}_S$ is an increasingly monotonic and additive univariate set function with $P(\emptyset) = 0$ and $P(S) = 1$.

A bivariate set function Ω on $\mathcal{B}_S \times \mathcal{B}_S$ is *skew-monotonic* if it is decreasingly monotonic in its first argument and increasingly monotonic in its second argument, i.e., for all $A, B, C \in \mathcal{B}_S$ for which $A \supseteq B$,

$$\Omega(B, C) \geq \Omega(A, C) \text{ and } \Omega(C, A) \geq \Omega(C, B).$$

Note that $\Omega(\emptyset, S) \geq \Omega(A, B) \geq \Omega(S, \emptyset)$ for all $A, B \in \mathcal{B}_S$.

This section considers bivariate forms of threshold functions which lead to the following general threshold representation for $\succ$: for all $A, B \in \mathcal{B}_S$,

$$A \succ B \Longleftrightarrow P(A) > P(B) + \Omega(A, B).$$

To formulate necessary and sufficient axioms, we define an indicator function I on $S \times \mathcal{B}_S$ by

$$I(s; A) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem says that the one-way representation axiomatized by Adams and Fishburn is equivalent to the strictly agreeing representation with bivariate threshold.

Theorem 2.1 (Adams, 1965; Fishburn, 1969) *There exist a probability measure P on $\mathcal{B}_S$ and a bivariate set function $\Omega \geq 0$ on $\mathcal{B}_S \times \mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \Omega(A, B)$$

if and only if the following axiom holds for all $A_1, A_2, \ldots, B_1, B_2, \ldots \in \mathcal{B}_S$ *and all integers* $m \geq 1$,

A2.1 *if for all* $s \in S$,

$$\sum_{i=1}^{m} I(s; A_i) \leq \sum_{i=1}^{m} I(s; B_i),$$

then it is false that $A_i \succ B_i$ *for* $i = 1, \ldots, m$.

Proof. Since axiom A2.1 is necessary and sufficient for the one-way representation, it suffices to prove that the one-way representation can be translated into a bivariate threshold model. We define

$$\Omega(A, B) = \begin{cases} 0 & \text{if } A \succ B, \\ |P(A) - P(B)| & \text{if } A \sim B. \end{cases}$$

Then it easily follows that for all $A, B \in \mathcal{B}_S$,

$$A \succ B \Longleftrightarrow P(A) > P(B) + \Omega(A, B).$$

This completes the proof. □

Fishburn (1969) noted that A2.1 implies that $A \supseteq B \Longrightarrow \text{not}(B \succ A)$, and that $\succ$ is irreflexive, asymmetric, and acyclic, i.e., $(A_1 \succ A_2, A_2 \succ A_3, \ldots, A_{m-1} \succ A_m) \Longrightarrow A_1 \succeq A_m$. Also, A2.1 does not imply that $S \succ \emptyset$, or that $\succ$ is transitive, or that $A \succ C$ when $A \succ B$ and $B \supset C$.

The inclusion monotonicity is defined in two ways as follows. We say that $\succ$ is *upward inclusion monotonic* if, for all $A, B, C \in \mathcal{B}_S$,

$$A \supset B \text{ and } B \succ C \Longrightarrow A \succ C;$$

and *downward inclusion monotonic* if, for all $A, B, C \in \mathcal{B}_S$,

$$A \succ B \text{ and } B \supset C \Longrightarrow A \succ C.$$

Fishburn (1969) showed that the downward inclusion monotonicity and axiom A2.2 below, which replaces inequality in A2.1 by equality, are sufficient for the representation of Theorem 2.1, and noted that we could add the upward inclusion monotonicity although the dwonward inclusion monotonicity and A2.2 do not imply the upward inclusion monotonicity. However, the following theorem shows that the downward and upward inclusion monotonicities together with A2.2 impose skew-monotonicity on the threshold function Ω.

Theorem 2.2 *There exist a probability measure* P *on* $\mathcal{B}_S$ *and a skew-monotonic bivariate set function* $\Omega \geq 0$ *on* $\mathcal{B}_S \times \mathcal{B}_S$ *such that for all* $A, B \in \mathcal{B}_S$,

$$A \succ B \Longleftrightarrow P(A) > P(B) + \Omega(A, B)$$

if and only if $\succ$ *is upward and downward inclusion monotonic and the following axiom holds, for all* $A_1, A_2, \ldots, B_1, B_2, \ldots \in \mathcal{B}_S$ *and all integers* $m \geq 1$,

A2.2 *if for all* $s \in S$,

$$\sum_{i=1}^{m} I(s; A_i) = \sum_{i=1}^{m} I(s; B_i),$$

then it is false that $A_i \succ B_i$ *for* $i = 1, \ldots, m$.

It easily follows from skew-monotonicity of Ω that the upward and downward inclusion monotonicities are necessary for the representation of the theorem. To see the necessity of A2.2, suppose that the hypotheses of the axiom holds and $A_i \succ B_i$ for $i = 1, \ldots, m$. Then $\sum_{i=1}^{m} P(A_i) = \sum_{i=1}^{m} P(B_i)$, since $\sum_{i=1}^{m} I(s; A_i) = \sum_{i=1}^{m} I(s; B_i)$. It follows from the representation that $\sum_{i=1}^{m} P(A_i) > \sum_{i=1}^{m} P(B_i) + \sum_{i=1}^{m} \Omega(A_i, B_i)$, so $0 > \sum_{i=1}^{m} \Omega(A_i, B_i)$, which contradicts nonnegativity of Ω. Therefore, A2.2 is necessary for the representation.

If we add the condition of $S \succ \emptyset$, i.e., nonemptyness of $\succ$, then we must have that $\Omega(S, \emptyset) < 1$. Clearly, A2.2 itself is not sufficient for the representation of Theorem 2.1, but A2.2 together with the downward inclusion monotonicity implies A2.1. Since A2.1 implies A2.2, the upward and downward inclusion monotonicities require that threshold function Ω in Theorem 2.1 be skew-monotonic. This suggests that when $\succ$ ageeing with the representation of Theorem 2.1 satisfies the upward and downward monotonicities, the threshold function can be always modifies to be skew-monotonic. To see this, consider the following example.

Example **2.1** Suppose that $\succ$ agrees with the representation of Theorem 2.1, i.e. for all $A, B \in \mathcal{B}$,

$$A \succ B \iff \Phi(A, B) > 0,$$

where $\Phi(A, B) = P(A) - P(B) - \Omega(A, B)$. Suppose also that Φ is skew-monotonic. Thus $\succ$ is upward and downward inclusion monotonic. We note that Ω is not necessarily skew-monotonic. We show below that Ω can be modified to be skew-monotonic. To see this, let $\epsilon = \max\{\Omega(A, B) : \Phi(A, B) \leq 0 \text{ and } (A, B) \in \mathcal{B}_S \times \mathcal{B}_S\}$, and define a bivariate set function Ω' on $\mathcal{B}_S \times \mathcal{B}_S$ as follows: for all $A, B \in \mathcal{B}_S$,

$$\Omega'(A, B) = \begin{cases} \epsilon & \text{if } \Phi(A, B) \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definition of Ω' and the upward and downward monotonicities that Ω' is skew-monotonic. We obtain that, for all $A, B \in \mathcal{B}_S$,

$$\begin{aligned} \Phi(A, B) > 0 &\implies P(A) - P(B) > 0 \\ &\implies P(A) - P(B) > \Omega'(A, B), \\ \Phi(A, B) \leq 0 &\implies P(A) - P(B) \leq \Omega(A, B) \leq \epsilon \\ &\implies P(A) - P(B) \leq \Omega'(A, B), \end{aligned}$$

so that the representation of Theorem 2.2 holds.

3 Separable Thresholds

In the rest of the paper, we shall consider (additive) separability of the threshold function Ω in Theorem 2.1, i.e., there are two univariate set functions, σ^+ and σ^-, on $\mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$,

$$\Omega(A, B) = \sigma^-(A) + \sigma^+(B).$$

Thus the numerical representation for $\succ$ that we shall examine is given by

$$A \succ B \Longleftrightarrow P(A) - \sigma^-(A) > P(B) + \sigma^+(B).$$

Letting $Q = P - \sigma^-$, this representation is rearranged to give

$$A \succ B \Longleftrightarrow Q(A) > Q(B) + \sigma^-(B) + \sigma^+(B),$$

where $\sigma^-(B) + \sigma^+(B) = \Omega(B, B) \geq 0$. Since the threshold function turns out to be univariate, separable thresholds require that $\succ$ be an interval order. We note, however, that Q may not be a probability measure. Recall that $\succ$ on $\mathcal{B}_S$ is an *interval order* if it is irreflexive and for all $A, B, C, D \in \mathcal{B}_S$, $A \succ D$ or $C \succ B$ whenever $A \succ B$ and $C \succ D$. Asymmetry and transitivity of $\succ$ follow from the definition of interval orders, so that $\succ$ is a strict partial order.

Since Ω is nonnegative, we have that $\min \sigma^- + \min \sigma^+ \geq 0$, where for $* \in \{+, -\}$, $\min \sigma^*$ denotes $\min_{A \in \mathcal{B}_S} \sigma^*(A)$ for short. We show that σ^- and σ^+ can be respectively modified to ω^- and ω^+ in such a way that $\omega^- \geq 0$, $\omega^+ \geq 0$, and for all $A, B \in \mathcal{B}_S$,

$$A \succ B \Longleftrightarrow P(A) - \omega^-(A) > P(B) + \omega^+(B).$$

To see this, we define, for all $A \in \mathcal{B}_S$,

$$\omega^+(A) = \begin{cases} \sigma^+(A) + \min \sigma^- & \text{if } \min \sigma^+ > 0 > \min \sigma^-, \\ \sigma^+(A) - \min \sigma^+ & \text{if } \min \sigma^- > 0 > \min \sigma^+, \\ \sigma^+(A) & \text{otherwise,} \end{cases}$$

$$\omega^-(A) = \begin{cases} \sigma^-(A) - \min \sigma^- & \text{if } \min \sigma^+ > 0 > \min \sigma^-, \\ \sigma^-(A) + \min \sigma^+ & \text{if } \min \sigma^- > 0 > \min \sigma^+, \\ \sigma^-(A) & \text{otherwise,} \end{cases}$$

so that $\omega^+ \geq 0$ and $\omega^- \geq 0$. It readily follows from the definitions of ω^+ and ω^- that for all $A, B \in \mathcal{B}_S$,

$$\begin{aligned} & P(A) - \sigma^-(A) > P(B) + \sigma^+(B) \\ \Longleftrightarrow \quad & P(A) - \omega^-(A) > P(B) + \omega^+(B). \end{aligned}$$

The most general representation that yields additively separable threshold in this paper is given by the following theorem, which says that if $\succ$ agreeing with the representation of Theorem 2.1 is an interval order, then the threshold function Ω in Theorem 2.1 can be additively separable.

Theorem 3.1 *There exist a probability measure P on $\mathcal{B}_S$ and set functions, $\omega^+ \geq 0$ and $\omega^- \geq 0$, on $\mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega^-(A) + \omega^+(B)$$

if and only if $\succ$ is an interval order and A2.1 *holds.*

Clearly, it is necessary that $\succ$ is an interval order. To see the necessity of A2.1, suppose that $\sum_{i=1}^m I(s; A_i) \leq \sum_{i=1}^m I(s; B_i)$ and $A_i \succ B_i$ for $i = 1, \ldots, m$. Then $\sum_{i=1}^m P(A_i) - \sum_{i=1}^m P(B_i) \leq 0$, and the representation requires that $P(A_i) > P(B_i) + \omega^-(A_i) + \omega^+(B_i)$ for $i = 1, \ldots, m$. Summing over i yields $\sum_i^m P(A_i) - \sum_{i=1}^m P(B_i) > \sum_{i=1}^m (\omega^-(A_i) + \omega^+(B_i)) \geq 0$. which leads to $0 > 0$, a contradiction. Hence A2.1 must be necessary for the representation.

Now we consider the effect of inclusion monotonicity on the representation of Theorem 3.1. It turns out that the resulting additively separable threshold is not necessarily skew-monotonic as shown by the following theorem.

Theorem 3.2 (Fishburn, 1986) *There exist a probability measure P on $\mathcal{B}_S$ and set functions, $\omega^- \geq 0$ and $\omega^+ \geq 0$, on $\mathcal{B}_S$ such that $P - \omega^-$ and $P + \omega^+$ are increasingly monotonic, and for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega^-(A) + \omega^+(B)$$

if and only if $\succ$ is an interval order and upward and downward inclusion monotonic, and A2.2 *holds.*

This theorem says that if $\succ$ agreeing with the representation of Theorem 2.2 is an interval order, then the threshold function Ω in Example 2.1 in place of the one in Theorem 2.2 can be additively separable. Fishburn (1986) called the threshold model of the theorem an additive interval model, and showed that the representation of Theorem 3.2 obtains if A2.1 in place of A2.2 holds. However, the weaker A2.2 suffices, since A2.2 together with the downward inclusion monotonicity implies A2.1.

The following theorem provides necessary and sufficient axioms for the skew-monotonic threshold function of Theorem 2.2 to be additively separable.

Theorem 3.3 *There exist a probability measure P on $\mathcal{B}_S$, a decreasingly monotonic set function $\omega^- \geq 0$ on $\mathcal{B}_S$, and an increasingly monotonic set function $\omega^+ \geq 0$ on $\mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega^-(A) + \omega^+(B)$$

if and only if $\succ$ is an interval order and upward and downward inclusion monotonic, and the following axiom holds for all $A_1, A_2, \ldots, B_1, B_2, \ldots, C_1, C_2, \ldots, D_1, D_2, \ldots \in \mathcal{B}_S$, all integers $m \geq 1$, and all integers $0 \leq \ell \leq m$,

A3.3 *if* $D_i \subseteq B_i$, $A_{\pi(i)} \subseteq C_i$, *and not*$(C_i \succ D_i)$ *for* $i = 1, \ldots, \ell$ *and some permutation* π *on* $\{1, \ldots, m\}$, *and for all* $s \in S$,

$$\sum_{i=1}^{m} I(s; A_i) + \sum_{i=1}^{\ell} I(s; D_i) = \sum_{i=1}^{m} I(s; B_i) + \sum_{i=1}^{\ell} I(s; C_i),$$

then it is false that $A_i \succ B_i$ *for* $i = 1, \ldots, m$.

We note that A3.3 is equivalent to A2.2 when $\ell = 0$. To show the necessity of A3.3, suppose that $D_i \subseteq B_i$, $A_{\pi(i)} \subseteq C_i$, not$(C_i \succ D_i)$ for $i = 1, \ldots, \ell$ and some permutation π on $\{1, \ldots, m\}$, $A_i \succ B_i$ for $i = 1, \ldots, m$, and for all $s \in S$, $\sum_{i=1}^{m} I(s; A_i) + \sum_{i=1}^{\ell} I(s; D_i) = \sum_{i=1}^{m} I(s; B_i) + \sum_{i=1}^{\ell} I(s; C_i)$. Then for $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$, we get

$$\begin{aligned} P(A_i) - \omega^-(A_i) &> P(B_i) + \omega^+(B_i), \\ P(D_j) + \omega^+(D_j) &\geq P(C_j) - \omega^-(C_j). \end{aligned}$$

Summing over i and j, we get

$$\begin{aligned} &\sum_{i=1}^{m}(P(A_{\pi(i)}) - \omega^-(A_{\pi(i)})) + \sum_{j=1}^{\ell}(P(D_j) + \omega^+(D_j)) \\ &> \sum_{i=1}^{m}(P(B_i) + \omega^+(B_i)) + \sum_{j=1}^{\ell}(P(C_j) - \omega^-(C_j)). \end{aligned}$$

Using $\sum_{i=1}^{m} I(s; A_{\pi(i)}) + \sum_{i=1}^{\ell} I(s; D_i) = \sum_{i=1}^{m} I(s; B_i) + \sum_{i=1}^{\ell} I(s; C_i)$, we get

$$\sum_{j=1}^{\ell} \omega^+(D_j) - \sum_{i=1}^{m} \omega^-(A_{\pi(i)}) > \sum_{i=1}^{m} \omega^+(B_i) - \sum_{j=1}^{\ell} \omega^-(C_j).$$

Since $\omega^- \geq 0$, $\omega^+ \geq 0$, $\omega^-(A_{\pi(i)}) \geq \omega^-(C_i)$ and $\omega^+(D_i) \leq \omega^+(B_i)$ for $i = 1, \ldots, \ell$, we get a contradition. Hence A2.3 is necessary for the representation.

The following example shows that the conditions of Theorem 3.2 do not imply A3.3, so that ω^- and ω^+ in Theorem 3.2 cannot be modified to be respectively decreasingly and increasingly monotonic. As discussed in the preceding section, those ω^- and ω^+ can be combined into a skew-monotonic bivariate set function Ω without affecting the representability of $\succ$. In the sequel, given a univariate set function Q on $\mathcal{B}_S$, we shall write $Q(s)$ and $Q(s,t)$ in place of $Q(\{s\})$ and $Q(\{s,t\})$, respectively.

Example 3.1 Let $S = \{s_1, s_2, s_3, s_4\}$ with $P(s_1) = \frac{2}{9}$, $P(s_2) = P(s_3) = \frac{3}{9}$, $P(s_4) = \frac{1}{9}$ and $\omega^- = \omega^+ = 0$ except $\omega^-(s_1, s_2) = \frac{2}{9}$. Then $P - \omega^-$ and $P + \omega^+$ are increasingly monotonic. We define $\succ$ according to the representation of Theorem 3.2. Let $m = 2$, $\ell = 1$, $A_1 = \{s_2, s_4\}$, $A_2 = \{s_1\}$, $B_1 = \{s_3\}$, $B_2 = \{s_4\}$, $C_1 = \{s_1, s_2\}$, and $D_1 = \{s_3\}$ for A3.3. Then $D_1 \subseteq B_1$, $A_2 \subseteq C_1$ and $I(s; A_1) + I(s; A_2) + I(s; D_1) = I(s; B_1) + I(s; B_2) + I(s; C_1)$. However, the representation of Theorem 3.2 requires that $A_1 \succ B_1$, $A_2 \succ B_2$, and $C_1 \sim D_1$. This contradicts A3.3.

4 Univariate Thresholds

As discussed in the previous section, numerical representations with separable thresholds are reduced to the representations with univariate thresholds, but we may not have probability measures. This section considers the existence of probability measures with univariate thresholds.

To formulate necessary and sufficient axioms, we shall use a binary relation $\succ^*$ on $\mathcal{B}_S$ defined from $\succ$ as follows: for all $A, B \in \mathcal{B}_S$,

$$A \succ^* B \Longleftrightarrow A \succ C \sim B \text{ for some } C \in \mathcal{B}_S.$$

Note that $\succ^*$ is an asymmetric weak order (see Fishburn, 1984).

The most general univariate threshold model is given by

Theorem 4.1 *There exist a probability measure P on $\mathcal{B}_S$ and a set function $\omega \geq 0$ on $\mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega(B)$$

if and only if $\succ$ is an interval order, and the following axiom holds, for all $A_1, A_2, \ldots, B_1, B_2, \ldots \in \mathcal{B}_S$ and all integers $m \geq 1$,

A4.1 *if for all $s \in S$,*

$$\sum_{i=1}^{m} I(s; A_i) \leq \sum_{i=1}^{m} I(s; B_i),$$

then it is false that $A_i \succ^* B_i$ for $i = 1, \ldots, m$.

Fishburn (1969) showed that the conditions in Theorem 4.2 below are sufficient for the representation of Theorem 4.1, since the downward inclusion monotonicity is not necessary. To illustrate this, consider the following example.

Example 4.1 Let $S = \{s_1, s_2\}$ with $P(s_1) = 0.4$, $P(s_2) = 0.6$, $\omega(\emptyset) = 0.7$, $\omega(s_1) = 0.1$, $\omega(s_2) = 0.1$, and $\omega(S) = 0$, and define $\succ$ according to the representation of the theorem. Then $\{s_2\} \succ \{s_1\}$ since $P(s_2) > P(s_1) + \omega(s_1)$, but not$(\{s_2\} \succ \emptyset)$ since $P(\emptyset) + \omega(\emptyset) > P(s_2)$. Thus the downward inclusion monotonicty fails to hold. Note that ω is not increasingly monotonic.

To show the necessity of A4.1, suppose that the representation of the theorem and the hypotheses of A4.1 hold. Assume that $A_i \succ^* B_i$ for $i = 1, \ldots, m$. Then for each i, there is a $C_i \in \mathcal{B}_S$ such that $A_i \succ C_i \sim B_i$, so that $P(A_i) > P(C_i) + \omega(C_i)$ and $P(C_i) + \omega(C_i) \geq P(B_i)$. Thus $P(A_i) > P(B_i)$. Summing over i and using $\sum_{i=1}^{m} I(s; A_i) \leq \sum_{i=1}^{m} I(s; B_i)$, we get $0 > 0$. Hence A4.1 is necessary for the representation. We note that A4.1 implies A2.1.

The upward and downward inclusion monotonicities require that $P + \omega$ in Theorem 4.1 be increasingly monotonic as shown by the following theorem.

Theorem 4.2 *There exist a probability measure P on $\mathcal{B}_S$ and a set function $\omega \geq 0$ on $\mathcal{B}_S$ such that $P+\omega$ is increasingly monotonic, and for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega(B)$$

if and only if $\succ$ is an interval order and upward and downdard inclusion monotonic, and the following axiom holds, for all $A_1, A_2, \ldots, B_1, B_2, \ldots \in \mathcal{B}_S$ and all integers $m \geq 1$,

A4.2 *if for all $s \in S$,*

$$\sum_{i=1}^{m} I(s; A_i) = \sum_{i=1}^{m} I(s; B_i),$$

then it is false that $A_i \succ^* B_i$ for $i = 1, \ldots, m$.

The necessities of A4.2, interval order, and upward and downward inclusion monotonicities easily follow. We note that A4.2 implies A2.2. To see that the conditions of Theorem 4.2 implies A4.1, assume that for all $s \in S$, $\sum_{i=1}^{m} I(s; A_i) \leq \sum_{i=1}^{m} I(s; B_i)$. Then there are $D_1, \ldots, D_m \in \mathcal{B}_S$ such that $B_i \supseteq D_i$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} I(s; A_i) = \sum_{i=1}^{m} I(s; D_i)$. By A4.2, not$(A_k \succ^* D_k)$ for some $i \leq k \leq m$. Assume $A_k \succ^* B_k$, so $A_k \succ C \sim B_k$ for some $C \in \mathcal{B}_S$. Thus $A_k \succ C \sim B_k \supseteq D_k$. By the upward inclusion monotonicity, not$(D_k \succ C)$. Thus $A_k \succ C \succeq D_k$, so $A_k \succ^* D_k$, a contradiction. Hence it is false that $A_i \succ^* B_i$ for $i = 1, \ldots, m$, so A4.1 obtains.

The following theorem requires that ω itself in Theorem 4.2 be increasingly monotonic.

Theorem 4.3 *There exist a probability measure P on $\mathcal{B}_S$ and an increasingly monotonic set function $\omega \geq 0$ on $\mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega(B)$$

if and only if not$(\emptyset \succ \emptyset)$, $\succ$ is upward and downward inclusion monotonic, and the following axiom holds, for all $A_1, A_2, \ldots, B_1, B_2, \ldots, C_1, C_2, \ldots, D_1, D_2, \ldots \in \mathcal{B}_S$ and all integers $m \geq 1$,

A4.3 *if $D_i \subseteq B_i$ and not$(C_i \succ D_i)$ for $i = 1, \ldots, m$, and for all $s \in S$,*

$$\sum_{i=1}^{m} I(s; A_i) + \sum_{i=1}^{m} I(s; D_i) = \sum_{i=1}^{m} I(s; B_i) + \sum_{i=1}^{m} I(s; C_i),$$

then it is false that $A_i \succ B_i$ for $i = 1, \ldots, m$.

It is easy to see that the upward and downward inclusion monotonicities are necessary for the representation. We show the necessity of A4.3. Suppose that $\sum_{i=1}^{m} I(s; A_i) + \sum_{i=1}^{m} I(s; D_i) = \sum_{i=1}^{m} I(s; B_i) + \sum_{i=1}^{m} I(s; C_i)$, $A_i \succ B_i$, $D_i \subseteq B_i$, and not($C_i \succ D_i$) for $i = 1, \ldots, m$. Then for each i, the representation of the theorem gives that

$$\begin{aligned} P(A_i) &> P(B_i) + \omega^+(B_i), \\ P(D_i) + \omega(D_i) &\geq P(C_i). \end{aligned}$$

Summing over i, we get

$$\sum_{i=1}^{m} P(A_i) + \sum_{i=1}^{m} P(D_i) + \sum_{i=1}^{m} \omega(D_i) > \sum_{i=1}^{m} P(B_i) + \sum_{i=1}^{m} P(C_i) + \sum_{i=1}^{m} \omega(B_i).$$

Using $\sum_{i=1}^{m} I(s; A_i) + \sum_{i=1}^{m} I(s; D_i) = \sum_{i=1}^{m} I(s; B_i) + \sum_{i=1}^{m} I(s; C_i)$, we get

$$\sum_{i=1}^{m} \omega(D_i) > \sum_{i=1}^{m} \omega(B_i).$$

On the other hand, $\omega(B_i) \geq \omega(D_i)$ for all i since ω is increasingly monotonic. This is a contradiction. Hence A4.3 is necessary for the representation.

We note that when $\sum_{i=1}^{m} I(s; A_i) + \sum_{i=1}^{m} I(s; D_i) \leq \sum_{i=1}^{m} I(s; B_i) + \sum_{i=1}^{m} I(s; C_i)$ in A4.3, we can drop upward and downward inclusion monotonicities. It easily follows that that A4.3 implies A3.3. To see this, simply adding $m - \ell$ times $I(s; \emptyset)$ to the both sides of the equation in the hypotheses of A3.3 and applying A4.3 yield the desired result.

The conditions of Theorem 4.3 imply those of Theorem 4.2. To see this, suppose that $\sum_{i=1}^{m} I(s; A_i) = \sum_{i=1}^{m} I(s; B_i)$ and $A_i \succ^* B_i$ for $i = 1, \ldots, m$. Then there are $C_1, \ldots, C_m$ such that $A_i \succ C_i \sim B_i$ for $i = 1, \ldots, m$. Since $\sum_{i=1}^{m} I(s; A_i) + \sum_{i=1}^{m} I(s; C_i) = \sum_{i=1}^{m} I(s; C_i) + \sum_{i=1}^{m} I(s; B_i)$, it follows from A4.3 that not($A_k \succ C_k$) for some k. This is a contradiction. Hence A4.2 holds.

To check that $\succ$ is an interval order, let $m = 1$ and $(A_1, D_1) = (B_1, C_1) = (A, \emptyset)$ for A4.3. Then A4.3 and not($\emptyset \succ \emptyset$) imply not($A \succ A$), so $\succ$ is irreflexive. We assume next that $A \succ B$ and $C \succ D$, but not($A \succ D$) and not($C \succ B$). Let $m = 2$, $(A_1, A_2, D_1, D_2) = (A, C, D, B)$, and $(B_1, B_2, C_1, C_2) = (B, D, A, C)$ for A4.3. Then by A4.3, it is false that $A \succ B$ and $C \succ D$. This is a contradiction. Hence, $A \succ D$ or $C \succ B$, so that $\succ$ is an interval order.

To show that the threshold function ω in Theorem 4.2 cannot always be modified to be increasingly monotonic, consider the following example.

Example 4.2 Let $S = \{s_1, s_2, s_3\}$ with $P(s_1) = P(s_3) = 0.25$, $P(s_2) = 0.5$, $\omega(\emptyset) = \omega(S) = 0$, $\omega(s_1) = \omega(s_2) = 0.05$, $\omega(s_3) = 0.3$, and $\omega(s_1, s_2) = \omega(s_1, s_3) = \omega(s_2, s_3) = 0.2$. Since $P + \omega$ is increasingly monotonic, we define $\succ$ according to the representation of Theorem 4.2. Then $\{s_1, s_2\} \succ \{s_1, s_3\}$, since $P(s_1, s_2) = 0.75 > P(s_1, s_3) + \omega(s_1, s_3) = 0.7$, and $s_2 \sim s_3$, since $P(s_2) = 0.5 < P(s_3) + \omega(s_3) = 0.55$ and $P(s_3) = 0.25 < P(s_2) + \omega(s_2) = 0.55$. On the other hand, let $m = 1$, $A_1 = \{s_1, s_2\}$, $B_1 = \{s_1, s_3\}$, $C_1 = \{s_2\}$, and $D_1 = \{s_3\}$ for A4.3. Since $D_1 \subsetneq B_1$, not($C_1 \succ D_1$), and $I(s; A_1) + I(s; D_1) = I(s; B_1) + I(s; C_1)$, A4.3 implies not($A_1 \succ B_1$). This is a contradiction.

Although $\succ$ in the above example does not agree with the representation of Theorem 4.3, it agrees with the representation of Theorem 3.3. To illustrate this, we construct an increasingly monotonic σ^+ and a decreasingly monotonic σ^- for which for all $A, B \in \mathcal{B}_S$,

$$A \succ B \Longleftrightarrow P(A) > P(B) + \sigma^-(A) + \sigma^+(B).$$

Let $\sigma^+(\emptyset) = 0$, $\sigma^+(s_1) = \sigma^+(s_2) = 0.05$, $\sigma_+(s_3) = 0.2$, $\sigma^+(s_1, s_2) = \sigma(s_1, s_3) = \sigma(s_2, s_3) = 0.2, \sigma^+(S) = 0.2, \sigma^-(\emptyset) = 0.2, \sigma^-(s_1) = 0.2, \sigma^-(s_2) = 0.1, \sigma^-(s_3) = 0.2$, $\sigma(s_1, s_3) = 0.1$, and $\sigma^-(s_1, s_2) = \sigma^-(s_2, s_3) = \sigma^-(S) = 0$. Therefore, $\succ$ in the example could have both representations of Theorems 3.3 and 4.2 which are not reduced to the representation of Theorem 4.3. However, the following two examples show that the representations of Theorems 3.3 and 4.2 do not generally imply each other.

Example 4.3 Let $S = \{s_1, s_2, s_3, s_4\}$. Suppose that $\succ$ on $\mathcal{B}_S$ agrees with the representation of Theorem 4.2 with $P(s_1) = 0.3, P(s_2) = 0.2, P(s_3) = 0.1, P(s_4) = 0.4$, and $\omega(A) = 0$ for all $A \in \mathcal{B}_S$ except $\omega(s_1) = 0.2, \omega(s_2) = \omega(s_3) = \omega(s_4) = 0.1, \omega(s_1, s_2) = 0.05, \omega(s_1, s_3) = \omega(s_2, s_4) = \omega(s_3, s_4) = 0.1$. Let $A_1 = \{s_2, s_4\}, A_2 = \{s_3\}, B_1 = \{s_1, s_2\}, B_2 = \emptyset, D_1 = \{s_1\}$, and $C_1 = \{s_3, s_4\}$. Then $P + \omega$ is increasingly monotonic. It follows that $D_1 \subset B_1, A_2 \subset C_1, D_1 \sim C_1$ and $I(s; A_1) + I(s; A_2) + I(s; D_1) = I(s; B_1) + I(s; B_2) + I(s; C_1)$ for all $s \in S$, so that the hypotheses of A3.3 for $m = 2$ and $\ell = 1$ hold. However, we have $A_1 \succ B_1$ and $A_2 \succ B_2$. Hence A3.3 fails to hold.

Example 4.4 Let $S = \{s_1, s_2, s_3, s_4, s_5\}$. Suppose that $\succ$ on $\mathcal{B}_S$ agrees with the representation of Theorem 3.3 with $P(s_1) = 0.2, P(s_2) = 0.2, P(s_3) = 0.15, P(s_4) = 0.3, P(s_5) = 0.15$, and $\omega^+(A) = \omega^-(A) = 0$ for all $A \in \mathcal{B}_S$ except $\omega^-(\emptyset) = 0.15$, $\omega^-(s_1) = 0.15, \omega^-(s_2) = \omega^-(s_3) = \omega^-(s_2, s_3) = 0.125$, and $\omega^-(s_4) = \omega^-(s_5) = \omega^-(s_4, s_5) = \omega^-(s_1, s_3) = \omega^-(s_1, s_5) = \omega^-(s_3, s_5) = \omega^-(s_1, s_3, s_5) = 0.05$. Then ω^+ is increasingly monotonic and ω^- is decreasingly monotonic. Let $A_1 = \{s_1, s_3, s_5\}$, $A_2 = \{s_2\}$, $A_3 = \{s_4\}$, $B_1 = \{s_4, s_5\}$, $B_2 = \{s_1\}$, and $B_3 = \{s_2, s_3\}$. Then $I(s; A_1) + I(s; A_2) + I(s; A_3) = I(s; B_1) + I(s; B_2) + I(s; B_3)$ for all $s \in S$, so that the hypotheses of A4.2 for $m = 3$ hold. However, we have $A_i \succ^* B_i$ for $i = 1, 2, 3$. Hence A4.2 fails to hold.

5 Additive Thresholds

This section is concerned with additive univariate threshold functions. Recall that a univariate set function ω on $\mathcal{B}_S$ is additive if, for all disjoint $A, B \in \mathcal{B}_S$, $\omega(A \cup B) + \omega(\emptyset) = \omega(A) + \omega(B)$. Then $\omega(A) = \sum_{i=1}^n \omega(s_i) - (n-1)\omega(\emptyset) = \sum_{i=1}^n (\omega(s_i) - \omega(\emptyset)) + \omega(\emptyset)$ whenever $A = \{s_1, \ldots, s_n\}$. Therefore, given an additive univariate set function $\omega \geq 0$, we define additive univariate set functions, $\sigma_\omega^- \geq 0$ and $\sigma_\omega^+ \geq 0$, on

$\mathcal{B}_S$ and a constant ϵ as follows: $\epsilon = \omega(\emptyset)$, $\sigma_\omega^-(\emptyset) = \sigma_\omega^+(\emptyset) = 0$, and for all $s \in S$,

$$\sigma_\omega^-(s) = \begin{cases} \epsilon - \omega(s) & \text{if } \omega(s) < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

$$\sigma_\omega^+(s) = \begin{cases} \omega(s) - \epsilon & \text{if } \omega(s) > \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\omega(A) = \sigma_\omega^+(A) - \sigma_\omega^-(A) + \epsilon$ for all $A \in \mathcal{B}_S$. Clearly, ω is increasingly (respectively, decreasingly) monotonic if $\sigma_\omega^- = 0$ (respectively, $\sigma_\omega^+ = 0$) and $\epsilon \geq 0$.

The case that σ_ω^- and σ_ω^+ vanish (i.e., a constant threshold model) is developed by Fishburn (1969). We examine three cases below. The first case, which is dealt with in the following theorem, is that σ_ω^- and σ_ω may not vanish, but it ratains the inclusion monotonicity. The second and third are respectively concerned with $\sigma_\omega^- = 0$ and $\sigma_\omega^- = \epsilon = 0$.

Theorem 5.1 *There exist a probability measure P on $\mathcal{B}_S$ and an additive set function $\omega \geq 0$ on $\mathcal{B}_S$ such that $P + \omega$ is increasingly monotonic, and for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \iff P(A) > P(B) + \omega(B)$$

if and only if $\succ$ is irreflexive and upward and downward inclusion monotonic, and the following axiom holds for all $A_1, A_2, \ldots, B_1, B_2, \ldots, C_1, C_2, \ldots, D_1, D_2, \ldots \in \mathcal{B}_S$ and all integers $m \geq 1$,

A5.1 *if not($C_i \succ D_i$) for $i = 1, \ldots, m$, and for all $s \in S$,*

$$\sum_{i=1}^m I(s; A_i) = \sum_{i=1}^m I(s; C_i),$$
$$\sum_{i=1}^m I(s; B_i) = \sum_{i=1}^m I(s; D_i),$$

then it is false that $A_i \succ B_i$ for $i = 1, \ldots, m$.

We do not explicitly assume that $\succ$ is an interval order. To see that $\succ$ is an interval order, suppose that $A \succ B$, $C \succ D$, not($A \succ D$), and not($C \succ B$). Let $(A_1, A_2) = (A, C)$, $(B_1, B_2) = (B, D)$, $(C_1, C_2) = (A, C)$, and $(D_1, D_2) = (D, B)$ for A5.1, so that the hypotheses of A5.1 hold. Hence A5.1 requires that both of $A \succ B$ and $C \succ D$ do not hold, a contradiction. Hence $\succ$ must be an interval order, since $\succ$ is irreflexive.

It is easy to see that irreflexivity and upward and downward inclusion monotonicities are necessary for the representation. To see that A5.1 is also necessary, suppose that $\sum_{i=1}^m I(s; A_i) = \sum_{i=1}^m I(s; C_i)$, $\sum_{i=1}^m I(s; B_i) = \sum_{i=1}^m I(s; D_i)$, and not($C_i \succ D_i$) and $A_i \succ B_i$ for $i = 1, \ldots, m$. For each i, $P(A_i) > P(B_i) + \omega(B_i)$

and $P(D_i) + \omega(D_i) \geq P(C_i)$. Summing over i, we get

$$\sum_{i=1}^{m} P(A_i) > \sum_{i=1}^{m} P(B_i) + \sum_{i=1}^{m} \omega(B_i),$$
$$\sum_{i=1}^{m} P(D_i) + \sum_{i=1}^{m} \omega(D_i) \geq \sum_{i=1}^{m} P(C_i),$$

which are combined to give

$$\sum_{i=1}^{m} P(A_i) + \sum_{i=1}^{m} P(D_i) + \sum_{i=1}^{m} \omega(D_i) > \sum_{i=1}^{m} P(B_i) + \sum_{i=1}^{m} \omega(B_i) + \sum_{i=1}^{m} P(C_i).$$

Since P and ω are additive, using $\sum_{i=1}^{m} I(s; A_i) = \sum_{i}^{m} I(s; C_i)$ and $\sum_{i=1}^{m} I(s; B_i) = \sum_{i=1}^{m} I(s; D_i)$, we have

$$\begin{aligned} \sum_{i=1}^{m} P(A_i) &= \sum_{i=1}^{m} P(C_i), \\ \sum_{i=1}^{m} P(B_i) + \sum_{i=1}^{m} \omega(B_i) &= \sum_{i=1}^{m} P(D_i) + \sum_{i=1}^{m} \omega(D_i), \end{aligned}$$

which contradict the above strict inequality. Hence A5.1 is necessary for the representation.

We note that A5.1 implies A4.2. To see this, assume that A5.1 and the hypotheses of A4.2 hold, i.e., $\sum_{i=1}^{m} I(s; A_i) = \sum_{i=1}^{m} I(s; B_i)$ for all $s \in S$. Suppose that there are $C_1, \ldots, C_m \in \mathcal{B}_S$ such that $A_i \succ C_i \sim B_i$ for $i = 1, \ldots, m$, i.e., $A_i \succ^* B_i$. Since $\sum_{i=1}^{m} I(s; C_i) = \sum_{i=1}^{m} I(s; C_i)$ for all $s \in S$, it is false by A5.1 that $A_i \succ C_i$ for $i = 1, \ldots, m$. This is a contradiction, so that A4.2 holds.

Note that the conditions of Theorem 5.1 excluding the downward inclusion monotonicity are necessary and sufficient for the representation of Theorem 5.1 without increasing monotonicity of $P + \omega$.

Although the conditions of Theorem 4.2 do not imply the representation of Theorem 3.3 as shown by Example 4.3, the representation of Theorem 5.1 can be modified to the representation of Theorem 3.3 as shown below. Suppose that Theorem 5.1 holds with an additive univariate threshold function ω. Let $\epsilon = \omega(\emptyset) \geq 0$. Since there are additive univariate set functions, $\sigma_\omega^- \geq 0$ and $\sigma_\omega^+ \geq 0$, on $\mathcal{B}_S$ such that $\sigma_\omega^- = \sigma_\omega^+(\emptyset) = 0$ and $\omega = \sigma_\omega^+ - \sigma_\omega^- + \epsilon$, it follows that, for all $A, B \in \mathcal{B}_S$,

$$\begin{aligned} & P(A) > P(B) + \omega(B) \\ \iff\ & P(A) > P(B) + \sigma_\omega^+(B) - \sigma_\omega^-(B) + \epsilon \\ \iff\ & P(A) - \sigma_\omega^-(A) > P(B) - \sigma_\omega^-(B) - \sigma_\omega^-(A) + \sigma_\omega^+(B) + \epsilon \\ \iff\ & Q(A) - \omega^-(A) > Q(B) + \omega^+(B), \end{aligned}$$

where $Q = P - \sigma_\omega^- + \epsilon$, $\omega^- = \epsilon - \sigma_\omega^-$, and $\omega^+ = \sigma_\omega^+$. By the definitions of σ_ω^- and σ_ω^+, we have that $Q \geq 0$, $\omega^- \geq 0$, $\omega^+ \geq 0$, ω^- is decreasingly monotonic, and

ω^+ is increasingly monotonic. By an appropriate normalization of Q, we obtain the representation of Theorem 3.3.

The case that σ_ω^- vanishes is covered by the following theorem.

Theorem 5.2 *There exist a probability measure P on $\mathcal{B}_S$ and an increasingly monotonic additive set function $\omega \geq 0$ on $\mathcal{B}_S$ such that for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega(B)$$

if and only if not($\emptyset \succ \emptyset$), and the following axiom holds for all $A_1, A_2, \ldots,\ B_1, B_2, \ldots,\ C_1, C_2, \ldots,\ D_1, D_2, \ldots \in \mathcal{B}_S$ and all integers $m \geq 1$,

A5.2 *if not($C_i \succ D_i$) for $i = 1, \ldots, m$, and for all $s \in S$,*

$$\begin{aligned} \sum_{i=1}^m I(s; A_i) + \sum_{i=1}^m I(s; D_i) &\leq \sum_{i=1}^m I(s; B_i) + \sum_{i=1}^m I(s; C_i), \\ \sum_{i=1}^m I(s; B_i) &\geq \sum_{i=1}^m I(s; D_i), \end{aligned}$$

then it is false that $A_i \succ B_i$ for $i = 1, \ldots, m$.

We note that the conditions of Theorem 5.2 are sufficient for the representation of Theorem 5.1 as shown below. It easy to see that A5.2 implies A5.1. Let $m = 1$, $A_1 = A$, $B_1 = A$, $C_1 = \emptyset$, and $D_1 = A$ for A5.2. Since not($\emptyset \succ \emptyset$), A5.2 implies not($A \succ A$), so $\succ$ is irreflexive. To see that $\succ$ is upward inclusion monotonic, suppose that $A \supset B$, $B \succ C$, and not($A \succ C$). Then let $m = 1$, $A_1 = B$, $B_1 = D_1 = C$, and $C_1 = A$ for A5.2. Then by A5.2, not($B \succ C$), a contradiction. To see that $\succ$ is downward inclusion monotonic, suppose that $A \succ B$, $B \supset C$, and not($A \succ C$). Then let $m = 1$, $A_1 = C_1 = A$, $B_1 == B$, and $D_1 = C$ for A5.2. Then by A5.2, not($A \succ B$), a contradiction.

The following two examples show that the representations of Theorem 5.1 and 4.3 do not generally imply each other.

Example 5.1 Let $S = \{s_1, s_2, s_3\}$ with $P(s_1) = 0.2$, $P(s_2) = P(s_3) = 0.4$, $\omega(\emptyset) = \omega(s_1) = 0.3$, $\omega(s_2) = 0.1$, and $\omega(s_3) = 0.4$. Since $P + \omega$ is increasingly monotonic, we define $\succ$ on $\mathcal{B}_S$ according to the representation of Theorem 5.1. Then $\{s_2, s_3\} \succ \{s_1, s_2\}$ since $P(s_2, s_3) = 0.8 > P(s_1, s_2) + \omega(s_1, s_2) = 0.7$, and $\{s_1\} \sim \{s_3\}$ since $P(s_1) = 0.2 < P(s_3) + \omega(s_3) = 0.8$ and $P(s_3) = 0.4 < P(s_1) + \omega(s_1) = 0.5$. On the other hand, let $m = 1$, $A_1 = \{s_2, s_3\}$, $B_1 = \{s_1, s_2\}$, $C_1 = \{s_3\}$, and $D_1 = \{s_1\}$ for A4.3. Since not($C_1 \succ D_1$), $I(s; A_1) + I(s; D_1) = I(s; B_1) + I(s; C_1)$, and $D_1 \subseteq B_1$, A4.3 implies not($A_1 \succ B_1$). This is a contradiction.

Example 5.2 Suppose that the representation of Theorem 4.3 holds with a probability measure P and an increasingly monotonic set function $\omega \geq 0$. Let $S = \{s_1, \ldots, s_6\}$ with $P(s_1) = P(s_2) = P(s_3) = 0.1$, $P(s_4) = P(s_5) = 0.2$, $P(s_6) = 0.3$, $\omega(s_1) = 0.05$, $\omega(s_2) = 0.1$, $\omega(s_1, s_3) = 0.3$, and $\omega(s_2, s_3) = 0.1$. Let $A_1 = \{s_4\}$,

$A_2 = \{s_5, s_6\}$, $C_1 = \{s_5\}$, $C_2 = \{s_4, s_6\}$, $B_1 = \{s_1\}$, $B_2 = \{s_2, s_3\}$, $D_1 = \{s_2\}$, and $D_2 = \{s_1, s_3\}$. Then we have

$$\begin{aligned} P(A_1) = 0.2 &> P(B_1) + \omega(B_1) = 0.15, \\ P(A_2) = 0.5 &> P(B_2) + \omega(B_2) = 0.3, \\ P(D_1) + \omega(D_1) = 0.2 &\geq P(C_1) = 0.2, \\ P(D_2) + \omega(D_2) = 0.5 &\geq P(C_2) = 0.5, \end{aligned}$$

so that $A_1 \succ B_1$, $A_2 \succ B_2$, not$(C_1 \succ D_1)$, and not$(C_2 \succ D_2)$. However, $I(s; A_1) + I(s; A_2) = I(s; C_1) + I(s; C_2)$ and $I(s; B_1) + I(s; B_2) = I(s; D_1) + I(s; D_2)$ for all $s \in S$. This violates Axiom A5.1.

The last case deals with $\sigma_\omega^+ = \epsilon = 0$ as follows.

Theorem 5.3 *There exist a probability measure P and an additive set function $\omega \geq 0$ on $\mathcal{B}_S$ such that $\omega(\emptyset) = 0$ and for all $A, B \in \mathcal{B}_S$,*

$$A \succ B \Longleftrightarrow P(A) > P(B) + \omega(B),$$

if and only if $\succ$ is irreflexive and upward inclusion monotonic, and the following axiom holds for all $A_1, A_2, \ldots, B_1, B_2, \ldots, C_1, C_2, \ldots, D_1, D_2, \ldots \in \mathcal{B}_S$, all integers $m \geq 1$, and all integers $\ell \geq 1$,

A5.3 *if not$(C_i \succ D_i)$ for $i = 1, \ldots, \ell$, and for all $s \in S$,*

$$\begin{aligned} \sum_{i=1}^{m} I(s; A_i) &= \sum_{i=1}^{\ell} I(s; C_i), \\ \sum_{i=1}^{m} I(s; B_i) &= \sum_{i=1}^{\ell} I(s; D_i), \end{aligned}$$

then it is false that $A_i \succ B_i$ for $i = 1, \ldots, m$.

It is easy to see that A5.3 implies A5.2. The downward inclusion monotonicity follows from A5.3. To see this, suppose $A \succ B$, $B \supseteq C$, and not$(A \succeq C$. Let $A_1 = A$, $B_1 = B$, $(C_1, C_2) = (A, \emptyset)$, and $(D_1, D_2) = (C, B \setminus C)$ for A5.3. By the upward inclusion monotonicity, not$(C_2 \succ D_2)$. Since not$(C_1 \succ D_1)$, A5.3 implies not$(A \succ B)$, a contradiction. Hence the downward inclusion monotonicity holds.

6 Sufficiency Proofs

This section provides sufficiency proofs of the theorems in the preceding sections except Theorems 2.1 and 3.2. The necessities of the axioms were noted in those sections. Our sufficiency proofs use the following version of the familiar lemma for the existence of a solution to a finite system of linear inequalities (see Fishburn, 1970). When $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ are N dimensional vectors of real numbers, denote the inner product by $a \cdot b = \sum_{i=1}^{N} a_i b_i$. A real vector is called *rational* if each component is a rational number, and is called *integral* if each of its components is an integer.

Lemma 1 *Let $a^1, \ldots, a^M$ be N dimensional rational vectors and $1 \leq K \leq M$. Then either there is an N dimensional integral vector ρ such that*

$$\begin{aligned} \rho \cdot a^k &> 0 \quad \text{for } k = 1, \ldots, K, \\ \rho \cdot a^k &\geq 0 \quad \text{for } k = K+1, \ldots, M, \end{aligned}$$

or else there are nonnegative integers $\alpha_1, \ldots, \alpha_M$, with $\alpha_k > 0$ for some $k \leq K$, such that

$$\sum_{k=1}^{M} \alpha_k a_j^k = 0 \text{ for } j = 1, \ldots, N.$$

Note that the last equations in the lemma are described in the vector form by

$$\sum_{k=1}^{M} \alpha_k a^k = 0,$$

where 0 is an N dimensional zero vector. Since this equation says that some of $a^1, \ldots, a^M$ are linearly dependent, we shall call it the *linearly dependent* (LD) equation.

Throughout the proofs we shall let n be the number of states in S with $S = \{s_1, \ldots, s_n\}$ and $\mathcal{B}_S = \{S_1, \ldots, S_{2^n}\}$. Let $\mathcal{B}_S^2$ be the set of all subsets of $\mathcal{B}_S$ and define indicator functions $\tilde{I}$ on $\mathcal{B}_S \times \mathcal{B}_S^2$ and $\tilde{J}$ on $\mathcal{B}_S \times \mathcal{B}_S \times \mathcal{B}_S^2 \times \mathcal{B}_S^2$ by

$$\begin{aligned} \tilde{I}(A; \tilde{A}) &= \begin{cases} 1 & \text{if } A \in \tilde{A} \\ 0 & \text{otherwise.} \end{cases} \\ \tilde{J}(A, B; \tilde{A}, \tilde{B}) &= \begin{cases} 1 & \text{if } (A, B) \in \tilde{A} \times \tilde{B} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For a univariate set function Q on $\mathcal{B}_S$ and a bivariate set function R on $\mathcal{B}_S \times \mathcal{B}_S$, we define an n dimensional row vector ρ_Q, a 2^{2n} dimensional row vector $\tilde{\rho}_R$, and a 2^n dimensional row vector $\tilde{\rho}_Q$ by

$$\begin{aligned} \rho_Q &= (Q(\{s_1\}), \ldots, Q(\{s_n\})), \\ \tilde{\rho}_R &= (R(S_1, S_1), \ldots, R(S_1, S_{2^n}), \ldots, R(S_{2^n}, S_1), \ldots, R(S_{2^n}, S_{2^n})), \\ \tilde{\rho}_Q &= (Q(S_1), \ldots, Q(S_{2^n})). \end{aligned}$$

For $A, B \in \mathcal{B}_S$, we define three column vectors, $\theta(A)$ with dimension n, $\tilde{\theta}(A)$ with dimension 2^n, and $\tilde{\tau}(A, B)$ with dimension 2^{2n} by

$$\begin{aligned} \theta_i(A) &= I(s_i; A) && \text{for } i \leq i \leq n, \\ \tilde{\theta}_i(A) &= \tilde{I}(S_i; \{A\}) && \text{for } 1 \leq i \leq 2^n, \\ \tilde{\tau}_{2^n(j-1)+k}(A, B) &= \tilde{J}(S_j, S_k; \{A\}, \{B\}) && \text{for } 1 \leq j \leq 2^n \text{ and } 1 \leq k \leq 2^n. \end{aligned}$$

Note that $\tilde{\theta}$ and $\tilde{\tau}$ are unit vectors. Given $\succ$ on $\mathcal{B}_S$, let

$$\begin{aligned}
\mathcal{P}_{\succ} &= \{(A,B) \in \mathcal{B}_S \times \mathcal{B}_S : A \succ B\},\\
\mathcal{P}_{\sim} &= \{(A,B) \in \mathcal{B}_S \times \mathcal{B}_S : A \sim B\},\\
S^0 &= \{s \in S : \{s\} \sim \emptyset\},\\
\mathcal{P}_{\supset} &= \{(A,B) \in \mathcal{B}_S \times \mathcal{B}_S : A \supset B\},
\end{aligned}$$

and enumerate $\mathcal{P}_{\succ}$ as $(A_1,B_1),\ldots,(A_{L_1},B_{L_1})$, half of $\mathcal{P}_{\sim}$ as $(C_1,D_1),\ldots,(C_{L_2},D_{L_2})$ by using only one of (A,B) and (B,A) when $A \sim B$, S^0 as $s^1,\ldots,s^{L_3}$, and $\mathcal{P}_{\supset}$ as $(E_1,F_1),\ldots,(E_{L_4},F_{L_4})$.

Sufficiency Proof of Theorem 2.2 We assume that $\succ$ is upward and downward inclusion monotonic and axiom A2.2 holds.

To specify our system of linear inequalities, suppose that the representation of the theorem holds with a probability meausre P on $\mathcal{B}_S$ and a skew-monotone bivariate set funciton $\Omega \geq 0$ on $\mathcal{B}_S \times \mathcal{B}_S$ satisfying that

(1a) $P(A) - P(B) - \Omega(A,B) > 0$ for all $A, B \in \mathcal{B}_S$ such that $A \succ B$,
(1b) $P(B) - P(A) + \Omega(A,B) \geq 0$ and $P(A) - P(B) + \Omega(B,A) \geq 0$ for all $A, B \in \mathcal{B}_S$ such that $A \sim B$.

Then letting $\rho = (\rho_P, \tilde{\rho}_\Omega)$ be an $(n + 2^{2n})$-dimensional row vector, our system of linear inequalities is stated as follows.

(a) $$\rho \cdot \begin{bmatrix} \theta(A_i) - \theta(B_i) \\ -\tilde{\tau}(A_i, B_i) \end{bmatrix} > 0 \qquad \text{for } i = 1,\ldots,L_1,$$

(b) $$\rho \cdot \begin{bmatrix} \theta(D_i) - \theta(C_i) \\ \tilde{\tau}(C_i, D_i) \end{bmatrix} \geq 0 \qquad \text{and}$$

$$\rho \cdot \begin{bmatrix} \theta(C_i) - \theta(D_i) \\ \tilde{\tau}(D_i, C_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1,\ldots,L_2,$$

(c) $$\rho \cdot \begin{bmatrix} \theta(\{s^i\}) \\ 0 \end{bmatrix} \geq 0 \qquad \text{for } i = 1,\ldots,L_3,$$

(d) $$\rho \cdot \begin{bmatrix} 0 \\ \tilde{\tau}(F_i, S_j) - \tilde{\tau}(E_i, S_j) \end{bmatrix} \geq 0 \qquad \text{and}$$

$$\rho \cdot \begin{bmatrix} 0 \\ \tilde{\tau}(S_j, E_i) - \tilde{\tau}(S_j, F_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1,\ldots,L_4, \text{ and } j = 1,\ldots,2^n,$$

(e) $$\rho \cdot \begin{bmatrix} 0 \\ \tilde{\tau}(S, \emptyset) \end{bmatrix} \geq 0.$$

Inequalities (a) and (b) follow from (1a) and (1b), respectively. Nonnegativity of P is reflected in (c), since the case of $\{s\} \succ \emptyset$ is covered by (a). Inequalities (d) follow from skew-monotonicity of Ω. Nonnegativity of Ω follows from skew-monotonicity and (e), since $\Omega(\emptyset, S) \geq \Omega(A, B) \geq \Omega(S, \emptyset)$ for all $A, B \in \mathcal{B}_S$.

The sufficiency proof is completed by establishing that the system of linear inequalities (a) through (e) has a ρ solution. Suppose on the contrary that there is no ρ solution. Then it follows from Lemma 1 that there are nonnegative integers, α_i for $i = 1, \ldots, L_1$, β_{ik} for $i = 1, \ldots, L_2$ and $k = 1, 2$, γ_i for $i = 1, \ldots, L_3$, δ_{ijk} for $i = 1, \ldots, L_4$, $j = 1, \ldots, 2^n$, and $k = 1, 2$, and ϵ such that $\alpha_k > 0$ for some $1 \leq k \leq L_1$, and

$$\sum_{i=1}^{L_1} \alpha_i \begin{bmatrix} \theta(A_i) - \theta(B_i) \\ -\bar{\tau}(A_i, B_i) \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i1} \begin{bmatrix} \theta(D_i) - \theta(C_i) \\ \bar{\tau}(C_i, D_i) \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i2} \begin{bmatrix} \theta(C_i) - \theta(D_i) \\ \bar{\tau}(D_i, C_i) \end{bmatrix}$$
$$+ \sum_{i=1}^{L_3} \gamma_i \begin{bmatrix} \theta(\{s^i\}) \\ 0 \end{bmatrix} + \sum_{i=1}^{L_4} \sum_{j=1}^{2^n} \delta_{ij1} \begin{bmatrix} 0 \\ \bar{\tau}(F_i, S_j) - \bar{\tau}(E_i, S_j) \end{bmatrix}$$
$$+ \sum_{i=1}^{L_4} \sum_{j=1}^{2^n} \delta_{ij2} \begin{bmatrix} 0 \\ \bar{\tau}(S_j, E_i) - \bar{\tau}(S_j, F_i) \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ \bar{\tau}(S, \emptyset) \end{bmatrix} = 0.$$

In what follows, we show that $\beta_{ik} = 0$ for $i = 1, \ldots, L_2$ and $k = 1, 2$. When this is the case, the first n rows of the LD equation give

$$\sum_{i=1}^{L_1} \alpha_i [\theta(A_i) - \theta(B_i)] + \sum_{i=1}^{L_3} \gamma_i \theta(\{s^i\}) = 0.$$

List the elements of $\mathcal{P}_{\succ}$ and S^0 with α_i repeats for (A_i, B_i) and γ_i repeats for s^i, and enumerate them as $(A_1^*, B_1^*), \ldots, (A_m^*, B_m^*)$ for $\mathcal{P}_{\succ}$ and $t^1, \ldots, t^K$ for S^0, where $m = \sum_{i=1}^{L_1} \alpha_i$ and $K = \sum_{i=1}^{L_3} \gamma_i$. Then we get

$$\sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{K} I(s; \{t^i\}) = \sum_{i=1}^{m} I(s; B_i^*).$$

Since, for each $i = 1, \ldots, K$, there is a $1 \leq k \leq m$ such that $t^i \in B_k^*$, we can get rid of one t^i at a time by reducing a B_k^* that contains the t^i to $B_k^* \setminus \{t^i\}$. Continuing this reduction process for all t^i, we arrive at a reduced B set $\{B_1', \ldots, B_m'\}$ for which $B_i^* \supseteq B_i'$ for $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} I(s; A_i^*) = \sum_{i=1}^{m} I(s; B_i').$$

By the downward inclusion monotonicity, $A_i \succ B_i'$ for $i = 1, \ldots, m$, which violates A2.2, so that there must be a ρ solution.

Suppose that $E \supseteq F$, $G \supseteq H$, $E' \supseteq F'$, and $G' \supseteq H'$. If $(F,G) = (E', H')$, then

$$\begin{bmatrix} 0 \\ \tilde{\tau}(F,G) - \tilde{\tau}(E,H) \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{\tau}(F',G') - \tilde{\tau}(E',H') \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ \tilde{\tau}(F',G') - \tilde{\tau}(E,H) \end{bmatrix},$$

where $E \supseteq F'$ and $G' \supseteq H$. Noting this fact, there are ℓ quadraples $(E_i^*, F_i^*, G_i^*, H_i^*) \in \mathcal{B}_S \times \mathcal{B}_S \times \mathcal{B}_S \times \mathcal{B}_S$ for $i = 1, \ldots, \ell$ such that for all i, $E_i^* \supseteq F_i^*$ and $G_i^* \supseteq H_i^*$, and

$$\sum_{i=1}^{L_4}\sum_{j=1}^{2^n} \delta_{ij1} \begin{bmatrix} 0 \\ \tilde{\tau}(F_i, S_j) - \tilde{\tau}(E_i, S_j) \end{bmatrix} + \sum_{i=1}^{L_4}\sum_{j=1}^{2^n} \delta_{ij2} \begin{bmatrix} 0 \\ \tilde{\tau}(S_j, E_i) - \tilde{\tau}(S_j, F_i) \end{bmatrix}$$
$$= \sum_{i=1}^{\ell} \begin{bmatrix} 0 \\ \tilde{\tau}(F_i^*, G_i^*) - \tilde{\tau}(E_i^*, H_i^*) \end{bmatrix}.$$

Since $\tilde{\tau}$ is a unit vector, it follows from the LD equation that for each $i = 1, \ldots, \ell$, there is an $1 \leq i^* \leq L_1$ such that $\alpha_{i^*} > 0$ and $(A_{i^*}, B_{i^*}) = (F_i^*, G_i^*)$; so that $F_i^* \succ G_i^*$.

Similaly, for each $i = 1, \ldots, \ell$, we have that either, for some $1 \leq j^* \leq L_2$ and some $k = 1, 2$, $\beta_{j^*k} > 0$ and

$$(E_i^*, H_i^*) = \begin{cases} (C_{j^*}, D_{j^*}) & \text{when } k = 1, \\ (D_{j^*}, C_{j^*}) & \text{when } k = 2, \end{cases}$$

or $(E_i^*, H_i^*) = (S, \emptyset)$. Suppose that $(E_i^*, H_i^*) \neq (S, \emptyset)$. Then $E_i^* \sim H_i^*$. On the other hand, since $E_i^* \supseteq F_i^* \succ G_i^* \supseteq H_i^*$, it follows from upward and downward inclusion monotonicities that $E_i^* \succ H_i^*$. This is a contradiction. Hence $(G_i, H_i) = (S, \emptyset)$ for all $i = 1, \ldots, \ell$. Thus we have $\epsilon = \ell$ and $\beta_{ik} = 0$ for $i = 1, \ldots, L_2$ and $k = 1, 2$. □

Sufficiency Proofs of Theorems 3.1 and 3.3 We assume that $\succ$ is an interval order, and either axiom A2.1 for Theorem 3.1 holds or upward and downward inclusion monotonicities and axiom A3.3 hold for Theorem 3.3.

To specify our system of linear inequalities, suppose that there exist a probability measure P on $\mathcal{B}_S$ and set functions, $\omega^+ \geq 0$ and $\omega^- \geq 0$, on $\mathcal{B}_S$ such that

(1a) $P(A) - P(B) - \omega^-(A) - \omega^+(B) > 0$ for all $A, B \in \mathcal{B}_S$ such that $A \succ B$,
(1b) $P(B) - P(A) + \omega^-(A) + \omega^+(B) \geq 0$ and $P(A) - P(B) + \omega^-(B) + \omega^+(A) \geq 0$ for all $A, B \in \mathcal{B}_S$ such that $A \sim B$.

Then letting $\rho = (\rho_P, \tilde{\rho}_{\omega^-}, \tilde{\rho}_{\omega^+})$ be an $(n+2^{n+1})$-dimensional row vector, our system of linear inequalities is stated as follows.

(a) $$\rho \cdot \begin{bmatrix} \theta(A_i) - \theta(B_i) \\ -\tilde{\theta}(A_i) \\ -\tilde{\theta}(B_i) \end{bmatrix} > 0 \qquad \text{for } i = 1, \ldots, L_1,$$

(b)

$$\rho \cdot \begin{bmatrix} \theta(D_i) - \theta(C_i) \\ \tilde{\theta}(C_i) \\ \tilde{\theta}(D_i) \end{bmatrix} \geq 0 \qquad \text{and}$$

$$\rho \cdot \begin{bmatrix} \theta(C_i) - \theta(D_i) \\ \tilde{\theta}(D_i) \\ \tilde{\theta}(C_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \dots, L_2,$$

(c)

$$\rho \cdot \begin{bmatrix} \theta(\{s^i\}) \\ 0 \\ 0 \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \dots, L_3,$$

Inequalities (a) and (b) follow from (1a) and (1b), respectively. Nonnegativity of P is reflected in (c), since the case of $\{s\} \succ \emptyset$ is covered by (a).

For Theorem 3.1, nonnegativities of ω^- and ω^+ are reflected by

(d)

$$\rho \cdot \begin{bmatrix} 0 \\ \tilde{\theta}(S_i) \\ 0 \end{bmatrix} \geq 0 \qquad \text{and}$$

$$\rho \cdot \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta}(S_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \dots, 2^n,$$

For Theorem 3.3, decreasing monotonicity of ω^- and increasing monotonicity of ω^+ are respectively reflected by

(e)

$$\rho \cdot \begin{bmatrix} 0 \\ \tilde{\theta}(F_i) - \tilde{\theta}(E_i) \\ 0 \end{bmatrix} \geq 0. \qquad \text{and}$$

$$\rho \cdot \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta}(E_i) - \tilde{\theta}(F_i) \end{bmatrix} \geq 0. \qquad \text{for } i = 1, \dots, L_4,$$

Together with (e), the following inequalities assure that ω^+ and ω^- are nonnegative.

(f)

$$\rho \cdot \begin{bmatrix} 0 \\ \tilde{\theta}(S) \\ 0 \end{bmatrix} \geq 0.$$

$$\rho \cdot \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta}(\emptyset) \end{bmatrix} \geq 0.$$

The proof is completed by establishing that the system of linear inequalities (a)–(d) for Theorem 3.1, and (a)–(c), (e), and (f) for Theorem 3.3 has a ρ solution. Suppose on the contrary that there is no ρ solution. Then it follows from Lemma 1 that there are nonnegative integers, α_i for $i = 1, \ldots, L_1$, β_{ij} for $i = 1, \ldots, L_2$ and $j = 1, 2$, γ_i for $i = 1, \ldots, L_3$, δ_{ij} for $i = 1, \ldots, 2^n$ and $j = 1, 2$, ϵ_{ij} for $i = 1, \ldots, L_4$ and $j = 1, 2$, and κ_j for $j = 1, 2$ such that $\alpha_k > 0$ for some $1 \leq k \leq L_1$, $\epsilon_{ij} = \kappa_j = 0$ for all i, j in case of Theorem 3.1, $\delta_{ij} = 0$ for all i, j in case of Theorem 3.3, and

$$\sum_{i=1}^{L_1} \alpha_i \begin{bmatrix} \theta(A_i) - \theta(B_i) \\ -\tilde{\theta}(A_i) \\ -\tilde{\theta}(B_i) \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i1} \begin{bmatrix} \theta(D_i) - \theta(C_i) \\ \tilde{\theta}(C_i) \\ \tilde{\theta}(D_i) \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i2} \begin{bmatrix} \theta(C_i) - \theta(D_i) \\ \tilde{\theta}(D_i) \\ \tilde{\theta}(C_i) \end{bmatrix}$$

$$+ \sum_{i=1}^{L_3} \gamma_i \begin{bmatrix} \theta(\{s^i\}) \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{2^n} \delta_{i1} \begin{bmatrix} 0 \\ \tilde{\theta}(S_i) \\ 0 \end{bmatrix} + \sum_{i=1}^{2^n} \delta_{i2} \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta}(S_i) \end{bmatrix}$$

$$+ \sum_{i=1}^{L_4} \epsilon_{i1} \begin{bmatrix} 0 \\ \tilde{\theta}(F_i) - \tilde{\theta}(E_i) \\ 0 \end{bmatrix} + \sum_{i=1}^{L_4} \epsilon_{i2} \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta}(E_i) - \tilde{\theta}(F_i) \end{bmatrix}$$

$$+ \kappa_1 \begin{bmatrix} 0 \\ \tilde{\theta}(S) \\ 0 \end{bmatrix} + \kappa_2 \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta}(\emptyset) \end{bmatrix} = 0.$$

List the elements of $\mathcal{P}_\succ$, $\mathcal{P}_\sim$, S^0, and $\mathcal{P}_\sqsupset$ with α_i repeats for (A_i, B_i), β_{i1} repeats for (C_i, D_i), β_{i2} repeats for (D_i, C_i), γ_i repeats for s^i, ϵ_{i1} repeats for (F_i, E_i), and ϵ_{i2} repeats for (E_i, F_i), and then enumerate them as $(A_1^*, B_1^*), \ldots, (A_{m_1}^*, B_{m_1}^*)$ for $\mathcal{P}_\succ$, $(C_1^*, D_i^*), \ldots, (C_{m_2}^*, D_{m_2}^*)$ for $\mathcal{P}_\sim$, $t^1, \ldots, t^K$ for S^0, $(G_1^*, H_1^*), \ldots, (G_{\ell_1}^*, H_{\ell_1}^*)$ for reversely ordered pairs of $\mathcal{P}_\sqsupset$, and $(E_1^*, F_1^*), \ldots, (E_{\ell_2}^*, F_{\ell_2}^*)$ for $\mathcal{P}_\sqsupset$, where $m_1 = \sum_{i=1}^{L_1} \alpha_i$, $m_2 = \sum_{i=1}^{L_2} (\beta_{i1} + \beta_{i2})$, $K = \sum_{i=1}^{L_3}$, $\ell_1 = \sum_{i=1}^{L_4} \epsilon_{i1}$, and $\ell_2 = \sum_{i=1}^{L_4} \epsilon_{i2}$.

First we show a sufficiency proof of Theorem 3.1, so that $\epsilon_{ij} = \kappa_j = 0$ for all i, j. Suppose that $m_2 > 0$. Then it follows from the LD equation that $m_1 > 1$. For each $1 \leq k \leq m_2$, there are $1 \leq i \leq m_1$ and $1 \leq j \leq m_1$ such that $i \neq j$, $A_i^* = C_k^*$, and $B_j^* = D_j^*$, so that

$$\begin{bmatrix} \theta(A_i^*) - \theta(B_i^*) \\ -\tilde{\theta}(A_i^*) \\ -\tilde{\theta}(B_i^*) \end{bmatrix} + \begin{bmatrix} \theta(A_j^*) - \theta(B_j^*) \\ -\tilde{\theta}(A_j^*) \\ -\tilde{\theta}(B_j^*) \end{bmatrix} + \begin{bmatrix} \theta(D_k^*) - \theta(C_k^*) \\ -\tilde{\theta}(C_k^*) \\ -\tilde{\theta}(D_k^*) \end{bmatrix}$$

$$= \begin{bmatrix} \theta(A_j^*) - \theta(B_i^*) \\ -\tilde{\theta}(A_j^*) \\ -\tilde{\theta}(B_i^*) \end{bmatrix}.$$

Since $\succ$ is an interval order, we have that $A_j^* \succ B_i^*$. Thus we delete the pairs (A_i^*, B_i^*), (A_j^*, B_j^*), and (C_k^*, D_k^*) from the original sequence of the (A, B)- pairs. Then we add the pair (A_j^*, B_i^*) as a new (A, B)-pair, and reenumerate the remaining

(A, B)-pairs and (C, D)-pairs. This process continues to delete all (C, D)-pairs, so that the remaining (A, B)-pairs are enumerated as $(A'_1, B'_1), \ldots, (A'_m, B'_m)$, where $m = m_1 - m_2$. Hence it follows from the first n rows of the LD equation that

$$\sum_{i=1}^{m}[\theta(A'_i) - \theta(B'_i)] + \sum_{i=1}^{K}[\theta(\{t^i\})] = 0,$$

which is equivalent to

$$\sum_{i=1}^{m} I(s; A'_i) \leq \sum_{i=1}^{m} I(s; B'_i).$$

Since $A'_i \succ B'_i$ for $i = 1, \ldots, m$, A2.1 is violated. Hence, there must be a ρ solution.

Now we show a sufficiency proof of Theorem 3.3, so that $\delta_{ij} = 0$ for all i, j. By a similar argument to the preceding paragraph, we assume with no loss of generality that, for every $1 \leq k \leq m_2$, there exist no i, j such that $A^*_i = C^*_k$ and $B^*_j = D^*_k$. Furthermore, we assume by appropriate cancellation that $G^*_i = H^*_j$ for no i, j, $E^*_i = F^*_j$ for no i, j, $H^*_i \neq S$ for all i, and $F^*_i \neq \emptyset$ for all i.

It follows from the LD equation and renumbering the pairs (A^*_i, B^*_i) for $i = 1, \ldots, m_1$ and the pairs (C^*_i, D^*_i) for $i = 1, \ldots, m_2$ that $\ell_1 + \ell_2 \geq m_2$, $\kappa_1 = \kappa_2$, $m_1 = m_2 + \kappa_1$, and there are permutations π_1 on $\{1, \ldots, m_1\}$ and π_2 on $\{1, \ldots, m_2\}$ such that

$$\begin{array}{lcll}
A^*_{\pi_1(i)} & = & G^*_i & \text{for } i = 1, \ldots, \ell_1, \\
C^*_{\pi_2(i)} & = & H^*_i & \text{for } i = 1, \ldots, \ell_1, \\
A^*_{\pi_1(i)} & = & C^*_{\pi_2(i)} & \text{for } i = \ell_1 + 1, \ldots, m_2, \\
A^*_{\pi_1(i)} & = & S & \text{for } i = m_2 + 1, \ldots, m_2 + \kappa_1, \\
B^*_i & = & E^*_i & \text{for } i = 1, \ldots, \ell_2, \\
D^*_i & = & F^*_i & \text{for } i = 1, \ldots, \ell_2, \\
B^*_i & = & D^*_i & \text{for } i = \ell_2 + 1, \ldots, m_2, \\
B^*_i & = & \emptyset & \text{for } i = m_2 + 1, \ldots, m_2 + \kappa_2,
\end{array}$$

so that $A^*_k \succ B^*_k$ for $k = 1, \ldots, m_1$, and $C^*_i \sim D^*_i$, $D^*_i \subseteq B^*_i$, and $A^*_{\pi(i)} \subseteq C^*_i$ for $i = 1, \ldots, m_2$ and some permutation π on $\{1, \ldots, m_1\}$.

By the first n rows of the LD equation, we get

$$\sum_{i=1}^{m_1} I(s; A^*_i) + \sum_{i=1}^{m_2} I(s; D^*_i) + \sum_{i=1}^{K} I(s; \{t^i\}) = \sum_{i=1}^{m_1} I(s; B^*_i) + \sum_{i=1}^{m_2} I(s; C^*_i).$$

Since, for each $i = 1, \ldots, K$, there is a $1 \leq k \leq m_2$ such that either $t^i \in B^*_k \setminus D^*_k$ or $t^i \in C^*_k \setminus A^*_{\pi(k)}$, we can get rid of one t^i at a time by reducing either B^*_k for which $t^i \in B^*_k \setminus D^*_k$ to $B^*_k \setminus \{t^i\}$ or C^*_k for which $t^i \in C^*_k \setminus A^*_{\pi(k)}$ to $C^*_k \setminus \{t^i\}$. Continuing this reduction process for all t^i, we arrive at a reduced B set $\{B'_1, \ldots, B'_{m_1}\}$ and a reduced C set $\{C'_1, \ldots, C'_{m_2}\}$ for which $D^*_i \subseteq B'_i$ and $A^*_{\pi(i)} \subseteq C_i$ for $i = 1, \ldots, m_2$, and

$$\sum_{i=1}^{m_1} I(s; A^*_i) + \sum_{i=1}^{m_2} I(s; D^*_i) = \sum_{i=1}^{m_1} I(s; B'_i) + \sum_{i=1}^{m_2} I(s; C'_i).$$

Since $A_i^* \succ B_i^*$ for $i = 1, \ldots, m_1$ and $B_j^* \supseteq B_j'$ for $j = 1, \ldots, m_2$, and $B_k^* = B_k'$ for $k = m_2 + 1, \ldots, m_1$, it follows from downward inclusion monotonicity that $A_i^* \succ B_i'$ for $i = 1, \ldots, m_1$. Since $D_i^* \sim C_i^*$ and $C_i^* \supseteq C_i'$ for $i = 1, \ldots, m_2$, it follows from upward inclusion monotonicity that not$(C_i' \succ D_i^*)$ for $i = 1, \ldots, m_2$. Hence A3.3 is violated, so that there must be a ρ solution. □

Sufficiency Proofs of Theorems 4.1, 4.2, and 4.3 Assume that the conditions of each of Theorems 4.1, 4.2, and 4.3 hold.

To specify our system of linear inequalities, suppose that there exist a probability measure P on $\mathcal{B}_S$ and a univariate set function $\omega \geq 0$ on $\mathcal{B}_S$ such that

(1a) $P(A) - P(B) - \omega(B) > 0$ for all $A, B \in \mathcal{B}_S$ such that $A \succ B$,

(1b) $P(B) - P(A) + \omega(B) \geq 0$ and $P(A) - P(B) + \omega(A) \geq 0$ for all $A, B \in \mathcal{B}_S$ such that $A \sim B$.

Then letting $\rho = (\rho_P, \tilde{\rho}_\omega)$ be an $(n + 2^n)$-dimensional row vector, our system of linear inequalities is stated as follows.

(a) $$\rho \cdot \begin{bmatrix} \theta(A_i) - \theta(B_i) \\ -\tilde{\theta}(B_i) \end{bmatrix} > 0 \qquad \text{for } i = 1, \ldots, L_1,$$

(b) $$\rho \cdot \begin{bmatrix} \theta(D_i) - \theta(C_i) \\ \tilde{\theta}(D_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_2,$$

$$\rho \cdot \begin{bmatrix} \theta(C_i) - \theta(D_i) \\ \tilde{\theta}(C_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_2,$$

(c) $$\rho \cdot \begin{bmatrix} \theta(\{s^i\}) \\ 0 \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_3,$$

Inequalities (a) and (b) follow from (1a) and (1b), respectively. Nonnegativity of P is reflected in (c), since the case of $\{s\} \succ \emptyset$ is covered by (a). Nonnegativity of ω is covered by (b), since $A \sim A$ implies $\omega(A) \geq 0$.

For Theorem 4.2, we add

(d) $$\rho \cdot \begin{bmatrix} \theta(E_i \setminus F_i) \\ \tilde{\theta}(E_i) - \tilde{\theta}(F_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_4,$$

which cover increasing monotonicity of $P + \omega$. For Theorem 4.3, we add

(e) $$\rho \cdot \begin{bmatrix} 0 \\ \tilde{\theta}(E_i) - \tilde{\theta}(F_i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_4,$$

which covers monotonicity of ω.

The sufficiency proof is completed by establishing that the system of linear inequalities (a)–(c) for Theorem 4.1, (a)–(d) for Theorem 4.2, and (a)–(c) and (e) for Theorem 4.3 has a ρ solution. Suppose on the contrary that there is no ρ solution. Then it follows from Lemma 1 that there are nonnegative integers, α_i for $i = 1, \ldots, L_1$, β_{ij} for $i = 1, \ldots, L_2$ and $j = 1, 2$, γ_i for $i = 1, \ldots, L_3$, and δ_{ij} for $i = 1, \ldots, L_4$ and $j = 1, 2$ such that $\delta_{ij} = 0$ for all i, j in case of Theorem 4.1, $\delta_{i1} = 0$ for all i in case of Theorem 4.2, $\delta_{i2} = 0$ for all i in case of Theorem 4.3, $\alpha_k > 0$ for some $1 \le k \le L_1$, and

$$\sum_{i=1}^{L_1}\alpha_i\begin{bmatrix}\theta(A_i)-\theta(B_i)\\-\tilde{\theta}(B_i)\end{bmatrix}+\sum_{i=1}^{L_2}\beta_{i1}\begin{bmatrix}\theta(D_i)-\theta(C_i)\\\tilde{\theta}(D_i)\end{bmatrix}+\sum_{i=1}^{L_2}\beta_{i2}\begin{bmatrix}\theta(C_i)-\theta(D_i)\\\tilde{\theta}(C_i)\end{bmatrix}$$

$$+\sum_{i=1}^{L_3}\gamma_i\begin{bmatrix}\theta(\{s^i\})\\0\end{bmatrix}+\sum_{i=1}^{L_4}\delta_{i1}\begin{bmatrix}\theta(E_i\setminus F_i)\\\tilde{\theta}(E_i)-\tilde{\theta}(F_i)\end{bmatrix}+\sum_{i=1}^{L_4}\delta_{i2}\begin{bmatrix}0\\\tilde{\theta}(E_i)-\tilde{\theta}(F_i)\end{bmatrix}=0.$$

List the elements of $\mathcal{P}_\succ, \mathcal{P}_\sim$, S^0, and $\mathcal{P}_\supset$ with α_i repeats for (A_i, B_i), β_{i1} repeats for (C_i, D_i), β_{i2} repeats for (D_i, C_i), γ_i repeats for s^i, δ_{i1} repeats for (E_i, F_i), and δ_{i2} repeats for (E_i, F_i), and then enumerate them as $(A_1^*, B_1^*), \ldots, (A_{m_1}^*, B_{m_1}^*)$ for $\mathcal{P}_\succ$, $(C_1^*, D_1^*), \ldots, (C_{m_2}^*, D_{m_2}^*)$ for $\mathcal{P}_\sim$, $t^1, \ldots, t^K$ for $\mathcal{S}^0$, $(E_1^*, F_1^*), \ldots, (E_{m_3}^*, F_{m_3}^*)$ for $\mathcal{P}_\supset$, where $m_1 = \sum_{i=1}^{L_1}\alpha_i$, $m_2 = \sum_{i=1}^{L_2}(\beta_{i1}+\beta_{i2})$, $K = \sum_{i=1}^{L_3}\gamma_i$, and $m_3 = \sum_{i=1}^{L_4}(\delta_{i1}+\delta_{i2})$.

First we show a sufficiency proof of Theorem 4.1, so that $\delta_{ij} = 0$ for all i, j. Therefore, it follows from the first n rows of the LD equation that

$$\sum_{i=1}^{m_1} I(s; A_i^*) + \sum_{i=1}^{m_2} I(s; D_i^*) + \sum_{i=1}^{K} I(s; \{t^i\}) = \sum_{i=1}^{m_1} I(s; B_i^*) + \sum_{i=1}^{m_2} I(s; C_i^*).$$

Since $\tilde{\theta}$ is a unit vector, it easily follows that $m_1 = m_2$ and $B_i^* = D_{\sigma(i)}$ for $i = 1, \ldots, m_1$ and some permutation σ on $\{1, \ldots, m_1\}$, so that $\sum_{i=1}^{m_1} I(s; B_i^*) = \sum_{i=1}^{m_2} I(s; D_i^*)$. With no loss of generality, we assume that $B_i^* = D_i^*$ for $i = 1, \ldots, m_1$. Hence, letting $m = m_1$, we obtain that $A_i^* \succ B_i^* = D_i^* \sim C_i^*$ (i.e., $A_i^* \succ^* C_i^*$) for $i = 1, \ldots, m$, and

$$\sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{K} I(s; \{t^i\}) = \sum_{i=1}^{m} I(s; C_i^*).$$

which violate A4.1, so that there must be a ρ solution.

Next we show a sufficiency proof of Theorem 4.2, so that $\delta_{i2} = 0$ for all i. Suppose that $F_i^* = E_j^*$. Then we have

$$\begin{bmatrix}\theta(E_i^*\setminus F_i^*)\\\tilde{\theta}(E_i^*)-\tilde{\theta}(F_i^*)\end{bmatrix}+\begin{bmatrix}\theta(E_j^*\setminus F_j^*)\\\tilde{\theta}(E_j^*)-\tilde{\theta}(F_j^*)\end{bmatrix}=\begin{bmatrix}\theta(E_i^*\setminus F_j^*)\\\tilde{\theta}(E_i^*)-\tilde{\theta}(F_j^*)\end{bmatrix},$$

so that two pairs, (E_i^*, F_i^*) and (E_j^*, F_j^*), are reduced to generate a new pair (E_i, F_j) with $E_i \supseteq F_j$. This process continues until such reduction becomes impossible. With no loss of generality, a sequence of pairs, $(E_1^*, F_i^*), \ldots, (E_{m_3}^*, F_{m_3}^*)$ is assumed to have no reduction.

It follows from the LD equation and appropriately renumbering (A, B)-pairs and (C, D)-pairs that $m_1 = m_2 \geq m_3$, $B_i^* = E_i^*$, $F_i^* = D_i^*$ for $i = 1, \ldots, m_3$, and $B_j^* = D_j^*$ for $j = m_3 + 1, \ldots, m_1$. Therefore, letting $m = m_1$, it follows from the first n rows of the LD equation that $A_i^* \succ B_i^* \supseteq D_i^* \sim C_i^*$ for $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{m_3} I(s; D_i^*) + \sum_{i}^{K} I(s; \{t^i\}) + \sum_{i=1}^{m_3} I(s; E_i \setminus F_i)$$

$$= \sum_{i=1}^{m_3} I(s; B_i^*) + \sum_{i=1}^{m} I(s; C_i^*),$$

which is rearranged to give

$$\sum_{i=1}^{m} I(s; A_i^*) + \sum_{i}^{K} I(s; \{t^i\}) = \sum_{i=1}^{m} I(s; C_i^*).$$

Since, for each $i = 1, \ldots, K$, there is a $1 \leq k \leq m$ such that $t^i \in C_k$, we can get rid of one t^i at a time by reducing a C_k that contains the t^i to $C_k \setminus \{t^i\}$. Continuing this rsduction process for all t^i, we arrive at a reduced C set $\{C_1', \ldots, C_m'\}$ for which $C_i \supseteq C_i'$ for $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} I(s; A_i^*) = \sum_{i=1}^{m} I(s; C_i').$$

By the upward inclusion monotonicity, $D_i^* \succeq C_i'$ for $i = 1, \ldots, m$. Since $A_i^* \succ B_i^* \supset D_i^*$, the downward inclusion monotonicity implies that $A_i^* \succ D_i^*$. Hence $A_i^* \succ^* C_i'$ for $i = 1, \ldots, m$, which violate A4.2, so that there must be a ρ solution.

Last we show a sufficiency proof of Theorem 4.3, so that $\delta_{i1} = 0$ for all i. By a similar argument to the proof of Theorem 4.2, we arrive at

$$\sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{m} I(s; D_i^*) + \sum_{i}^{K} I(s; \{t^i\}) = \sum_{i=1}^{m} I(s; B_i^*) + \sum_{i=1}^{m} I(s; C_i^*),$$

where $m = m_1 = m_2$ and $A_i^* \succ B_i^* \supseteq D_i^* \sim C_i^*$ for $i = 1, \ldots, m$.

Since, for each $i = 1, \ldots, K$, there is a $1 \leq k \leq m$ such that $t^i \in C_k^*$ or $t^i \in B_k^*$, we can get rid of one t^i at a time by reducing C_k^* or B_k^* that contains the t^i to $C_k^* \setminus \{t^i\}$ or $B_k^* \setminus \{t^i\}$, respectively. We assume that $0 \leq K' \leq K$, $t^1, \ldots, t^{K'}$ are contained in some C sets, and $t^{K'+1}, \ldots, t^K$ are not contained in C sets but in some B sets. Continuing this reduction process for $t^1, \ldots, t^{K'}$, we arrive at a reduced C set $\{C_1', \ldots, C_m'\}$ for which $C_i \supseteq C_i'$ for $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{m} I(s; D_i^*) + \sum_{i=K'+1}^{K} I(s; \{t^i\}) = \sum_{i=1}^{m} I(s; B_i^*) + \sum_{i=1}^{m} I(s; C_i').$$

Since $t^{K'+1}, \ldots, t^K$ are contained in some $B_i^* \setminus D_i^*$, we also get a reduced B set $\{B_1', \ldots, B_m'\}$ for which $B_i^* \supseteq B_i' \supseteq D_i^*$ for $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{m} I(s; D_i^*) = \sum_{i=1}^{m} I(s; B_i') + \sum_{i=1}^{m} I(s; C_i').$$

By the upward inclusion monotonicity, $D_i^* \succeq C_i'$ for $i = 1, \ldots, m$. By the downward inclusion monotonicity, $A_i^* \succ B_i'$ for all i, which violate A4.3, so that there must be a ρ solution. □

Sufficiency Proofs of Theorems 5.1, 5.2, and 5.3 We assume that the conditions of each of Theorems 5.1, 5.2, and 5.3 hold.

To specify our system of linear inequalities, suppose that there exist a probability measure P, an additive set function $\omega \geq 0$ on $\mathcal{B}_S$, and a nonnegative constant ϵ such that $\omega(\emptyset) = 0$, and

(1a) $P(A) - P(B) - \omega(B) - \epsilon > 0$ for all $A, B \in \mathcal{B}_S$ such that $A \succ B$,

(1b) $P(B) - P(A) + \omega(B) + \epsilon \geq 0$ and $P(A) - P(B) + \omega(A) + \epsilon \geq 0$ for all $A, B \in \mathcal{B}_S$ such that $A \sim B$.

Then letting $\rho = (\rho_P, \rho_\omega, \epsilon)$ be a $(2n+1)$-dimensional row vector, our system of linear inequalities is stated as follows.

(a) $$\rho \cdot \begin{bmatrix} \theta(A_i) - \theta(B_i) \\ -\theta(B_i) \\ -1 \end{bmatrix} > 0 \qquad \text{for } i = 1, \ldots, L_1,$$

(b) $$\rho \cdot \begin{bmatrix} \theta(D_i) - \theta(C_i) \\ \theta(D_i) \\ 1 \end{bmatrix} \geq 0 \qquad \text{and}$$

$$\rho \cdot \begin{bmatrix} \theta(C_i) - \theta(D_i) \\ \theta(C_i) \\ 1 \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_2,$$

(c) $$\rho \cdot \begin{bmatrix} \theta(\{s^i\}) \\ 0 \\ 0 \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_3,$$

(d) $$\rho \cdot \begin{bmatrix} \theta(\{s_i\}) \\ \theta(\{s_i\}) \\ 0 \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, n,$$

Inequalities (a) and (b) follows from (1a) and (1b), respectively. Nonnegativity of P is reflected in (c), since the case of $\{s\} \succ \emptyset$ is covered by (a). Nonnegativity of

$\omega + \epsilon$ follows from (b) since $A \sim A$ implies $\omega(A) + \epsilon \geq 0$. Increasing monotonicity of $P + \omega + \epsilon$ is reflected in (d).

For Theorem 5.2, we replace (d) by

$$\text{(e)} \qquad \rho \cdot \begin{bmatrix} 0 \\ \theta(\{s_i\}) \\ 0 \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, n,$$

which covers increasing monotonicity of $\omega + \epsilon$.

The proof is completed by establishing that the system of linear inequalities (a)–(c) for Theorem 5.3, (a)–(d) for Theorem 5.1, and (a)–(c), and (e) for Theorem 5.2 has a ρ solution. We shall omit the sufficiency proof for Theorem 5.3, since it is similar to the one for Theorem 5.1 by letting $\epsilon = 0$. Suppose on the contrary that there is no ρ solution. Then it follows from Lemma 1 that there are nonnegative integers, α_i for $i = 1, \ldots, L_1$, β_{ij} for $i = 1, \ldots, L_2$ and $j = 1, 2$, γ_i for $i = 1, \ldots, L_3$, and δ_{ij} for $i = 1, \ldots, n$ and $j = 1, 2$ such that $\delta_{i2} = 0$ for all i in case of Theorem 5.1, $\delta_{i1} = 0$ for all i in case of Theorem 5.2, $\alpha_k > 0$ for some $1 \leq k \leq L_1$, and

$$\sum_{i=1}^{L_1} \alpha_i \begin{bmatrix} \theta(A_i) - \theta(B_i) \\ -\theta(B_i) \\ -1 \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i1} \begin{bmatrix} \theta(D_i) - \theta(C_i) \\ \theta(D_i) \\ 1 \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i2} \begin{bmatrix} \theta(C_i) - \theta(D_i) \\ \theta(C_i) \\ 1 \end{bmatrix}$$

$$+ \sum_{i=1}^{L_3} \gamma_i \begin{bmatrix} \theta(\{s^i\}) \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{n} \delta_{i1} \begin{bmatrix} \theta(\{s_i\}) \\ \theta(\{s_i\}) \\ 0 \end{bmatrix} + \sum_{i=1}^{n} \delta_{i2} \begin{bmatrix} 0 \\ \theta(\{s_i\}) \\ 0 \end{bmatrix} = 0.$$

List the elements of $\mathcal{P}_\succ$, $\mathcal{P}_\sim$, and S^0 with α_i repeats for (A_i, B_i), β_{i1} repeats for (C_i, D_i), β_{i2} repeats for (D_i, C_i), γ_i repeats for s^i, δ_{i1} repeats for s_i, and δ_{i2} repeats for s_i, and enumerate them as $(A_1^*, B_1^*), \ldots, (A_m^*, B_m^*)$ for $\mathcal{P}_\succ$, $(C_1^*, D_1^*), \ldots, (C_m^*, D_m^*)$ for $\mathcal{P}_\sim$, $t^1, \ldots, t^K$ for S^0, and $r^1, \ldots, r^L$ for S, where $m = \sum_{i=1}^{L_1} \alpha_i = \sum_{i=1}^{L_2} (\beta_{i1} + \beta_{i2})$, $K = \sum_{i=1}^{L_3} \gamma_i$, and $L = \sum_{i=1}^{n} (\delta_{i1} + \delta_{i2})$.

Fisrt we show a sufficiency proof of Theorem 5.1, so that $\delta_{i2} = 0$ for all i. Therefore, it follows from the LD equation that

$$\begin{aligned} \sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{m} I(s; D_i^*) + \sum_{i=1}^{K} I(s; \{t^i\}) + \sum_{i=1}^{L} I(s; r^i) &= \sum_{i=1}^{m} I(s; B_i^*) + \sum_{i=1}^{m} I(s; C_i^*), \\ \sum_{i=1}^{m} I(s; D_i^*) + \sum_{i=1}^{L} I(s; r^i) &= \sum_{i=1}^{m} I(s; B_i^*). \end{aligned}$$

Subtracting the second from the first and reducing C sets by getting rid of one t^i at a time, we get a reduced C set $\{C_1', \ldots, C_m'\}$ for which $C_i \supseteq C_i'$ for $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} I(s; A_i^*) = \sum_{i=1}^{m} I(s; C_i').$$

By the upward inclusion monotonicty, not($C_i' \succ D_i^*$) for $i = 1, \ldots, m$. Similarly, reducing B sets by getting rid of one r^i at a time, we get a reduced B set $\{B_1', \ldots, B_m'\}$ for which $B_i^* \supseteq B_i'$ for $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} I(s; D_i^*) = \sum_{i=1}^{m} I(s; B_i').$$

By the downward inclusion monotonicity, $A_i^* \succ B_i'$ for $i = 1, \ldots, m$ which violate A5.1. Hence there must be a ρ solution.

Next we show a sufficiency proof of Theorem 5.2, so that $\delta_{i1} = 0$ for all i. Thus it follows from the LD equation that

$$\begin{aligned} \sum_{i=1}^{m} I(s; A_i^*) + \sum_{i=1}^{m} I(s; D_i^*) + \sum_{i=1}^{K} I(s; \{t^i\}) &= \sum_{i=1}^{m} I(s; B_i^*) + \sum_{i=1}^{m} I(s; C_i^*), \\ \sum_{i=1}^{m} I(s; D_i^*) + \sum_{i=1}^{L} I(s; \{r^i\}) &= \sum_{i=1}^{m} I(s; B_i^*). \end{aligned}$$

Since not($C_i^* \succ D_i^*$) and $A_i^* \succ B_i^*$ for $i = 1, \ldots, m$, A5.2 is violated. Hence there must be a ρ solution. □

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