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A New View on Statistical Inference
Part IV
Tow Action Problems

by

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Abstract.

Let $I_A(x)$ be an indicator function such that for any set A $I_A(x)=1$, if $x \in A$; $=0$, if $x \notin A$.

We consider two action problems under the densiry

$$f(x|\theta)=c^{-1}I_{[\theta+\eta 1,\ \theta+\eta 2)}(x), \qquad \forall\theta$$

where $c=\eta_2-\eta_1(>0)$ and real numbers η_1 and η_2.

§ 1. Two action problem.

Let $\Omega=(-\infty,\infty)$ be a parameter space and let Ω_0 and Ω_1 be the subspace of Ω such that $\Omega_0 \cup \Omega_1=\Omega$ and $\Omega_0 \cap \Omega_1=\phi$. In this paper we consider the two-action problem of testing hypotheses $H_0:\theta\varepsilon\Omega_0$ versus $H_1:\theta\varepsilon\Omega_1$, provided the random observations X_1, ..., X_n from the population with density $f(x|\theta)$. Assume that a_i denotes the action deciding in favor of H_i $(i=0,1)$.
Let $L(\theta,a)$ be the loss function such that

(1) $$L(\theta,a_0)=\begin{cases}1, & \text{if } \theta\varepsilon\Omega_1\\ 0, & \text{if } \theta\varepsilon\Omega_0\end{cases}$$

and $L(\theta,a_1)=1-L(\theta,a_0)$.

Let $Y\doteq Y(X_1,\dots,X_n)$ be the underlined statistic with density $g(y|\theta)$. We denote by $\delta(y,\cdot)$ a decision rule showing the probability distribution on $A=\{a_0,a_1\}$. Namely,

(2) $$\delta(y,a_0)+\delta(y,a_1)=1.$$

For given y, the loss function of δ is given by

(3) $$L(\theta,\delta(y,\cdot)) = E_{\delta(y,\cdot)}(L(\theta,a)) = \int L(\theta,a)\ d\delta(y,a), \qquad \forall\theta.$$

The risk function $R(\theta,\delta)$ is of form

(4) $$R(\theta,\delta)=E_{Y|\theta}(L(\theta,\delta(Y,\cdot))=\iint L(\theta,a)\ d\delta(y,a)\ g(y|\theta)\ dy.$$

Namely, for $\theta\varepsilon\Omega_0$

(5) $$R(\theta,\delta)=E_{Y|\theta}(\delta(Y,a_0)L(\theta,a_0)+\delta(Y,a_1)L(\theta,a_1)) = \int \delta(y,a_1)\ g(y|\theta)\ dy\doteq E_\theta(\delta(Y,a_1)).$$

In the same way, for $\theta\varepsilon\Omega_1$

(6) $$R(\theta,\delta)=E_{Y|\theta}(L(\theta,\delta(Y,\cdot)))=\int \delta(y|a_0)\ g(y|\theta)\ dy\doteq E_\theta(\delta(Y,a_0)).$$

Hereafter, we let $\Omega_0=\{\theta_0\}$ with real number θ_0, and $\Omega_1=\{\theta:\theta\neq\theta_0\}$. We also let $\delta(y,a_i)$ abbreviate to $\delta_i(y)$ $(i=0,1)$.

In this paper we consider the decision rules of form

(7) $\delta_1(y) = \begin{cases} 1, & \text{if } y \le y_1 \text{ or } y \ge y_2 \\ 0, & \text{if } y_1 < y < y_2 \end{cases}$

and $\delta_0(y) = 1 - \delta_1(y)$, where y_1 and y_2 are real numbers such that $y_1 < y_2$.

In the next section we consider the distribution with density of form $f(x|\theta) = c^{-1} I_{(\theta+\eta_1,\ \theta+\eta_2)}(x)$.

§2. $f(x|\theta) = c^{-1} I_{(\theta+\eta_1,\ \theta+\eta_2)}(x)$--Case.

In this section we consider the population with density

(8) $f(x|\theta) = c^{-1} I_{(\theta+\eta_1,\ \theta+\eta_2)}(x)$, for $\theta \in (-\infty, \infty)$

where η_1 and η_2 are real numbers with $c = \eta_2 - \eta_1 (>0)$.

Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$, taken randomly from the population with density $f(x|\theta)$. In this section we use an unbiased estimator $Y = (X_{(1)} + X_{(n)} - \eta_0)/2$ $(\eta_0 = \eta_1 + \eta_2)$ to get the optimal decision $\delta_0(y)$ for deciding a_0. Applying variable transformations $Y = (X_{(1)} + X_{(n)} - \eta_0)/2$ and $Z = X_{(1)}$ to the joint density of $(X_{(1)}, X_{(n)})$ and taking marginal probabiltiy density function (p.d.f.) of Y as follows:

(9) $g(y|\theta) = \begin{cases} nc^{-n}(c - 2|y-\theta|)^{n-1}, & \text{for } -c/2 < y-\theta < c/2 \\ 0, & \text{elsewhere.} \end{cases}$

To get the optimal interval (y_1, y_2) for deciding a_0 we minimize $y_2 - y_1 (>0)$, provided that for a real number α with $0 < \alpha < 1$ and for all $\theta \in (-\infty, \infty)$,

(10) $E_\theta(\delta_0(Y)) = P_\theta[y_1 < Y < y_2] = \int_{y_1}^{y_2} g(y|\theta)\, dy = 1 - \alpha.$

Since $g(y|\theta)$ is symmetric at θ, we take $y_1 = \theta - r$ and $y_2 = \theta + r$, and obtain

(11) $r = c(1 - \alpha^{1/n})/2$.

(We note that the optimal region $(Y-r, Y+r)$ satisfies that for all $\theta \in (-\infty, \infty)$,

(12) $P_\theta[\theta \in (Y-r, Y+r)] = 1 - \alpha$.)

Letting

(13) $$\delta_1^{\theta}(y)=\begin{cases}1, & \text{if } y<\theta-r \text{ or } \theta+r<y,\\ 0, & \text{if } \theta-r<y<\theta+r,\end{cases}$$

and $\delta_0^{\theta}(y)=1-\delta_1^{\theta}(y)$, we get

(14) $$R(\theta,\delta^{\theta})=E_{\theta}(\delta_1^{\theta}(Y))=\alpha, \qquad \theta\varepsilon\Omega_0$$

$$R(\theta,\delta^{\theta})=E_{\theta}(\delta_0^{\theta}(Y))=1-E_{\theta}(\delta_1^{\theta}(Y))=1-\alpha, \qquad \theta\varepsilon\Omega_1.$$

If we especially let

(15) $$\delta_1^{*}(y)=\delta_1^{\theta_0}(y) \quad \text{and} \quad \delta_0^{*}(y)=1-\delta_1^{\theta_0}(y),$$

then

(16) $$R(\theta_0,\delta^{*})=E_{\theta_0}(\delta_1^{*}(Y))=\alpha,$$

and

(17) $$R(\theta,\delta^{*})=E_{\theta}(\delta_0^{*}(Y))=1-E_{\theta}(\delta_1^{*}(Y))\leq 1-\alpha\leq E_{\theta}(1-\delta_1(Y))=R(\theta,\delta), \qquad \text{for } \theta\neq\theta_0,$$

because $E_{\theta}(\delta_1^{*}(Y))\geq\alpha$. So, if $D=\{\delta: R(\theta_0,\delta)=\alpha$ and $\delta_1(y)$ is of form (7)$\}$, then δ^{*} is of best among all $\delta\varepsilon D$.