On the Existence of an Optimal Income Tax Schedule*)

by

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April 1980

No.75 (80-13)

Abstract: We model an optimal income taxation problem in the context of general equilibrium analysis. We define an optimal tax schedule as one which gives a competitive equilibrium with the maximum value of a social welfare function. The purpose of this paper is to prove the existence of an optimal tax schedule.

*). The author wishes to express his thanks to Professors Yozo Ito and Toru Maruyama for valuable discussions and comments.

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1. Introduction

Since the pioneering work of Mirrlees [3] , many authors have considered the nonlinear optimal taxation problem in models with two kinds of goods - labor and consumption - in terms of the variational method or the maximum principle. They have been, however, done without any verification of the existence of an optimal tax schedule. That is, under the assumption of the existence, several necessary conditions have been derived and investigated. On the other hand, Kaneko [1] proved the existence of an optimal tax schedule in the class of progressive (convex) ones, which is narrower than that employed in the preceding works. Although the progressiveness of tax schedules is a natural condition to consider the optimal tax problem, it is better to be derived as a result in a more general theory which permits nonprogressive tax schedules. Hence we have two problems - one is the existence in the classs which permits nonprogressive tax schedules and another is the progressiveness of the optimal tax schedule as the result. In this paper we consider only the existence problem.

In this paper we model the optimal taxation problem in the context of general equilibrium analysis. We consider an economy with a finite number of individuals, a finite number of firms and government, where a finite number of consumption goods and one public good are produced by the firms and the government using labor as input. An optimal tax schedule means one which gives a competitive equilibrium with the maximum value of a social welfare function. Thus the model of this paper is much wider than those of the preceding studies. The purpose of this paper is to prove the existence of an optimal tax schedule.

2. Tax Schedule and Competitive Equilibrium

We consider an economy consisting of a finite number of individuals, a finite number of firms and an agent called"government". Let $N = \{1,2,\ldots,n\}$ be the set all individuals. We assume that leisure, c-kinds of consumption goods and a public good enter the individuals'utility functions $U^i(t,x,Q)$ ($i \in N$), where t denotes leisure time, x a level of consumption goods and Q a level of the public good supplied by the government. Every U^i is defined on $Y = [0,L] \times E_+^{c+1}$, where $L > 0$ is the initial endowment of leisure time and E_+^{c+1} the nonnegative orthant of the c+1-dimensional Euclidean space E^{c+1}. We assume:

(A): For all $i \in N$, $U^i(t,x,Q)$ is a monotonically increasing, continuous and quasi-concave function of (t,x,Q).

The following argument can be directly applicable to the case of more than one public goods. But we discuss the economy with one public good for notational simplicity.

Each individual i owns a labor production function $f^i(h)$. That is, if he works for h-hours, he can provide a quantity $f^i(h)$ of service called "labor". For example, labor may be measured in terms of the unit of man-power/hour. We assume:

(B): For all $i \in N$, $f^i(h)$ is a continuous and concave function of $h \in [0,L]$ with $f^i(0) = 0$ and $f^i(h) > 0$ for some $h > 0$.

We assume that all the individuals are endowed with no consumption goods. The consumption goods are produced by firms. There are m firms in the economy. Firm j ($j = 1,\ldots,m$) has a production set Z^j. We assume:

(C): $Z^j \subset E_- \times E^c$ for all $j = 1,\ldots,m$.

(D): Z^j is a closed convex cone for all $j = 1,\ldots,m$.

(E): There are $z^1 = (z_0^1, z_C^1) \in Z^1, \ldots, z^m = (z_0^m, z_C^m) \in Z^m$ such that $\sum_{j=1}^{m} z_C^j > 0$ but there are no $z^1 \in Z^1, \ldots, z^m \in Z^m$ such that $\sum_{j=1}^{m} z^j \geq 0$.

(F): For all $j = 1,\ldots,m$, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ for any $z^j \in Z^j$ and k_1, k_2 $(1 \leqq k_1, k_2 \leqq c)$ with $z_{k_1}^j > 0$ such that $z^j - \varepsilon_1 e^{k_1} + \varepsilon_2 e^{k_2} \in Z^j$.

Here E_- is the set of all nonpositive real numbers and $z_C^j = (z_1^j, \ldots, z_c^j) > 0$ means $z_k^j > 0$ for all $k = 1,\ldots,c$, and $z \geq 0$ does $z \geqq 0$ but $z \neq 0$, and e^k is the unit vector in E^{c+1} with $e_k^k = 1$.

Assumption (C) means that labor can not be produced by the firms. We would not need to explain Assumption (D), but this is a crucial assumption in this paper. Assumption (E) means that all the consumption goods can be produced by some firm and the impossibility of free production. Assumption (F) means that when a firm produces a positive amount of a consumption good, it can produce a positive amount of any other good by decreasing the production level of the consumption good.

Each firm j has a profit-share vector $d^j = (d_1^j, \ldots, d_n^j)$ with $d_i^j \geqq 0$ for all $i \in N$ and $\sum_{i \in N} d_i^j = 1$.

The government produces and supplies the public good using labor and consumption goods as inputs. The government has a production set Z^0 .

We assume:

(G): $Z^0 \subset E_- \times E_-^c \times E_+$ and $0 \in Z^0$.

(H): Z^0 is a closed convex set.

(I): There is no $z^0 \in Z^0$ such that $z^0 \geq 0$.

Here E_-^c is the nonpositive orthant of E^c .

Assumption (G) means that labor can not be produced and the public good can not be used as inputs. Further it excludes the possibility that the government produces consumption goods. We would not need to explain Assumption (H). Assumption (I) means the impossibility of free production.

We are now in a position to define our optimal taxation problem. A <u>tax function</u> T is a function from [0,M] to E which satisfies

$$T(y) \text{ is a continuous and nondecreasing function of } y \text{ with } T(y) \leqq y \text{ for all } y \in [0,M] . \tag{2.1}$$

Here M is a positive real number with $M \geqq \max_{i \in N} \max_{h} f^i(h)$. We assume that the government imposes taxes on the individuals' incomes measured in terms of labor. That is, a tax function T means that when an individual i works for h-hours and earns income $f^i(h)$, he must pay an income tax $T(f^i(h)) = Tf^i(h)$ to the government. Hence a tax function must satisfy $T(y) \leqq y$ for all $y \in [0,M]$. We denote by $\mathcal{T}$ the set of all tax functions.

Kaneko [1] assumed the convexity besides (2.1) on tax functions. The significance of the convexity (progressiveness) in the optimal taxation theory is clear and we do not need any discussion here. Mathematically, however, the convexity is a kind of uniformness condition and is a strong one. It permits a great deal of variety on tax functions to dispose the convexity. But we will show in the next section that we can restrict our consideration to a certain narrower class than , which plays the same role with the convexity in th existence proof of Kaneko [1] in the domain of progressive tax functions.

We say that $((t^1,x^1,Q),\ldots,(t^n,x^n,Q),z^0,z^1,\ldots,z^m)$ is an <u>allocation</u> iff

$$(t^i,x^i,Q) \in Y \text{ for all } i \in N ,\ z^j \in Z^j \text{ for all } j = 0,1,\ldots,m \text{ and } z^0_{c+1} = Q , \tag{2.2}$$

$$\sum_{i=1}^{n} f^i(L-t^i) + \sum_{j=0}^{m} z_0^j = 0 , \tag{2.3}$$

$$\sum_{i=1}^{n} x^i - \sum_{j=0}^{m} z_C^j = 0 . \tag{2.4}$$

It is easily verified that under our assumptions the set of all allocations is a compact set.

The economy works as follows. The government plans to employ a tax function $T \in \mathcal{T}$ and a production schedule $z^0 \in Z^0$. We call (T,z^0) a <u>tax schedule</u> and denote by $\mathcal{S}$ the set of all tax schedules. The government announces the tax schedule employed to the individuals. Under the tax schedule, the individuals and firms behave as price takers, and the prices of consumption goods and labor are determined by market mechanism. Thus we get the following definition.

<u>Definition 1</u>. $\gamma = (p,(t^1,x^1,Q),\ldots,(t^1,x^1,Q),z^1,\ldots,z^m)$ is said to be a <u>competitive equilibrium under a tax schedule</u> $\tau = (T,z^0)$ iff

$$p = (p_0,p_C) \in E_+^{c+1} \text{ with } p_0 > 0 \text{ and } ((t^1,x^1,Q),\ldots,(t^n,x^n,Q),z^0,z^1,\ldots,z^m) \text{ is an allocation,} \tag{2.5}$$

$$p_C x^i \leqq p_0(1-T)[f^i(L-t^i)+ \sum_{j=1}^{m} d_i^j pz^j/p_0] \quad \text{for all } i \in N, \tag{2.6}$$

$$\text{for all } i \in N,\ U^i(t^i,x^i,Q) \geqq U^i(t,x,Q) \quad \text{for all } (t,x) \text{ such that } p_C x \leqq p_0(1-T)[f^i(L-t)+ \sum_{j=1}^{m} d_i^j pz^j/p_0] , \tag{2.7}$$

$$\text{for all } j = 1,\ldots,m,\ pz^j \geqq pz \quad \text{for all } z \in Z^j . \tag{2.8}$$

We call $\tau = (T,z^0)$ a <u>feasible tax schedule</u> iff there exists a competitive equilibrium under τ which satisfies

$$-p(z_0^0, z_C^0) \leqq p_0 \sum_{i=1}^{n} T[f^i(L-t^i) + \sum_{j=1}^{m} d_i^j p z^j / p_0] \quad . \tag{2.9}$$

We denote by $\mathcal{S}_f$ the set of all feasible tax schedules and by $C(\tau)$ the set of all competitive equilibria under τ with (2.9).

Condition (2.5) means the positiveness of the price of labor and the coincidence of the total demands and the total supplies of consumption goods and labor. Condition (2.6) is the individuals' budget constraint and (2.7) is the individuals' utility maximization under the budget constraint. Condition (2.8) is the firms' profit maximization. Condition (2.9) means that the government's expenditure for the prodution of the public good does not exceed the revenue.

Let $(T^o,0)$ be the <u>trivial</u> tax schedule, i.e., $T^o(y) = 0$ for all $y \in [0,M]$ and $z^0 = 0$. That is, the government does nothing. If there exists a competitive equilibrium under $(T^o,0)$, then (2.9) is clearly satisfied by it. The definition of competitive equilibrium under $(T^o,0)$ is reduced to the standard one, so the existence of a competitive equilibrium in this case can be proved, slightly modifying the standard existence proof, e.g., Nikaido [4]. Thus we get the following theorem.

<u>Theorem I.</u> There exists a feasible tax schedule τ , i.e, $\mathcal{S}_f \neq \phi$.

Under our assumptions, we can easily prove:

<u>Lemma 1</u>. $p > 0$ for any competitive equilibrium $(p,(t^1,x^1,Q),\ldots,(t^n,x^n,Q), z^1,\ldots,z^m) \in C(\tau)$ and any $\tau \in \mathcal{S}_f$.

Since Z^j is a convex cone for all $j = 1,\ldots,m$, firm j's profit pz^j is always zero in equilibria for all $j = 1,\ldots,m$. In the following, we will use this fact and Lemma 1 without any remark.

The government employs a social welfare function such that

$$\sum_{i=1}^{n} G^i[U^i(\,.\,)] ,$$

where G^i is a monotone increasing and continuous function on $[U^i(L,0,0),+\infty)$. For example, if $G^i[U^i(\,.\,)] = \log[U^i(\,.\,) - U^i(0,0,0)]$ for all $i \in N$, then this welfare function is the Nash social welfare function of Kaneko and Nakamura [2]. When the government employs a feasible tax schedule $\tau = (T,z^0)$ and so a competitive equilibrium $\gamma = (p,(t^1,x^1,Q),\ldots,(t^n,x^n,Q),z^1,\ldots,z^m)$ results, the social welfare function is represented as

$$W(\tau,\gamma) = \sum_{i=1}^{n} G^i[U^i(t^i,x^i,Q)] . \tag{2.10}$$

The government maximizes the social welfare function over the feasible tax schedules.

Definition 2. $\tilde{\tau} \in \mathcal{S}_f$ is said to be an optimal tax schedule iff for some $\tilde{\gamma} \in C(\tilde{\tau})$,

$$W(\tilde{\tau},\tilde{\gamma}) = \sup_{\substack{\tau \in \mathcal{S}_f \\ \gamma \in C(\tau)}} W(\tau,\gamma) . \tag{2.11}$$

We would need no explanation for this definition. The purpose of this paper is to prove the existence of an optimal tax schedule. Although the meaning of an optimal tax schedule is clear, the definition deserves a special critical comment. The government has tax functions $T \in \mathcal{T}$ and production schedules $z^0 \in Z^0$ as the controllable variables. The other participants,i.e, the individuals and firms pursue independently their utilities and profits under no constraint other than the income taxation upon the individuals. Then the prices are determined by market mechanism. Although the government has some effect upon the prices by manipulating its production schedule z^0, i.e., it can behave as a monopolist with quantity control, it can not directly manipulate the market prices. That is, the government can not choose any competitive equilibrium from $C(\tau)$ after he does $\tau \in \mathcal{S}_f$. In the case of multi-equilibria in $C(\tau)$, the government can not expect a priori which equilibrium will result. It is determined historically by market mechanism. From the viewpoint of planning, it may be better in this case to define an optimal tax schedule $\tilde{\tau} \in \mathcal{S}_f$ by

$$W(\tilde{\tau},\tilde{\gamma}) = \max_{\tau \in \mathcal{S}_f} \min_{\gamma \in C(\tau)} W(\tau,\gamma) \quad \text{and} \quad \tilde{\gamma} \in C(\tilde{\tau}) . \tag{2.12}$$

This definition requires that the gurantee level of social welfare be maximized. Regretfully the author has not succeeded in proving the existence of an optimal tax schedule in the sense of (2.12).

3. The Restriction Theorem

Before we state and show the existence of an optimal tax schedule, we need to show the theorem which makes us enable to restrict our consideration within a narrower class, $\mathcal{T}_o$, than $\mathcal{T}$. $\mathcal{T}_o$ designates the set of all tax functions $T \in \mathcal{T}$ having the following property (3.1):

$$\frac{T(y_1)-T(y_2)}{y_1 - y_2} \leqq 1 \quad \text{for all } y_1, y_2 \in [0,M] \text{ with } y_1 \neq y_2 . \tag{3.1}$$

A tax function $T \in \mathcal{T}_o$ has the property that its marginal tax rate is not greater than 100 % at everywhere.

Let (T_1, z^0) and (T_2, z^0) be feasible tax schedules. Then (T_1, z^0) is said to be equivalent to (T_2, z^0) iff $C(T_1, z^0) = C(T_2, z^0)$. The equivalence of (T_1, z^0) and (T_2, z^0) means that even if the government employs either, the same equilibria can be expected to result. Of course, this relation is an "equivalence relation" on $\mathcal{S}_f$.

Theorem II. (The Restriction Theorem): For any feasible tax schedule (T, z^0), there exists a tax function T_o in $\mathcal{T}_o$ such that (T, z^0) is equivalent to (T_o, z^0) .

Theorem II says that for every feasible tax schedule, we can curve the tax function so that the new tax function has the same equilibria but has also the marginal tax rates not greater than 100 % . Hence this theorem ensures that we can restrict our consideration to $\mathcal{T}_o$ instead of $\mathcal{T}$.

In the rest of this section we prove this theorem. If $T \in \mathcal{T}_o$, then we need to prove nothing. Suppose $T \notin \mathcal{T}_o$ in the following. We define T_o by

$$T_o(y) = \min[T(y), \min_{0 \leq y_1 \leq y} (y-y_1)+T(y_1)] \quad \text{for all } y \in [0,M] . \qquad (3.2)$$

Lemma 2. T_o is a nondecreasing and continuous function having property (3.1), i.e., $T_o \in \mathcal{T}_o$.

Proof. Since $\min_{0 \leq y_1 \leq y} (y-y_1)+T(y_1)$ is a continuous function of y, $T_o(y)$ is also a continuous function of y. If $(T_o(y_1)-T_o(y_2))/(y_1-y_2) > 1$ for some y_1, y_2 with $y_1 > y_2$, then

$$T_o(y_1) > (y_1-y_2) + T_o(y_2) .$$

But it follows from (3.2) that $T_o(y_2) = T(y_2)$ or $T_o(y_2) = (y_2-y_3) + T(y_3)$ for some $y_3 (< y_2)$. This implies

$$T_o(y_1) > (y_1-y_2) + T(y_2)$$

or

$$T_o(y_1) > (y_1-y_2) + (y_2-y_3) + T(y_3) = (y_1-y_3) + T(y_3) ,$$

which is a contradiction to (3.2). Hence T_o has property (3.1) .

We show the monotonicity. Let $y_1 > y_2$. When $T(y_1) = T_o(y_1)$, we have, by the monotonicity of T, $T_o(y_2) \leq T(y_2) \leq T(y_1) = T_o(y_1)$. Hence we can suppose that $T(y_1) > T_o(y_1)$. Since $T(y_1) > T_o(y_1)$, there is an $x_1 < y_1$ such that $T_o(y_1) = (y_1-x_1) + T(x_1)$. When $x_1 \geq y_2$, we have

$$T_o(y_2) \leq T(y_2) \leq T(x_1) < (y_1-x_1) + T(x_1) = T_o(y_1) .$$

When $x_1 < y_2$, it holds by (3.2) that

$$T_o(y_2) \leq (y_2-x_1) + T(x_1) .$$

Hence we have

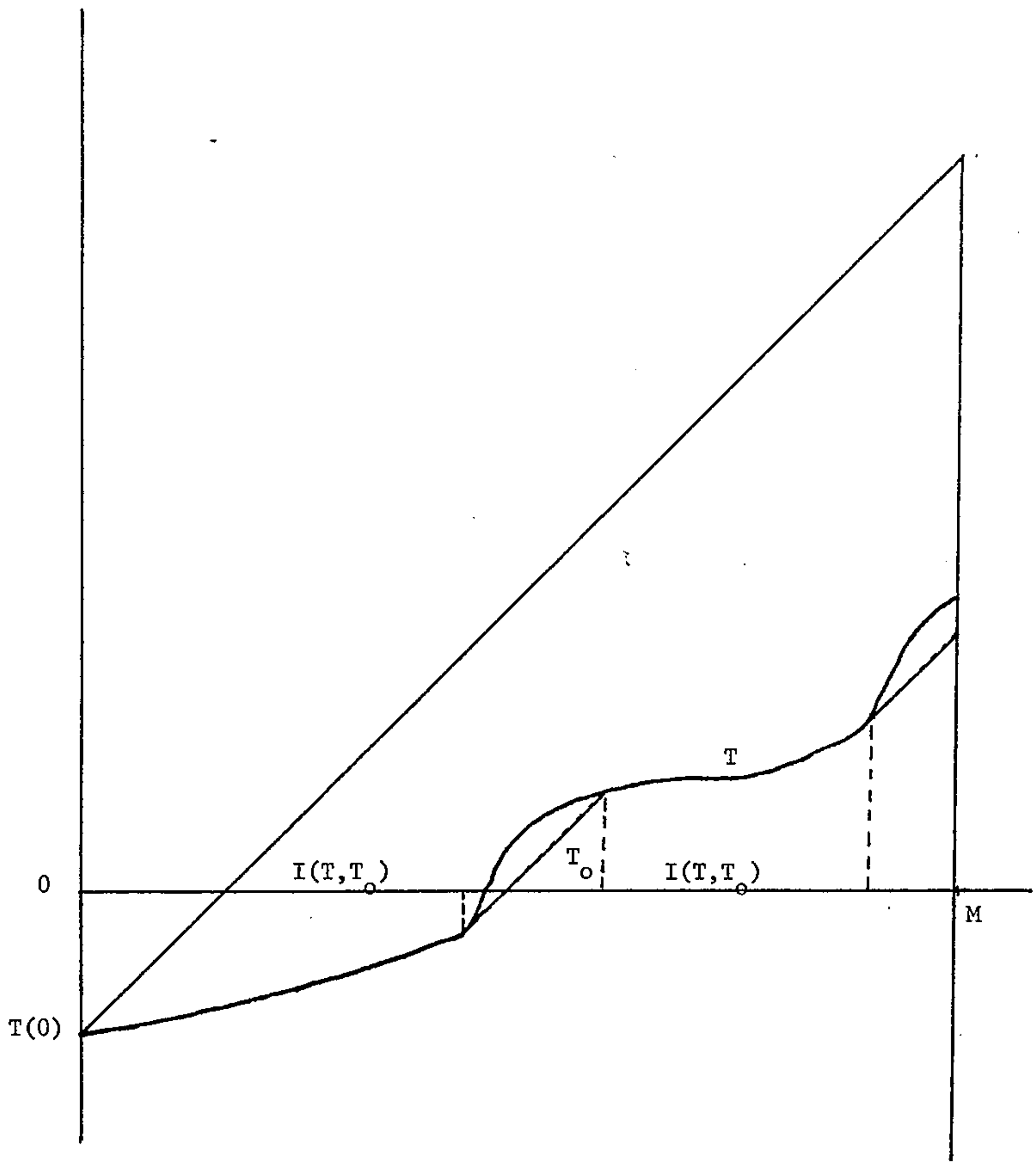

Figure 1.

$$T_o(y_2) \leq (y_2-x_1) + T(x_1) < (y_1-x_1) + T(x_1) = T_o(y_1) .$$

Q.E.D.

We define $I(T,T_o)$ by

$$I(T,T_o) = \{ y \in [0,M] : T(y) = T_o(y) \} . \tag{3.3}$$

Lemma 3.(i). If $\gamma = (p,(t^1,x^1,Q),\ldots,(t^n,x^n,Q),z^1,\ldots,z^n) \in C(T,z^0)$, then $f^i(L-t^i) \in I(T,T_o)$ for all $i \in N$.

(ii). If $\gamma \in C(T_o,z^0)$, then $f^i(L-t^i) \in I(T,T_o)$ for all $i \in N$.

Proof. We prove only (i), but can prove (ii) analogously. Individual i's gross income is $f^i(L-t^i)$ when we put $p_0 = 1$. Suppose $f^i(L-t^i) \notin I(T,T_o)$. Then $T(y) > T_o(y)$, where $y = f^i(L-t^i)$. So there is a $y_1 < y$ such that $T_o(y) = (y-y_1) + T(y_1)$. If he works for $L-t_1$ hours such that $f^i(L-t_1) = y_1$ and $t_1 > t^i$, then the disposable income is greater than $y-T(y)$, because $y-T(y) < y-T_o(y) = y - (y-y_1) - T(y_1) = y_1 - T(y_1)$. This means that the individual can increase his utility level. This is a contradiction to the supposition that γ is a competitive equilibrium under (T,z^0) . Q.E.D.

Lemma 4. $\gamma \in C(T,z^0)$ iff $\gamma \in C(T_o,z^0)$.

Proof. We prove that if $\gamma \in C(T,z^0)$, then $\gamma \in C(T_o,z^0)$. It is sufficient to show that γ satisfies (2.6),(2.7) and (2.9) under (T_o,z^0). It follows from Lemma 3 that γ satisfies (2.6) and (2.9). Let (2.7) be not satisfied. Then there is a (t,x) such that $f^i(L-t) \notin I(T,T_o)$, $U^i(t,x,Q) > U^i(t^i,x^i,Q)$ and $p_C x \leq p_0(1-T_o)f^i(L-t)$. Since $f^i(L-t) \equiv y \notin I(T,T_o)$, there is a $y_1 < y$ such that $T_o(y) = (y-y_1) + T(y_1)$. Let t_1 be the real number such that $y_1 = f^i(L-t_1)$ and $t_1 > t$. Then $f^i(L-t_1) - Tf^i(L-t_1) = f^i(L-t) - (f^i(L-t)-$

$f^i(L-t_1)) - Tf^i(L-t_1) = f^i(L-t) - T_o f^i(L-t)$. In sum, we have

$$f^i(L-t_1) \in I(T,T_o) \ , \ U^i(t_1,x,Q) > U^i(t,x,Q) > U^i(t^i,x^i,Q)$$
$$\text{and } p_C x \leqq p_0(1-T_o)f^i(L-t) = p_0(1-T)f^i(L-t_1) \ .$$

This is a contradiction to the supposition $\gamma \in C(T,z^0)$.

We can analogously prove the converse part of this lemma.

Q.E.D.

4. The Existence Theorem

We are in a position to state the main theorem of this paper.

Theorem III. (The Existence Theorem): There exists an optimal tax schedule.

Let $\mathcal{S}_o = \{ (T,z^0) \in \mathcal{S}_f : T \in \mathcal{T}_o \}$. Then it follows from Theorem II that

$$\sup_{\substack{\gamma \in C(\tau) \\ \tau \in \mathcal{S}_o}} W(\tau,\gamma) = \sup_{\substack{\gamma \in C(\tau) \\ \tau \in \mathcal{S}_f}} W(\tau,\gamma) \ . \tag{4.1}$$

Since U^i and G^i are continuous functions for all $i \in N$ and the set of all allocations is a compact set, we have $\sup W(\tau,\gamma) < +\infty$. Hence there is a sequence $\{(\tau^s,\gamma^s)\} = \{((T^s,z^{0s}),(p^s,(t^{1s},x^{1s},Q^s),\ldots,(t^{ns},x^{ns},Q^s),z^{1s},\ldots,z^{ms}))\}$ such that $\gamma^s \in C(\tau^s)$ & $\tau^s \in \mathcal{S}_o$ for all $s \geqq 1$ and $\lim_{s \to \infty} W(\tau^s,\gamma^s) = \sup W(\tau,\gamma)$. The outline of the following proof is that we can choose a convergence subsequence $\{(\tau^{s^\nu},\gamma^{s^\nu})\}$ from $\{(\tau^s,\gamma^s)\}$ and then the limit point (τ,γ) really attains (4.1) .

Lemma 5. $\inf T^s(0) > -\infty$.

Proof. Suppose $\inf\limits_s T^s(0) = -\infty$. Since each T^s belongs to $\mathcal{T}_o$, $T^s(y) \leqq T^s(0) + y$ for all $y \in [0,M]$ and all s. Hence there is an s_o such that $T^s(y) \leqq T^s(0) + M < 0$ for all $y \in [0,M]$ and all $s \geqq s_o$. This means that for large s, the government's revenue is negative. This is a contradiction to the supposition that $\gamma^s \in C(\tau^s)$ for all s . Q.E.D.

Let $K = \inf\limits_s T^s(0)$. We define C[0,M] by

$$C[0,M] = \{ t : t \text{ is a continuous nondecreasing function with } (t(y_1)-t(y_2))/(y_1-y_2) \leqq 1 \text{ for all } y_1,y_2 \ (y_1 \neq y_2) \text{ and } K \leqq t(y) \leqq y \text{ for all } y \in [0,M] \} \tag{4.2}$$

Lemma 6. C[0,M] is a compact set with respect to the topology of uniform convergence.

Proof. By Ascoli's theorem (Simmons [5 , Section 25,Theorem C]), it is sufficient to show that C[0,M] is closed, bounded and equicontinuous.

It is easily verified that C[0,M] is closed and bounded. By (4.2) it holds that for all $t \in C[0,M]$,

$$|y_1 - y_2| \leqq \delta \quad \text{implies} \quad |t(y_1) - t(y_2)| \leqq \delta .$$

This means that C[0,M] is equicontinuous. Q.E.D.

We can normalize $\{p^s\}$ without loss of generality such that each p^s belongs to $P = \{ p \in E_+^{c+1} : \sum_{k=0}^{c} p_k = 1 \}$. Let A be the set of all allocations. We reorder $(\tau,\gamma) = ((T,z^0),(p,(t^1,x^1,Q),\ldots,(t^n,x^n,Q),z^1,\ldots,z^m)$ such that $(T,p,(t^1,x^1,Q),\ldots,(t^n,x^n,Q),z^0,z^1,\ldots,z^m)$ and regard it as the same with (τ,γ).

Then the sequence $\{(\tau^s,\gamma^s)\}$ is in $C[0,M]\times P\times A$. Clearly $C[0,M]\times P\times A$ is a compact set, so there is a subsequence $\{(\tau^{s^\nu},\gamma^{s^\nu})\}$ of $\{(\tau^s,\gamma^s)\}$ which converges to $(\tau^*,\gamma^*) = ((T^*,z^{0*}),(p^*,(t^{1*},x^{1*},Q^*),\ldots,(t^{n*},x^{n*},Q^*),z^{1*},\ldots,z^{m*})$ in the sense of the product topology.

We assume for notational simplicity that $\{(\tau^s,\gamma^s)\}$ itself converges to (τ^*,γ^*). Then since U^i and G^i are continuous functions for all $i\in N$, we have

$$\sup W(\tau,\gamma) = \lim_{s\to\infty} \sum_{i\in N} G^i(U^i(t^{is},x^{is},Q^s)) = \sum_{i\in N} G^i(U^i(t^{i*},x^{i*},Q^*)).$$

Further it holds that $T^*\in C[0,M]\subset \mathcal{T}_o$. Hence it is sufficient to show that γ^* is a competitive equilibrium under τ^* with (2.9).

Lemma 7. $p_0^* > 0$.

Proof. Let $p_0^* = 0$. By (E) there are $z^1\in Z^1,\ldots,z^m\in Z^m$ such that $\sum_{j=1}^m z_C^j > 0$. Then for some s_o, $p^s\cdot\sum_{j=1}^m z^j > 0$ for all $s \geqq s_o$. That is, for all $s\geqq s_o$, there is a j such that $p^s z^j > 0$. Then $p^s z^{js} \geqq p^s z^j > 0$ by (2.8). This is impossible. Q.E.D.

Suppose:

$$f^i(L-t^{i*}) - T^* f^i(L-t^{i*}) = 0 \quad \text{for all } i\in N. \tag{4.3}$$

Then it is easily verified that if $p_k^* > 0$, then $\lim_{s\to\infty} x_k^{is} = 0$ for all $i\in N$ because of Lemma 7 and (4.3). Let $P_C^* = 0$. Then since $\left\{p_C^s\cdot\sum_{j=1}^m z_C^{js}\right\}$ converges to 0 by the compactness of the set of allocations, $p^s\cdot\sum_{j=1}^m z^{js} = p_0^s\cdot\sum_{j=1}^m z_0^{js} + p_C\cdot\sum_{j=1}^m z_C^{js}$ $= 0$ for all s imply $\lim_{s\to\infty}\sum_{j=1}^m z_0^{js} = 0$, i.e., $z_0^{j*} = 0$ for all j because of Lemma 7. This implies $z_C^{j*} = 0$ for all $j = 1,\ldots,m$ by (E), and so, $x^{i*} = 0$ for all $i\in N$. Finally let us consider the case where $p^*_{k_1} = 0$ and $p^*_{k_2} > 0$ for some k_1, k_2 $(1 \leq k_1, k_2 \leq c)$. In this case, if $z^{j*}_{k_1} > 0$ for some j, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by (F) such that $z^{j*} - \varepsilon_1 e^{k_1} + \varepsilon_2 e^{k_2}$ is in Z^j. Hence $p^s(z^{j*} - \varepsilon_1 e^{k_1} + \varepsilon_2 e^{k_2}) \longrightarrow p^*(z^{j*} - \varepsilon_1 e^{k_1} + \varepsilon_2 e^{k_2}) = p^*_{k_2}\varepsilon_2 > 0$, which is a contradiction to that $\max_{z \in Z^j} p^s z = 0$ for all s. Thus we have shown that $z^{j*}_{k_1} = 0$ for all j. Hence it follows that $x^{i*}_{k_1} = 0$ for all $i \in N$. In sum, it is always true that $x^{i*} = 0$ for all $i \in N$.

Since U^i and G^i are continuous functions for all $i \in N$, it follows from the above argument that $\lim_{s \to \infty} U^i(t^{is}, x^{is}, Q^s) = U^i(t^{i*}, 0, Q^*)$ for all $i \in N$. If $t^{i*} < L$, then for a sufficiently small $\varepsilon > 0$, there is an s_o such that $U^i(L,0,Q^*) - \varepsilon > U^i(t^{is}, x^{is}, Q^s)$ for all $s \geq s_o$. Further there is an s_1 such that $U^i(L,0,Q^s) > U^i(L,0,Q^*) - \varepsilon$ for all $s \geq s_1$. These imply $U^i(L,0,Q^s) > U^i(t^{is}, x^{is}, Q^s)$ for all $s \geq \max(s_o, s_1)$, which is a contradiction because (L,0) always satisfies his budget. Hence we have $t^{i*} = L$ for all $i \in N$. This also implies $Q^* = 0$. Hence we have

$$\sup W(\tau,\gamma) = \sum_{i \in N} G^i(U^i(L,0,0)) .$$

Let $\tau^o = (T^o, 0)$ be the trivial tax schedule and $\gamma^o = (p^o, (t^{1o}, x^{1o}, 0), \ldots, (t^{no}, x^{no}, 0), z^{1o}, \ldots, z^{mo}) \in C(\tau^o)$. The existence of such a γ^o has been stated in the preceding paragraph of Theorem I. It is clear that $U^i(t^{io}, x^{io}, 0) \geq U^i(L,0,0)$ for all $i \in N$. Hence we have

$$W(\tau^o, \gamma^o) = \sup W(\tau,\gamma) . \tag{4.4}$$

In the following we consider the case where

$$f^i(L-t^{i*}) - T^*f^i(L-t^{i*}) > 0 \text{ for some } i \in N . \qquad (4.5)$$

Lemma 8. $p^* > 0$.

Proof. Let $p_k^* = 0$ for some k $(1 \leqq k \leqq c)$. Then it is easily verified that $x_k^{is} \to \infty (s \to \infty)$ for all i with $f^i(L-t^{i*})-T^*f^i(L-t^{i*}) > 0$. This contradicts the impossibility of free production. Q.E.D.

It is easily verified that γ^* satisfies (2.5),(2.6) and (2.8) under τ^* . We show that γ^* satisfies (2.7). Suppose that there is a (t^o,x^o) such that $U^i(t^o,x^o,Q^*) > U^i(t^{i*},x^{i*},Q^*)$ and $p_C^*x^o \leqq p_0^*(1-T^*)f^i(L-t^o)$. Now let us show $(1-T^*)f^i(L-t^o) > 0$. On the contrary let $(1-T^*)f^i(L-t^o) = 0$. Then $x^o = 0$ because $p^* > 0$. Hence $U^i(L,0,Q^*) \geqq U^i(t^o,0,Q^*) > U^i(t^{i*},x^{i*},Q^*)$. Since $\{(t^{is}, x^{is},Q^s)\}$ converges to (t^{i*},x^{i*},Q^*), there is an s_o such that $U^i(L,0,Q^s) > U^i(t^{is}, x^{is},Q^s)$ for all $s \geqq s_o$. But (L,0) satisfies i's budget constraint under (T^s,z^s). This is a contradiction. Hence $(1-T^*)f^i(L-t^o) > 0$. Then there is another $(\tilde{t},\tilde{x})$ in a neiborhood of (t^o,x^o) by the continuity of U^i and Lemma 8 such that $U^i(\tilde{t},\tilde{x},Q^*) > U^i(t^{i*},x^{i*},Q^*)$ and $p_C^*\tilde{x} < p_0^*(1-T^*)f^i(L-\tilde{t})$. Since $\{T^s\}$ converges uniformly to T^* and $\{p^s\}$ converges to p^* , there is an s_1

such that $p_C^s\tilde{x} < p_0^s(1-T^s)f^i(L-\tilde{t})$ for all $s \geq s_1$. Since $\{(t^{is},x^{is},Q^s)\}$ converges to (t^{i*},x^{i*},Q^*), there is an s_2 for some $\varepsilon > 0$ such that $U^i(\tilde{t},\tilde{x},Q^*) - \varepsilon > U^i(t^{is},x^{is},Q^s)$ for all $s \geq s_2$. Since $\{Q^s\}$ converges to Q^*, there is an s_3 such that $U^i(\tilde{t},\tilde{x},Q^s) \geq U^i(\tilde{t},\tilde{x},Q^*) - \varepsilon$ for all $s \geq s_3$. Hence it holds that $U^i(\tilde{t},\tilde{x},Q^s) > U^i(t^{is},x^{is},Q^s)$ and $p_C^s\tilde{x} < p_0^s(1-T^s)f^i(L-\tilde{t})$ for all $s \geq \max(s_1,s_2,s_3)$. This is a contradiction to that γ^s is a competitive equilibrium under T^s for all $s \geq 1$. Thus we have proved (2.7).

Finally we show that γ^* satisfies (2.9) under T^*. Since $\{\gamma^s\}$ and $\{z^{0s}\}$ converges to γ^* and z^{0*} and since $\{T^s\}$ converges uniformly to T^*, $\{T^sf^i(L-t^{is})\}$ converges to $T^*f^i(L-t^{i*})$ for all i. Hence the condition that $-p^s(z_0^{0s},z_C^{0s}) \leq p_0^s \cdot \sum_{i=1}^{n} T^s[f^i(L-t^{is})]$ for all $s \geq 1$ implies $-p^*(z_0^{0*},z_C^{0*}) \leq p_0^* \sum_{i=1}^{n} T^*[f^i(L-t^{i*})]$.

5. Concluding Remarks

1). We have chosen labor as numeraire, and so we have assumed that the government imposes taxes upon the individuals' incomes measured in terms of labor. But it is also possible to choose any other good as numeraire. Or, althoguh the government does not measure the individuals' incomes by any one good, the argument of this paper remains true if we define appropriately the unit of incomes. For example, we always normalize price vector p such that $\sum_{k=0}^{c} p_k = 1$ and assume that incomes are measured in terms of the value defined by p, i.e., when individual i works for h hours, his gross income and disposable income are $p_0f^i(L-h)$ and $p_0f^i(L-h)-T[p_0f^i(L-h)]$. But since the individuals can get incomes only by selling thier labor in our economy, there is no substantial difference among these settings and that employed in this paper.

2). Kaneko [1] considers an optimal tax problem in an economy with a continuum of individuals and the class of progressive tax functions. The employment of a continuum of individuals is necessary and conveinient for us to consider limit properties of optimal tax functions as income level tends to infinity,e.g.,the limit marginal tax rate, the limit diposable income, etc. . In the model of Kankeo [1], an equilibrium is uniquely determined by a feasible tax schedule. This is caused by the assumptions of progressiveness and one consumption good. Using this fact, the existence proof of an optimal tax schedule is succeeded in Kaneko [1] . But it may yield multi-equilibria for a feasible tax schedule to dispose the progressiveness on tax functions or to permit more than one consumption goods. In the case of a continuum of individuals and multi-equilibria, the author has not succeeded in proving the existence of an optimal tax schedule. This is an important open problem.

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