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A New View on Statistical Inference.
Part II —(2)
——Positional Parameters——

by

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§0. Introductions.

In this paper we consider the following density

(1) $f(x)=f(x|\theta)=e^{-(x-\theta)}I_{(\theta,\infty)}(x)$, for real θ

where $I_A(x)$ is an indicator function such that for set A, $I_A(x)=1$, if $x\epsilon A$; $=0$, if $x\not\epsilon A$. We denote by RHS(A)(or LHS(A)) the right hand side of A (or the left hand side of A), respectively.

Let $X_1, \dots, X_n$ be a random sample of size n from the population with above density. For convenience we let $n=2m+1$ ($m(>0)$: an integer). Let $X_{(i)}$ be the i-th smallest observation of $X_1,\dots,X_n$ so that $X_{(1)}<\dots<X_{(n)}$. We use the sample median $X_{(m+1)}$ to get the optimal interval estimation for θ.

§1. Optimal interval estimation.

Let $Y=X_{(m+1)}$. We first find the p.d.f. of Y. To do so we introduce the c.d.f. of X as below.

(2) $F(x)=F(x|\theta)=(1-e^{-(x-\theta)})I_{(\theta,\infty)}(x)$.

Then, we get the p.d.f. of Y as follows:

(3) $g(y)=g(y|\theta)=k\quad F(y)^m(1-F(y))^m\quad f(y)$

$$=k\ (1-e^{-(y-\theta)})^m e^{-(m+1)\ (y-\theta)}I_{(\theta,\infty)}(y)$$

where $\theta\epsilon(-\infty,\infty)$ and

(4) $k=\Gamma(2m+2)/(\Gamma(m+1))^2$.

Since

(5) $$g'(y)=dg(y)/dy\begin{cases} =0, & \text{if } y=\theta+\ln((2m+1)/(m+1))(=y_\theta^*),\\ >0, & \text{if } y<y_\theta^*,\\ <0, & \text{if } y>y_\theta^*,\end{cases}$$

the p.d.f. $g(x)$ shows that Y has an unimodal distribution.

Let α be a real number such that $0<\alpha<1$. We want to find y_1 and y_2 which minimize $y_2-y_1(>0)$ and satisfy

(5) $$P_\theta[y_1<Y<y_2]=1-\alpha.$$

From the structure of the distribution of Y, it is sufficient to find r_1 and r_2 which minimize $r_2-r_1(>0)$ and satisfy

(6) $$P_\theta[r_1<Y-\theta<r_2]=1-\alpha.$$

Taking a variable transformation $t=e^{-(y-\theta)}$ we get the p.d.f. of T as follows:

(7) $$h_T(t)=k\ (1-t)^m t^m \quad \text{for } 0<t<1.$$

Using this p.d.f. we get

(8) $$(6)=P[r_1<\ln(1/T)<r_2]=P[\exp(-r_2)<T<\exp(-r_1)]$$

$$=1-\{\int_0^{\exp(-r_2)} h_T(t)\ dt + \int_{\exp(-r_1)}^{1} h_T(t)\ dt\}$$

Since the p.d.f. of T is symmetric at $t=1/2$, we take $\exp(-r_1)=1-\exp(-r_2)$ and then get with (6) that

(9) $$(\text{the extreme RHS}(7))=1-2\int_0^{\exp(-r_2)} k(1-t)^m t^m\ dt=1-\alpha,$$

or equivalently,

(10) $$\int_0^{\exp(-r_2)} k(1-t)^m t^m\ dt=\alpha/2.$$

Letting $\exp(-r_2)=\beta(\alpha/2)$ (i.e. $\beta(\alpha/2)$ is the $\alpha/2$ percentile point of Beta distribution $Be(m+1,m+1)$.), we obtain

(11) $$r_2^*=-\ln(\beta(\alpha/2)) \quad \text{and} \quad r_1^*=-\ln(1-\beta(\alpha/2)).$$

Thus, we finally obtain the optimal confidence interval

(12) $$(Y-r_2^*,Y-r_1^*)=(Y+\ln(\beta(\alpha/2)),Y+\ln(1-\beta(\alpha/2)))$$

such that for all $\theta\varepsilon(-\infty,\infty)$,

(13) $P_\theta[\theta \varepsilon (Y-r_2^*, Y-r_1^*)]=P_\theta[y_1^* < Y < y_2^*]=1-\alpha$

where $y_1^*=\theta+r_1^*$ and $y_2^*=\theta+r_2^*$.

§3. Optimal Property.

Assume that

(14) $$\phi_\theta(y)=\begin{cases}1, & \text{if } y \le y_1^\theta \text{ or } y_2^\theta \le y,\\ 0, & \text{if } y_1^\theta < y < y_2^\theta\end{cases},$$

for any real number $y_1(=y_1^\theta)$ and $y_2(=y_2^\theta)$ such that $y_1<y_2$.

Define $\beta(\theta)\doteq E_\theta(\phi_\theta(Y))$. Then, we try to choose y_1 and y_2 which satisfy the relations (15) and (16) below.

(15) $\beta(\theta)=E_\theta(\phi_\theta(Y))=\alpha,$ for all θ

and

(16) $\beta'(\theta)=d\beta(\theta)/d\theta=0.$

To choose y_1 and y_2 so that $\beta(\theta)=\alpha$ is equivalent to choose y_1 and y_2 so that

(17) $$\int_{y_1}^{y_2} g(y|\theta)\, dy=1-\alpha.$$

Because of the property of $g(y|\theta)$, we choose

(18) $y_i^\theta = \theta+r_i$ (i=1,2).

(Our interval (y_1^*, y_2^*) clearly satisfies (15).)

For y_i's in (18)

(19) $1-\beta(\theta)=P_\theta[y_1<Y<y_2]=P[\exp(-r_2)<T<\exp(-r_1)]=1-\alpha,$ for all θ.

Thus, $\beta'(\theta)\equiv 0$, for all real θ.

(Therefore, for y_1^* and y_2^*, $\beta'(\theta)\equiv 0$, for all θ.)

§4. Remarks.

We introduce other works under the distributions with positional parameters.

In Nogami(1997b), optimal interval estimation is made under the distribution with density

(20) $$f(x|\theta)= \begin{cases} (\delta_2-\delta_1)^{-1}, & \text{for } \theta+\delta_1<x<\theta+\delta_2 \\ 0, & \text{otherwise.} \end{cases}$$

where $\theta\varepsilon(-\infty,\infty)$ and δ_1 and δ_2 are the real numbers such that $\delta_1<\delta_2$.

There is another distribution with density

(20) $$f(x|\theta)=\pi^{-1}(1+(x-\theta)^2)^{-1}, \qquad -\infty<x<\infty \quad (-\infty<\theta<\infty)$$

for which the similar analysis applies (see Nogami(1997a).).

The discrete cases may be the interesting field to check on.

The author's work in this field started in 1992. It may be the time to put on the original name for the method (6).

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