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A new view on statistical inference.
---Part Ⅲ---
------ Case of the retreated distributions -----

by

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Abstract.

This is a modified version of some part of Nogami(1997,May). In Nogami(1997,May), the author uses the retreated distributions over the interval $[\theta+\delta_1,\theta+\delta_2)$ with density

(1) $$f(x|\theta)=\begin{cases} c^{-1}, & \text{for } \theta+\delta_1<x<\theta+\delta_2 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta\varepsilon(-\infty,\infty)$ and $c=\delta_2-\delta_1(>0)$ with real numbers δ_1 and δ_2.

Let $X_1,\ldots,X_n$ be a random sample of size n from $f(x|\theta)$. Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots,X_n$. In this paper, the author uses an unbiased estimator $Y\doteq(X_{(1)}+X_{(n)}-\delta_0)/2$ $(\delta_0=\delta_1+\delta_2)$ to find the optimal confidence interval (C.I.) for θ (see §2) and check its goodness by using the falsely-covering probability (see §3) and the optimal property of the shortest C. I. at $\theta=\theta_0$(see §4).

§1. Introduction.

This is the paper of a modified-version of some part of Nogami(1997, May). Let $I_A(x)$ be a indicator function such that $I_A(x)=1$, if $x \in A$; $=0$, if $x \notin A$, for a set A. In this paper the author considers the retreated distribution over the interval $[\theta+\delta_1, \theta+\delta_2)$ with density

(1) $f(x|\theta)=c^{-1}I_{[\theta+\delta_1,\ \theta+\delta_2)}(x),$ $\forall\theta$

where $\delta_i(i=1,2)$ are the real numbers and $c \doteq \delta_2-\delta_1(>0)$.

Let $X_1, \ldots, X_n$ be a random sample of size n from $f(x|\theta)$. Let $X_{(i)}$ (i=1,2, ...,n) be the i-th smallest observation such that $X_{(1)} < \ldots < X_{(n)}$. We try to construct an optimal confidence interval (C.I.) for θ.

As a literature the author's research in this direction was firstly explored in 1992 (see, for example,Nogami(1992)). In Nogami(1992) the author obtained the optimal C. I. for θ when the underlined-distribution is $f(x|\theta)$ with $\delta_1=0$ and $\delta_2=1$. Although at this time the author had been noticed that the unbiased estimator $Y=(X_{(1)}+X_{(n)}-1)/2$ for θ could be a good estimator(see p.1 in (1992)), miscalculation had prevented the author from reaching the right direction. Later(Nogami(1995)), this Y is used to obtain the optimal C. I. for θ. This paper is a generalized version of Nogami(1995) to $f(x|\theta)$ in (1). Let α be a real number such that $0<\alpha<1$.

In this paper the author uses an unbiased estimator $Y \doteq (X_{(1)}+X_{(n)}-\delta_0)/2$ $(\delta_0=\delta_1+\delta_2)$ to get the optimal shortest C. I.(Y-r,Y+r) with $r=c(1-\alpha^{1/n})/2$, where r is determined by

(2) $1-P_\theta[\theta \in (Y-r,Y+r)]=\alpha.$

(in Section 2). In Section 3, she computes the probability $\xi(\theta)$ of falsely covering $\theta_0(\neq\theta)$ when the true parameter is θ, as a function of θ, and checks that it is a concave (from below) function of θ. In Section 4, we check the optimal property for the C. I. (Y-r,Y+r) at $\theta=\theta_0$.

Let $\doteq$ be a defining property.

§2. Shortest C. I..

In this section we use the statistic $Y \doteq (X_{(1)}+X_{(n)}-\delta_0)/2$ $(\delta_0 \doteq \delta_1+\delta_2)$ to get the shortest C. I. for θ at confidence coefficient $1-\alpha(\doteq\gamma)$. We first find the probability density function (p.d.f.) of Y. Applying the variable transformations $Y=(X_{(1)}+X_{(n)}-\delta_0)/2$ and $Z=X_{(1)}$ to the joint density of $(X_{(1)},X_{(n)})$ and taking the marginal p.d.f. we obtain the p.d.f. of Y as follows:

$$g(y|\theta)(\doteq g(y-\theta))= \begin{cases} nc^{-n}(c-2|y-\theta|)^{n-1}, & \text{for } -c/2<y-\theta<c/2, \\ 0, & \text{elsewhere.} \end{cases} \tag{3}$$

To get the shortest C. I. for θ at confidence coefficient γ we minimize $y_2-y_1(>0)$ provided that for any $\theta\epsilon(-\infty,\infty)$

$$P_\theta[y_1<Y<y_2]=\int_{y_1}^{y_2} g(y|\theta)\ dy = \gamma \tag{4}$$

where y_1 and y_2 are real numbers such that $y_1<y_2$. To get such y_1 and y_2 we let

$$(L\doteq)L(y_1,y_2)\doteq\ y_2-y_1-\lambda\{\int_{y_1}^{y_2} g(y|\theta)\ dy-\gamma\}\ ,\text{for real } \lambda \tag{5}$$

and solve the equations

$$\partial L/\partial y_1 = -1+\lambda g(y_1|\theta)=0, \tag{6}$$

$$\partial L/\partial y_2 = 1-\lambda g(y_2|\theta)=0, \tag{7}$$

$$\partial L/\partial\lambda = \int_{y_1}^{y_2} g(y|\theta)\ dy-\gamma=0, \quad \forall\theta. \tag{8}$$

Since by (6) and (7), we obtain

$$g(y_1|\theta)=g(y_2|\theta)(=\lambda^{-1}), \quad \forall\theta, \tag{9}$$

we merely obtain y_1 and y_2 which satisfy (9) and (8), for any $\theta\epsilon(-\infty,\infty)$.

Hence, we obtain

(10) $y_1=\theta-r$ and $y_2=\theta+r$

where r is determined by

(11) $r \doteq c(1-\gamma^{1/n})/2.$

Hence, the shortest C. I. for θ is $(Y-r,Y+r)$ for which

(12) $P_\theta[\theta \varepsilon (Y-r,Y+r)]=\gamma.$

In the next section we compute the probability of falsely covering $\theta_0(\neq\theta)$ when the true parameter is θ and see that it is concave (from below) in θ.

§3. The Falsely covering probability.

Let

(13) $\xi(\theta) \doteq P_\theta[\theta_0 \varepsilon (Y-r,Y+r)],$ for real θ_0 with $\theta_0\neq\theta$

where r is given by (11). We shall call $\xi(\theta)$ as the probability of falsely covering $\theta_0(\neq\theta)$ when the true parameter is θ. We can easily calculate $\xi(\theta)$ as follows:

$$(14) \qquad \xi(\theta) = \begin{cases} 0, & \text{for } \theta \leq \theta_0-r-c/2, \\ \{1-2c^{-1}(\theta_0-r-\theta)\}^n/2, & \text{for } \theta_0-r-c/2 \leq \theta < \theta_0+r-c/2, \\ [\{1-2c^{-1}(\theta_0-r-\theta)\}^n-\{1-2c^{-1}(\theta_0+r-\theta)\}^n]/2, & \text{for } \theta_0+r-c/2 \leq \theta < \theta_0-r, \\ 1-[\{1-2c^{-1}(\theta-\theta_0+r)\}^n+\{1-2c^{-1}(\theta_0+r-\theta)\}^n]/2, & \text{for } \theta_0-r \leq \theta < \theta_0+r, \\ [\{1+2c^{-1}(\theta_0+r-\theta)\}^n - \{1-2c^{-1}(\theta-\theta_0+r)\}^n]/2, & \text{for } \theta_0+r \leq \theta < \theta_0-r+c/2, \\ \{1+2c^{-1}(\theta_0+r-\theta)\}^n/2, & \text{for } \theta_0-r+c/2 \leq \theta < \theta_0+r+c/2, \\ 0, & \text{for } \theta_0+r+c/2 \leq \theta. \end{cases}$$

(Here, we remark that for ($\alpha \geq 0.001$; $n \geq 10$), ($\alpha \geq 0.01$; $n \geq 7$), ($\alpha \geq 0.05$; $n \geq 5$), and ($\alpha \geq 0.10$; $n \geq 4$), we have that $\alpha^{1/n} \geq 1/2$. (See Table, below.) So, calculations led to (14) depend on α and n such that $\alpha^{1/n} \geq 2^{-1}$.)

Table. The Values of $\alpha^{1/n}$.

α	.10	.05	.01	.001
n=4	.56	.47	.32	.18
5	.63	.55	.40	.25
6	.68	.61	.46	.32
7	.72	.65	.52	.37
10	.79	.74	.63	.50
50	.95	.94	.91	.87
100	.98	.97	.96	.93

Since, $d\xi(\theta)/d\theta = \xi'(\theta) > 0$ for $\theta < \theta_0$, $\xi'(\theta) < 0$ for $\theta > \theta_0$ and $\xi'(\theta_0) = 0$, and since $\xi(\theta_0) = \alpha$ and $\xi(+\infty) = \xi(-\infty) = 0$, $\xi(\theta)$ is a concave (from below) function of θ. We also see that for all θ with $\theta \neq \theta_0$, $\xi(\theta) \downarrow 0$ as $n \uparrow \infty$.

In the next section we check the optimal property for the C. I. $(Y-r, Y+r)$ at $\theta = \theta_0$.

§4. Optimal property.

Assume, for simplicity, that

(15)
$$\phi(y) \doteq \begin{cases} 1, & \text{if } y_1 \geq y \quad \text{or } y_2 \leq y \\ 0, & \text{if } y_1 < y < y_2, \end{cases}$$

where y_1 and y_2 are real numbers such that $y_1 < y_2$.

Define $\beta(\theta) \doteq E_\theta(\phi(Y))$. Assume that $\theta = \theta_0$ is a true value. Then, we try to choose y_1 and y_2 which satisfy

(16)
$$\beta(\theta_0) = E_{\theta_0}(\phi(Y)) = \alpha$$

and

(17)
$$\beta'(\theta_0) = d\beta(\theta)/d\theta \Big|_{\theta=\theta_0} = g(y_2|\theta_0) - g(y_1|\theta_0) = 0.$$

Equations (16) and (17) are the same as (8) and (9) except for the value θ_0 of θ. Hence, the solution of (16) and (17) is (10) with θ replaced by θ_0.

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