No. 763

A New View on Statistical Inference
Subtitle: Statistical Inference based on Briefest Randomized Regions.
—Case of Uniform Distributions—

by

Yoshiko Nogami

December 1997

A New View on Statistical Inference

Subtitle: Statistical inference based on Briefest Randomized Regions.

---Case of Uniform Distributions ---

By Yoshiko Nogami

Abstract.

In this paper the author considers the uniform distributions over the interval $[\theta,\theta+1)$ with density

$$(0) \qquad f(x|\theta)=\begin{cases} 1, & \text{for } \theta<x<\theta+1, \quad \forall\theta \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density (0). In this paper we use an unbiased estimator for θ to construct the optimal randomized region(see §2) and apply this region for the tests of testing the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ (θ_0:a real number)(see §3 and §4).

The author wants to stress another point about Nogami's work since 1992(see,for example, Nogami(1992a,b)) through 1997 which is the first work to explore the optimal statistical inference for θ under the distributions with densities of form (0).

§1. Introduction

Let $I_A(x)$ be an indicator function of the set A such that $I_A(x)=1$ for $x \in A$; $=0$, for $x \notin A$. In this paper the author considers the uniform distributions over the interval $[\theta,\theta+1)$ with density

(1) $f(x|\theta)=I_{[\theta,\ \theta+1)}(x), \quad \forall\theta$

Let $X_1, \ldots ,X_n$ be a random sample of size n taken from the density (1). We note that this is especially the case that the likelihood ratio tests based on the minimal sufficient statistics are useless. (This statement is appeared in Nogami(1993). However, it may not be difficult to show this fact.) Let $X_{(i)}$ $(i=1,2,\ldots,n)$ be the i-th largest observation of $X_1, \ldots ,X_n$. Let α be a real number such that $0<\alpha<1$. We first construct the briefest randomized region with precision size $1-\alpha$ $(\doteq\gamma)$ (See (14)) based on the unbiased estimator $Y=(X_{(1)}+X_{(n)}-1)/2$.

Then, we get the tests with two-edged refusal region , for which we refuse the hypothesis H_0, by using this randomized region(R.R.) (in §3) and see in §4 that the refusal region C^* constructed in this way has the minimum probability with respect to (wrt) θ at $\theta=\theta_0$, provided that the probability of the refusal region under the hypothesis H_0 is a constant γ $(\doteq 1-\alpha)$.

§2. Randomized Region.

Let α be a real number such that $0<\alpha<1$. In this section we construct the briefest randomized region with precision size $1-\alpha$. (See (14) below.) Applying a variable transformation to the joint probability density of $(X_{(1)},X_{(n)})$ we get the density function $g(y|\theta)$ of $Y=(X_{(1)}+X_{(n)}-1)/2$ as below:

(3) $g(y|\theta)=n(1-2|y-\theta|)^{n-1} \qquad (-2^{-1}<y-\theta<2^{-1}).$

We may call $(\phi_1(Y),\phi_2(Y))$ to be the briefest randomized region (R.R.) for θ at precision size γ if

(4) $P_\theta[\theta\varepsilon(\phi_1(Y),\phi_2(Y))]=\gamma.$

To get the briefest R.R. for θ at precision size γ we minimize $y_2-y_1(>0)$ provided that for any $\theta\varepsilon(-\infty,\infty)$

(5) $P_\theta[y_1<Y<y_2]=\int_{y_1}^{y_2} g(y|\theta)\ dy=\gamma.$

To get such y_1 and y_2 we let

(6) $(L=)L(y_1,y_2)=y_2-y_1-\lambda\{\int_{y_1}^{y_2} g(y|\theta)\ dy-\gamma\}$ (for real λ)

and solve the equations

(7) $\partial L/\partial y_1=-1+\lambda g(y_1|\theta)=0$

(8) $\partial L/\partial y_2=1-\lambda g(y_2|\theta)=0$

(9) $\partial L/\partial\lambda=\int_{y_1}^{y_2} g(y|\theta)\ dy-\gamma=0,\quad \forall\theta.$

Since by (7) and (8) we obtain

(10) $g(y_1|\theta)=g(y_2|\theta)(=\lambda^{-1}),\quad \forall\theta$

we therefore get

(11) $y_1=\theta-r$ and $y_2=\theta+r$

where r is determined by

(12) $r=(1-\alpha^{1/n})/2.$

Hence, the briefest R.R. for θ is

(13) $(Y-r,Y+r)$

for which

(14) $P_\theta[\theta\varepsilon(Y-r,Y+r)]=\gamma.$

§3. Tests with Two-edged refusal region.

In this section we consider the problem of testing the hypothesis $H_0:\theta=\theta_0$ versus the alternative $H_1:\theta\neq\theta_0$,using the randomized region (13). The test is to refuse H_0 if $\theta_0 \notin (Y-r,Y+r)$ or equivalently, $Y\leq\theta_0-r$ or $\theta_0+r \leq Y$ and accept H_0 if $\theta_0 \in (Y-r,Y+r)$ or equivalently, $\theta_0-r<Y<\theta_0+r$. We call $(C^*=)\{y:y\leq\theta_0-r$ or $y\geq\theta_0+r\}$ as two-edged refusal region(cf.Nogami(1995,p.3))

. In the next section we show the optimality of this test.

§4. Probability of refusal region.

Let y_1 and y_2 be real numbers with $y_1<y_2$ and let $C=\{y:y\leq y_1$ or $y_2\leq y\}$ be a two-edged refusal region. Let α be a real number such that $0<\alpha<1$.

In this section we consider to minimize the probability $P_\theta(C)$ of refusal region C with respect to (wrt) θ provided that the probability $P_{\theta_0}(C)$ of the refusal region C under the hypothesis $H_0:\theta=\theta_0$ is a constant γ.

To do so we let $\pi_C(\theta)\doteq P_\theta(C)$. We choose y_1 and y_2 which satisfy

(15) $\quad \pi_C(\theta_0)=1 \; - P_{\theta_0}[y_1<Y<y_2]=\alpha$

and

(16) $\quad \pi'_C(\theta_0)=d\pi_C(\theta)/d\theta\Big|_{\theta=\theta_0} = g(y_2|\theta_0)-g(y_1|\theta_0)=0.$

Equations (15) and (16) are equivallent to (9) and (10) except for the value θ_0 of θ. Hence, the solution of (15) and (16) is (11) with θ replaced by θ_0.(cf. Nogami(1995,p.2(2), p.4))

Then, the test based on the refusal region $C^*=\{y:y\leq\theta_0-r$ or $y\geq\theta_0+r\}$ has the mininum probability of refusal region at $\theta=\theta_0$, among all $\theta\in(-\infty,\infty)$. (cf. Nogami(1995,p.4)) In Nogami(1995,p.3) the function $\pi_{C^*}(\theta)$ of θ is directly computed and it is convex from below.

REFERENCES.

[1] Y. Nogami(1992a) (Unpublished Paper)

[2]Y. Nogami(1992b). A Statistical Inference under the uniform distribution $U[\theta,\theta+1]$. Inst. Socio-Eco. Plan., Univ. of Tsukuba, D.P. 507,pp.1-9 (December)

[3]-----(1993). Statistical Inference Under the Uniform Distribution. (Unpublished Paper).

[4]------(1995). Statistical Inference Under Uniform Distribution$U[\theta,\theta+1]$. Part II--(in Japanese). Inst. Soci-Econ. Plan., U. of Tsukuba, D.P.627, pp.1-7(May).