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Choquet Expected Utility with Dense Capacities

by

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Abstract

Choquet expected utility (CEU) generalizes subjective expected utility (SEU) by relaxing additivity of state probabilities. There are only two axiomatizations by Gilboa (1987) and Sarin and Wakker (1992) to generalize Savage's SEU model. As in Savage's theory, they impose two restrictions on the representational form of CEU models, i.e., the boundedness of utility and the continuous divisibility of the state space. This paper establishes an axiomatic characterization without those restrictions in Savage's SEU framework. Our representational form yields a (not necessarily bounded) utility function and a dense capacity, the range of which forms a dense set in the unit interval.

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1 Introduction

Since Savage's (1954) axiomatization of subjective expected utility (SEU) model that represents numerically the personal beliefs (state probabilities) and preferences of a decision maker, numerous alternative theories in decision making under uncertainty have been proposed under various set-ups (see a survey by Fishburn (1981)). However, abundant evidence shows that people's carefully considered decisions often violate the underlying axioms of SEU models. One of the well-known violations was observed by Ellsberg (1961) who discussed decision situations under uncertainty that contradict additivity of state probabilities. It has not been recognized how to cope with capacities, i.e., nonadditive state probabilities, in SEU framework until Schmeidler (1984, 1989) first discovered that the Choquet (1953–54) integration provides an appropriate numerical representation, referred to here as Choquet expected utility (CEU).

Schmeidler's CEU model generalizes Anscombe and Aumann's (1963) SEU model which uses an auxiliary device, i.e., lottery-acts (functions from the state space into a mixture set of probability distributions on the consequence space). His model has motivated much research interest in axiomatic characterizations of CEU models in various set-ups, including Gilboa (1987), Chew and Karni (1994), Nakamura (1990, 1992), Sarin and Wakker (1992), and Wakker (1989, 1993).

Gilboa (1987) was the first to axiomatize a CEU model in Savage's SEU framework. As in Savage's theory, his axiomatization imposes two restrictions on the representational form of his CEU model. The first restriction is the boundedness of utility, and the second is the continuous divisibility of the state space, i.e., the set of state probability numbers is the unit interval. Furthermore, his axiomatization cannot draw directly on Savage's theory. A recent axiomatization by Sarin and Wakker (1992) is also confined to those two restrictions, since preferences for acts (i.e., functions from the state space into the consequence space) restricted to unambiguous events obey Savage's Theory. One of the aims of the paper is to give an axiomatic characterization of a CEU model without those restrictions in Savage's SEU framework, so that a utility function is not necessarily bounded and the set of state probability numbers forms a dense set in the unit interval.

Although additivity of the state probabilities contradicts Ellsberg's observation, another stream of the research to generalize Savage's theory was initiated by Machina and Schmeidler (1992) who demontrated that the existence of additive state probabilities is robust for less demanding preferences than in Savage's SEU model. When a decision maker's preferences for acts can be translated into risk preferences for probability measures over the consequence space induced by additive state probabilities through acts, they call those (transitive) preferences probabilistically sophisticated.

The preferences that satisfy Savage's SEU model are probabilistically sophisticated, so that the induced risk preferences obey von Neumann-Morgenstern expected utility (EU) model. As a generalization of EU models, Quiggin (1982) and Yaari (1987) independently investigated rank dependent expected utility (RDEU) models, which motivated much axiomatic refinements and generalizations, such as Chew

(1989), Green and Jullien (1988), Nakamura (1992, 1995a), Quiggin and Wakker (1994), Segal (1989), Wakker (1990), and others.

CEU and RDEU models have a common representational form, i.e., expected utility of a gamble or an act is calculated with respect to a (distorted) (de)cumulative distribution over the consequence space generated by that gamble or act. Nakamura (1995b) recently axiomatized a CEU model with probabilistically sophisticated preferences, so that the capacity is given by a strictly increasing transformation of an additive state probability measure, and the induced risk preferences obey an RDEU model. His model relaxed the boundedness of utility, but retained the continuous divisibility. The second aim of the paper is to axiomatize probabilistically sophisticated preferences for CEU models without the boundedness and the continuous divisibility. When the capacity is a linear transformation of an additive state probability measure, this model is reduced to an SEU model. Since the state space need not be continuously divisible, our SEU model has a more general representational form than Wakker's (1993) generalization of Savage's theory with an unbounded utility function.

The paper is organized as follows. Section 2 discusses a CEU model in our set-up and two specializations. Then Section 3 describes axioms and three representation theorems for simple acts. Section 4 provides axioms and theorems to cover all measurable acts. Section 5 proves the existence of a unique locally convex, and dense capacity. Then Section 6 proves the existence of a unique locally convex and finitely additive state probability measure for two specializations of our CEU model. Section 7 and 8 provide the proofs of three representation theorems respectively for all simple acts and all measurable acts.

2 Choquet Expected Utility Models

Let Γ_S denote a Boolean algebra of subsets of the state space S, i.e., Γ_S contains $\emptyset$ (empty set) and S, and is closed under finite unions and complementation. Elements of Γ_S are called *event.* By A^c we denote the complement $S\backslash A$ of an event A. A real valued function π on Γ_S is said to be a *capacity* if $\pi(S) = 1$, $\pi(\emptyset) = 0$, and for all $A, B \in \Gamma_S$, $\pi(A) \leq \pi(B)$ whenever $A \subseteq B$. A capacity π is *finitely additive* if for all disjoint $A, B \in \Gamma_S$, $\pi(A \bigcup B) = \pi(A) + \pi(B)$. In general a capacity is not necessarily additive.

Given a capacity π, the *range* of π is denoted by $R(\pi)$ and is defined to be the set $\{\pi(A) : A \in \Gamma_S\}$. A capacity π is said to be *locally convex* if for any $A, B \in \Gamma_S$ with $A \subset B$, and any $\alpha \in R(\pi)$ with $\pi(A) < \alpha < \pi(B)$, there is an event $C \in \Gamma_S$ such that $A \subset C \subset B$ and $\pi(C) = \alpha$. By $\overline{R(\pi)}$ we denote the closure of $R(\pi)$. We say that a capacity π is *dense* if $\overline{R(\pi)} = [0, 1]$ (the unit interval), and *continuous* if $R(\pi) = [0, 1]$. Capacities that we shall consider may not take any value in the unit interval, i.e., there may exist a number $0 < \alpha < 1$ such that $\pi(A) = \alpha$ for no event $A \in \Gamma_S$. However, we require that capacities be dense. When capacities are finitely additive, it is known that the range is either finite or dense (see Rao and Rao, 1983).

Acts are defined as functions from S into the consequence space Ω. A constant act is an act f such that $f(s) = x$ for all $s \in S$ and some $x \in \Omega$. Every $x \in \Omega$ will be identified with a constant act. Let $\preceq$ be the binary preference relation on a set $\mathcal{F}$ of acts including all constant acts. Let $\prec$ and $\sim$ be defined as usual, i.e., for $f, g \in \mathcal{F}$, $f \prec g$ if not$(g \preceq f)$, and $f \sim g$ if $f \preceq g$ and $g \preceq f$. We write $f \preceq g \preceq h$ when $f \preceq g$ and $g \preceq h$; $f \prec g \prec h$ when $f \prec g$ and $g \prec h$; $f \sim g \sim h$ when $f \sim g$ and $g \sim h$.

Following Fishburn (1982), a subset X of Ω is said to be a *preference interval* if $z \in X$ whenever $x, y \in X$ and $x \preceq z \preceq y$. Certain preference intervals will be denoted as follows.

$$\begin{aligned}(-\infty, a] &= \{x \in \Omega : x \preceq a\},\\ [a, +\infty) &= \{x \in \Omega : a \preceq x\},\\ [a, b) &= \{x \in \Omega : a \preceq x \prec b\}.\end{aligned}$$

Preference intervals such as $(-\infty, a), (a, +\infty), [a, b]$, and so forth are similarly defined.

By Γ_Ω we denote a Boolean algebra of subsets of Ω that contains all preference intervals. Given an act f, let $f^{-1}(X) = \{s \in S : f(s) \in X\}$ for $X \in \Gamma_\Omega$. For $x \in \Omega$, we shall write $f^{-1}(x)$ in place of $f^{-1}(\{x\})$. When $X = (-\infty, x]$, we shall write $f^{-1}(x_\downarrow)$ in place of $f^{-1}((-\infty, x]))$. We say that an act f is *measurable* if for every $X \in \Gamma_\Omega$, $f^{-1}(X)$ is in Γ_S. Note that all constant acts are measurable. In the sequel, we shall take $\mathcal{F}$ as the set of all measurable acts. Our basic structural assumptions for S, Ω, and $\mathcal{F}$ are stated as follows.

Assumption 1 *S is a nonempty set endowed with a Boolean algebra Γ_S of subsets of S. Γ_Ω is a Boolean algebra of subsets of Ω that contains all preference intervals. $\mathcal{F}$ is the set of all measurable acts.*

We say that $(\mathcal{F}, \preceq)$ has a CEU representation (u, π) if there exist a (not necessarily bounded) utility function u on Ω and a locally convex and dense capacity π on Γ_S such that for all $f, g \in \mathcal{F}$,

$$f \preceq g \Longleftrightarrow \int_S u(f(s))d\pi(s) \leq \int_S u(g(s))d\pi(s),$$

and u is unique up to a positive linear transformation and π is unique. The integration is defined as follows:

$$\begin{aligned}\int_S u(f(s))d\pi(s) &= \int_0^{+\infty} (1 - \pi(\{s : u(f(s)) \leq \tau\}))\, d\tau\\ &\quad - \int_{-\infty}^0 \pi(\{s : u(f(s)) \leq \tau\})d\tau,\end{aligned}$$

which is the Choquet integration meaning that the expectation is calculated with respect to (w.r.t.) cumulative distributions on the utility space induced by the capacity π through acts.

One of the aims of the paper is to provide an axiomatic characterization for the CEU representation (u, π). Gilboa (1987) presumed that $\Gamma_S = 2^S$ (the power set of S). His axiomatization yields a unique locally convex and continuous capacity π and a bounded utility function u as in Savage's SEU model. Sarin and Wakker (1992) presumed that Γ_S is a σ-algebra and consists of two types of events, unambiguous and ambiguous events, and that the set of unambiguous events forms a σ-subalgebra Γ_S^g of Γ_S. Their axiomatization yields a bounded utility function u on Ω and a unique capacity π on Γ_S such that π on Γ_S^g is locally convex, continuous, and finitely additive.

Gilboa (1985) considered the problem that given a locally convex and continuous capacity π on Γ_S, when there exists a function ϕ on $[0,1]$ and a locally convex, continuous, and finitely additive probability measure π^* on Γ_S such that $\pi = \phi(\pi^*)$, and provided necessary and sufficient conditions for that problem. Unfortunately, those conditions are not translated into preference-based conditions, i.e., conditions that are stated by $\preceq$. Recently, Nakamura (1995b) developed a preference-based axiomatization by applying the concept of probabilistic sophistication to a CEU model, and obtained a slightly stronger version of the RDEU representation defined below, since the capacity π is continuous.

The second aim of the paper is to provide axiomatic characterizations for the RDEU representation (u, π^*, ϕ) and the SEU representation (u, π^*) described in the sequel. We say that $(\mathcal{F}, \preceq)$ has an RDEU representation (u, π^*, ϕ) if it has a CEU representation (u, π) and there exist a locally convex, dense, and finitely additive probability measure π^* on Γ_S and a strictly increasing continuous function ϕ on $[0,1]$ such that for all $A \in \Gamma_S$,

$$\pi(A) = \phi(\pi^*(A)),$$

where ϕ is unique up to a positive linear transformation. The uniqueness of π^* follows from the uniqueness of π and ϕ. The integration is described as follows:

$$\begin{aligned} \int_S u(f(s))d\pi(s) &= \int_0^{+\infty} (\phi(1) - \phi(\pi^*(\{s : u(f(s)) \leq \tau\})))d\tau \\ &\quad - \int_{-\infty}^0 (\phi(\pi^*(\{s : u(f(s)) \leq \tau\})) - \phi(0))d\tau. \end{aligned}$$

The function ϕ may be called a *distortion function* of cumulative distributions. Utilities of acts are calculated w.r.t. distorted cumulative distributions on the utility space induced by an additive state probability measure π^* through acts. When ϕ is a linear function, the RDEU representation (u, π^*, ϕ) is reduced to an SEU representation (u, π^*).

3 Axioms and Theorems for Simple Acts

This section presents axioms and three representation theorems for CEU models dealing with all simple acts. The first theorem is concerned with a CEU representation (u, π). The last two theorems are concerned with specializations of the first, i.e., an RDEU representation (u, π^*, ϕ) and an SEU representation (u, π^*).

For $X \subseteq \Omega$ and $A \in \Gamma_S$, let $|X|$ denote the cardinality of X, and let $f(A) = \{f(s) : s \in A\}$. A simple (measurable) act f is an act $f \in \mathcal{F}$ for which $|f(A)|$ is finite. Let $\mathcal{F}^s$ be the set of all simple acts. Note that $\mathcal{F}^s \subseteq \mathcal{F}$. For $f, g \in \mathcal{F}$ and $A \in \Gamma_S$, an event-mixtue of f and g with respect to (w.r.t.) A, denoted by $f \bigcirc_A g$, is the act h such that $h(s) = f(s)$ for all $s \in A$, and $h(s) = g(s)$ for all $s \in A^c$. For example, given $f, g, h \in \mathcal{F}$ and $A, B \in \Gamma_S$, $(f \bigcirc_A g) \bigcirc_B h$ is an event-mixture of acts $f \bigcirc_A g$ and h w.r.t. B and $f \bigcirc_A (g \bigcirc_B h)$ is an event-mixture of acts f and $g \bigcirc_B h$ w.r.t A.

Any simple act can be represented as follows: for $A_1, \ldots, A_{n-1} \in \Gamma_S$ and $x_1, \ldots, x_n \in \Omega$,

$$f = (\cdots((x_1 \bigcirc_{A_1} x_2) \bigcirc_{A_2} x_3) \cdots \bigcirc_{A_{n-2}} x_{n-1}) \bigcirc_{A_{n-1}} x_n.$$

This means that for $i = 1, \ldots, n$, $f(s) = x_i$ if $s \in A_i \setminus A_{i-1}$, where $A_0 = \emptyset$ and $A_n = S$. When $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-1}$, we shall simply write a simple f by

$$x_1 \bigcirc_{A_1} x_2 \cdots x_{n-1} \bigcirc_{A_{n-1}} x_n$$

without parentheses.

We use seven axioms for $(\mathcal{F}^s, \preceq)$ to have a CEU representation (u, π). Six of them, which are understood as applying to all simple $f, g, h, f_1, f_2, g_1, g_2, h_1, h_2 \in \mathcal{F}^s$, all $x, y \in \Omega$, and all $A, B \in \Gamma_S$, are stated as follows.

A1. *$\prec$ on $\mathcal{F}$ is a weak order.*

A2. *If $f_i(s) \preceq g_i(s)$ for all $s \in S$ and $i = 1, 2$, and if $f_1 \sim f_2$, $g_1 \sim g_2$, and $(a \bigcirc_{g_i^{-1}(z_\downarrow)} b) \bigcirc_{f_i^{-1}(z_\downarrow)} c \sim a \bigcirc_{h_i^{-1}(z_\downarrow)} c$ for $i = 1, 2$, all $z \in \Omega$, and some $a, b, c \in \Omega$ with $a \prec b \prec c$, then $h_1 \sim h_2$.*

A3. *If $x \prec y$ and $A \subseteq B$, then $x \bigcirc_B y \preceq x \bigcirc_A y$.*

A4. *If $a \bigcirc_{f^{-1}(z_\downarrow)} b \preceq a \bigcirc_{g^{-1}(z_\downarrow)} b$ for all $z \in \Omega$ and some $a, b \in \Omega$ with $a \prec b$, then $f \preceq g$. If, in addition, $a \bigcirc_{f^{-1}(c_\downarrow)} b \prec a \bigcirc_{g^{-1}(c_\downarrow)} b$ for some $c \in \Omega$, then $f \prec g$.*

A5. *$a \prec b \prec c$ for some $a, b, c \in \Omega$.*

A6. *If $f \preceq h \preceq g$, then $h \sim f \bigcirc_C g$ for some $C \in \Gamma_S$.*

Axiom A1 says by definition that $\prec$ on $\mathcal{F}$ is asymmetric and negatively transitive. We note that axioms A2–A6 apply to only simple acts. To see the meaning of axiom A2, assuming that the CEU model holds, it may be useful to translate preference statements in the axiom into risk preferences for distributions on Ω induced by the capacity through each act. Under the hypotheses of the axiom, for $i = 1, 2$, the induced distributions of act h_i is identical to a unique common probability-mixture of the induced distributions of acts f_i and g_i that depends on the utility levels of a,

b, and c. Then the axiom requires that h_1 and h_2 be indifferent. Note that if the SEU model holds, then any common probability-mixture suffices to ensure that h_1 and h_2 are indifferent.

Axiom A3 says that preferences are monotonic w.r.t. event monotonicity. A dominance condition is stated in axiom A4 which is tantamount to the stochastic dominance of induced distributions on Ω. It is obvious to see that it implies Savage's P4 (Weak comparative probability axiom) which is stated as axiom $A4^*$ below, being understood as applying to all $x, y, z, w \in \Omega$ and all $A, B \in \Gamma_S$.

A4*. *If $x \prec y$, $z \prec w$, and $x \bigcirc_A y \prec x \bigcirc_B y$, then $z \bigcirc_A w \prec z \bigcirc_B w$.*

Axiom A5 is a structural property of Ω which says that Ω includes at least three consequences that are not mutually indifferent. Axiom A6 is a restricted solvability axiom, which seems weaker than Savage's P6 (Small event continuity axiom), since the local convexity and density of a capacity follows from A6, while P6 is crusial to obtain a locally convex and continuous state probability measure in Savage's SEU model. We note that if the CEU model (u, π) has a continuous utility function u, then A6 requires that π be continuous. On the other hand, if the range of u is a denumerable set, then π may not be continuous but dense.

To obtain a CEU representation, we need one more axiom, called Archimedean axiom, which guarantees the existence of a real valued function π on Γ_S. Throughout the paper M always stands for any set of consecutive integers. We define a *weak standard sequence* as a monotone sequence of events $\{A_i : i \in M\}$ for which there exist $x, y, z \in \Omega$ and $A, B \in \Gamma_S$ such that not$(x \bigcirc_A y \sim x \bigcirc_B y)$, either $x \prec y \prec z$ or $z \prec y \prec x$, and for all $i, i+1 \in M$, $A \cup B \subseteq A_i$ and

$$(x \bigcirc_A y) \bigcirc_{A_i} z \sim (x \bigcirc_B y) \bigcirc_{A_{i+1}} z.$$

Then our Archimedean axiom is stated as follows.

A7. *Every weak standard sequence is finite.*

We have the following CEU representation theorem for all simple acts. The existence of a unique locally convex and dense capacity will be proved in Section 5, and the complete proof of the theorem will be given in Section 7.

Theorem 1 *Suppose that Assumption 1 holds. Then axioms A1–A7 imply that $(\mathcal{F}^s, \preceq)$ has an CEU representation (u, π).*

We note that all the axioms in the theorem except A5 and A6 are necessary for the representation.

In what follows we shall consider two specializations of Theorem 1, i.e., an RDEU representation (u, π^*, ϕ) and an SEU representation (u, π^*). The both representations use restricted versions of Savage's P2 (Sure-thing principle), which are understood as applying to all simple $f, g, h, h' \in \mathcal{F}^s$, all $A \in \Gamma_S$, and all positive integers n.

A(n). *If* $|f(A) \cup g(A)| \leq n$ *and* $f \bigcirc_A h \preceq g \bigcirc_A h$, *then* $f \bigcirc_A h' \preceq g \bigcirc_A h'$.

This axiom says that if the number of consequences generated by both simple acts, f and g, when the event A obtains is at most n, then preferences between simple acts $f \bigcirc_A h$ and $g \bigcirc_A h$ do not depend upon the consequences the both acts yield when the complementary event A^c obtains. We note that axioms $A(n)$ holds for all $n > 0$ if and only if Savage's sure-thing principle P2 holds for all simple acts. Clearly, axiom $A(n+1)$ implies axiom $A(n)$. RDEU representations require only axiom $A(2)$. Although P2 is necessary for SEU representations, we shall use only axiom $A(3)$ for our SEU representation. Axioms $A(2)$ and Savage's weak comparative probability axiom $A4^*$ are combined to give an equivalent axiom to Machina and Schmeidler's (1992) strong comparative probability axiom, from which A4 follows. Therefore, we shall replace A4 by $A4^*$ for our RDEU and SEU representations.

Archimedean axiom A7 is not strong enough to make the capacity in Theorem 1 to be a strictly increasing transformation of a finitely additive probability measure, so that we need a stronger version of standard sequences in A7. We say that a monotone sequence of events $\{A_i : i \in M\}$ is a *strong standard sequence* if there exist $x, y, z \in \Omega$ and $A, B \in \Gamma_S$ such that not$(y \bigcirc_A z \sim y \bigcirc_B z)$, either $x \preceq y \prec z$ or $z \prec y \preceq x$, and for all $i, i+1 \in M$,

$$(x \bigcirc_A y) \bigcirc_{A \cup A_i} z \sim (x \bigcirc_B y) \bigcirc_{B \cup A_{i+1}} z.$$

In contrast with A7, we allow that for each i, A_i is disjoint from A and B, and either $x \sim y \prec z$ or $z \prec y \sim x$. These conditions are crusial to derive the existence of a finitely additive probability measure by applying additive conjoint measurement (see Krantz et al., 1971) as shown in Section 6. If we take a stong standard sequence $\{A_i\}$ to satisfy that $A \cup B \subseteq A_i$ in $A7^*$, the sequence is also weak whenever either $x \prec y \prec z$ or $z \prec y \prec x$.

Our Archimedean axiom is stated as follows.

A7*. *Every strong standard sequence is finite.*

We have the following RDEU representation (u, π^*, ϕ). The existence of a unique locally convex, dense, and finitely additive state probability measure will be proved in Section 6. The complete proof that implies the existence of a strictly increasing distortion function ϕ and a utility function u will be provided in Section 7.

Theorem 2 *Suppose that Assumption 1 holds. Then axioms A1, $A(2)$, A2, A3, $A4^*$, A5, A6, $A7^*$ imply that $(\mathcal{F}^s, \preceq)$ has an RDEU representation (u, π^*, ϕ).*

We note that $A4^*$ and $A7^*$ are necessary for the representation.

When axiom $A(3)$ is imposed on Theorem 2 in place of axiom $A(2)$, we obtain the following SEU representation (u, π^*), which generalizes Savage's SEU representation. The complete proof will be provided in Section 7.

Theorem 3 *Suppose that Assumption 1 holds. Then axioms A1, $A(3)$, A2, A3, $A4^*$, A5, A6, $A7^*$ imply that $(\mathcal{F}^s, \preceq)$ has an SEU representation (u, π^*).*

4 Axioms and Theorems for Nonsimple Acts

This section extends the results obtained in the preceding section to the set of all measurable acts. We shall do this in two steps. The first step is concerned with the set of all acts that are bounded in consequences introduced below. The second step considers the whole set $\mathcal{F}$.

We say that a preference interval $X \in \Gamma_\Omega$ is *bounded* if there are $a, b \in \Omega$ such that $a \preceq x \preceq b$ for all $x \in X$. If $X, Y \in \Gamma_\Omega$, then $X \prec Y$ means that $x \prec y$ for all $x \in X$ and all $y \in Y$. A *partition* of a preference interval X is a finite sequence of preference intervals that are mutually disjoint and whose union equals X. Thus if preference intervals Y and Z are in the partition, then either $Y \prec Z$ or $Z \prec Y$. Let $\mathcal{F}^b$ denote the set of all measurable acts that are bounded in consequences, i.e.,

$$\mathcal{F}^b = \{f \in \mathcal{F} : f^{-1}([a,b]) = S \text{ for some } a, b \in \Omega\}.$$

The following axioms apply to all $n > 1$, all $f, g, f_1, \ldots, f_n \in \mathcal{F}$, all $x, y \in \Omega$, all $A_1, \ldots, A_{n-1} \in \Gamma_S$, and all $X_1, \ldots, X_n \in \Gamma_\Omega$.

A3*. *If $f(s) \preceq g(s)$ for all $s \in S$, then $f \preceq g$.*

A8. *If $x \preceq f \prec g \preceq y$, then $f \prec x \bigcirc_A y \prec g$ for some $A \in \Gamma_S$.*

A9. *If $\{X_1, \ldots, X_n\}$ is a partition of some bounded preference interval, $X_1 \prec \cdots \prec X_n$, and $A_k = f^{-1}\left(\bigcup_{i=1}^{k} X_i\right)$ for $k = 1, \ldots, n$, then*

$$g \preceq f \quad \text{if} \quad g \prec x_1 \bigcirc_{A_1} x_2 \cdots x_{n-1} \bigcirc_{A_{n-1}} x_n \text{ for all } x_i \in X_i \text{ and } i = 1, \ldots, n;$$
$$f \preceq g \quad \text{if} \quad x_1 \bigcirc_{A_1} x_2 \cdots x_{n-1} \bigcirc_{A_{n-1}} x_n \prec g \text{ for all } x_i \in X_i \text{ and } i = 1, \ldots, n.$$

First we note that axioms *A3** and A8 may apply to acts that do not belong to $\mathcal{F}^b$, while axiom A9 applies to only bounded acts in $\mathcal{F}^b$. *A3** is a state-wise dominance axiom, which implies A3. Since our extended representations deal with nonsimple acts, we need to replace A3 by *A3**. Axiom A8 asserts that the set of all binary acts are dense in preference order. Since f and g are not necessarily simple, there may not exist binary acts, $x \bigcirc_B y$ and $x \bigcirc_C y$, such that $f \sim x \bigcirc_B y$ and $g \sim x \bigcirc_C y$ as shown by the following example.

Let Q be the set of all rational numbers in $I = [0, 1]$. Let $S = Q \cap [0, 1)$ and

$$\Gamma_S = \{\bigcup_{i=1}^{n} [a_i, b_i) \cap S : b_i \le a_j \text{ for } i < j,\ a_i \le b_i \text{ and } a_i, b_i \in Q \text{ for every } i, \text{ and } n \ge 1\}.$$

Then Γ_S is a Boolean algebra. For any set in Γ_S, let

$$\pi^*\left(\bigcup_{i=1}^{n} [a_i, b_i) \cap S\right) = \sum_{i=1}^{n} (b_i - a_i).$$

Then π^* is a locally convex, dense, and finitely additive probability measure with $R(\pi^*) = Q$. Let $\Omega = Q$ and $u(x) = x$ for all $x \in \Omega$. Then the set of expected utilities for all binary acts is given by Q. It suffices to show that there is an act whose expected utility is irrational.

Take any irrational number $0 < r < 0.1$, whose decimal expansion is given by

$$r = \sum_{i=1}^{\infty} \frac{a_i}{10^i},$$

where $a_1 = 0$ and $a_i \in \{0, 1, \ldots, 9\}$ for all $i > 1$. Then consider an act f defined as follows: for $i = 1, 2, \ldots,$

$$f(s) = \frac{a_i}{5^i} \text{ if } s \in \left[\frac{1}{2^i}, \frac{1}{2^{i-1}}\right).$$

Then the expected utility of this act is given by

$$\sum_{i=1}^{\infty} \pi^* \left(f^{-1}\left(\frac{a_i}{5^i}\right)\right) u\left(\frac{a_i}{5^i}\right) = \sum_{i=1}^{\infty} \frac{a_i}{10^i} = r,$$

which is irrational.

Axiom A9 is an event-wise dominance axiom, whose slightly different form was first investigated by Nakamura (1995b). The first part of A9 says that if the bounded act f is certain to yield outcomes in X_i when $A_i \setminus A_{i-1}$ obtains, and if every simple act that yield exactly one outcome x_i in each X_i for every state $s \in A_i \setminus A_{i-1}$ is strictly preferred to g, then f is weakly preferred to g. The second part has a similar interpretation.

Adding axioms A8 and A9 to each of Theorems 1, 2, and 3, we obtain the following representation theorem for all bounded acts in $\mathcal{F}^b$. The proof is deferred to section 8.

Theorem 4 *Suppose that Assumption 1 and axioms A8 and A9 hold.*

(1) *If Theorem 1 holds with A3 replaced by A3*, then $(\mathcal{F}^b, \preceq)$ has a CEU representation (u, π).*
(2) *If Theorem 2 holds with A3 replaced by A3*, then $(\mathcal{F}^b, \preceq)$ has an RDEU representation (u, π^*, ϕ).*
(3) *If Theorem 3 holds with A3 replaced by A3*, then $(\mathcal{F}^b, \preceq)$ has an SEU representation (u, π^*).*

We note that axioms $A3^*$, $A8$, and $A9$ are necessary for each representation of Theorem 4.

When $\mathcal{F} \setminus \mathcal{F}^b$ is not empty, Ω is unbounded, i.e., there exist no $a, b \in \Omega$ such that $a \preceq x \preceq b$ for all $x \in \Omega$. In this case, Wakker's (1989) truncation continuity axiom will suffice to derive CEU models. Following Wakker, we introduce the upper and lower truncations of measurable acts. Given $a \in \Omega$, the upper truncation of $f \in \mathcal{F}$, denoted f^a, is defined as follows. For all $s \in S$,

$$f^a(s) = \begin{cases} f(s) & \text{if } f(s) \prec a, \\ a & \text{if } a \preceq f(s). \end{cases}$$

The lower truncation of $f \in \mathcal{F}$, denoted f_a, is defined as follows.

$$f_a(s) = \begin{cases} f(s) & \text{if } a \prec f(s), \\ a & \text{if } f(s) \preceq a. \end{cases}$$

We note that for all $a \in \Omega$, f^a and f_a are measurable acts, so that f^a and f_a are in $\mathcal{F}$.

Wakker's truncation continuity axiom, which applies to all $f, g \in \mathcal{F} \backslash \mathcal{F}^b$, is stated as follows.

A10. *If $f \prec g$, then $f_a \prec g$ and $f \prec g^b$ for some $a, b \in \Omega$.*

Wakker (1993) examined this axiom to obtain unbounded utility representations in various set-ups including an extension of Savage's SEU theory. Also, in our set-up, the axiom ensures that the Choquet integration for $f \in \mathcal{F} \backslash \mathcal{F}^b$ is well defined even though a utility function might be unbounded.

Our representation theorem for all measurable acts is stated as follows. The proof is deferred to Section 8.

Theorem 5 *Suppose that Assumption 1 and axiom A10 hold.*

(1) *If Theorem 4(1) holds, then $(\mathcal{F}, \preceq)$ has a CEU representation (u, π).*
(2) *If Theorem 4(2) holds, then $(\mathcal{F}, \preceq)$ has an RDEU representation (u, π^*, ϕ).*
(3) *If Theorem 4(3) holds, then $(\mathcal{F}, \preceq)$ has an SEU representation (u, π^*).*

We note that axiom $A10$ is necessary for each representation in Theorem 5.

5 Dense Capacities

We shall assume throughout the section that Assumption 1 and axioms A1–A7 hold, i.e., the hypotheses of Theorem 1. Define a binary relation $\preceq^*$ on Γ_S as follows: for some $x, y \in \Omega$ and for all $A, B \in \Gamma_S$,

$$A \preceq^* B \Longleftrightarrow x \bigcirc_B y \preceq x \bigcirc_A y \text{ whenever } x \prec y.$$

Since A4 implies $A4^*$, the definition of $\preceq^*$ does not depend on the choice of x and y, so that $\preceq^*$ on Γ_S is well defined. By A1, it is a weak order. We say that a set function π on Γ_S *agrees with* $\preceq^*$ if for all $A, B \in \Gamma_S$,

$$A \preceq^* B \Longleftrightarrow \pi(A) \leq \pi(B).$$

The aim of the section is to show that there is a unique, locally convex, and dense capacity π on Γ_S agreeing with $\preceq^*$.

Define $\sim^*$ and $\prec^*$ in the usual way: $A\sim^*B$ if $A\preceq^*B$ and $B\preceq^*A$; $A\prec^*B$ if not$(B\preceq^*A)$. We write $A\preceq^*B\preceq^*C$ when $A\preceq^*B$ and $B\preceq^*C$. If $\Gamma_1, \Gamma_2 \subseteq \Gamma_S$, then $\Gamma_1\preceq^*\Gamma_2$ means that $A\preceq^*B$ for all $A \in \Gamma_1$ and all $B \in \Gamma_2$. Basic properties of $\preceq^*$ are given by the following lemma.

Lemma 1 (1) *If $A \subseteq B$, then $A\preceq^*B$.*
(2) *If $A\preceq^*B\preceq^*C$ and $A \subseteq C$, then there is a $D \in \Gamma_S$ such that $A \subseteq D \subseteq C$ and $D\sim^*B$.*
(3) *If $A_1\preceq^* \cdots \preceq^*A_n$, then there are $B_1, \ldots, B_n \in \Gamma_S$ such that $B_1 \subseteq \cdots \subseteq B_n$ and $A_i\sim^*B_i$ for $i = 1, \ldots, n$.*
(4) *If $x \prec y \preceq z$ and $A \cup B \subseteq C$, then*

$$x \bigcirc_A y \preceq x \bigcirc_B y \Longleftrightarrow (x \bigcirc_A y) \bigcirc_C z \preceq (x \bigcirc_B y) \bigcirc_C z.$$

(5) *If $x \preceq y \prec z$ and $C \subseteq A \cap B$, then*

$$y \bigcirc_A z \preceq y \bigcirc_B z \Longleftrightarrow x \bigcirc_C (y \bigcirc_A z) \preceq x \bigcirc_C (y \bigcirc_B z).$$

Proof. (1) This follows from A3.

(2) Suppose that $A \preceq^* B \preceq^* C$ and $A \subseteq C$. Let $a \prec b$. Then by definition, $a \bigcirc_C b \preceq a \bigcirc_B b \preceq a \bigcirc_A b$. It follows from A6 that there is an event $E \in \Gamma_S$ such that

$$a \bigcirc_B b \sim (a \bigcirc_C b) \bigcirc_E (a \bigcirc_A b).$$

Note $(a \bigcirc_C b) \bigcirc_E (a \bigcirc_A b) = a \bigcirc_{A\cup(C\cap E)} b$. By A1, $B \sim^* A \cup (C \cap E)$. Let $D = A \cup (C \cap E)$, so that $D \in \Gamma_S$, $A \subseteq D \subseteq C$, and $D \sim^* B$.

(3) This follows from (2), since $A_n \preceq^* S$ by (1).

(4) Suppose that $x \prec y \preceq z$ and $A \cup B \subseteq C$. If $x \bigcirc_A y \preceq x \bigcirc_B y$, then by A4, $(x \bigcirc_A y) \bigcirc_C z \preceq (x \bigcirc_B y) \bigcirc_C z$. Assume that $(x \bigcirc_A y) \bigcirc_C z \preceq (x \bigcirc_B y) \bigcirc_C z$. Then by A4, $(x \bigcirc_B y) \bigcirc_C z \prec (x \bigcirc_A y) \bigcirc_C z$ if $x \bigcirc_B y \prec x \bigcirc_A y$. Thus we must have that $x \bigcirc_A y \preceq x \bigcirc_B y$.

(5) Similar to (4). □

For $n \geq 2$, let

$$\Gamma_\uparrow^n = \{(A_1, \cdots, A_n) : A_1\preceq^* \ldots \preceq^*A_n \text{ and } A_i \in \Gamma_S \text{ for } i = 1, \ldots, n\},$$
$$\Omega_\uparrow^{n+1} = \{(x_1, \ldots, x_{n+1}) : x_1 \preceq \cdots \preceq x_{n+1} \text{ and } x_i \in \Omega \text{ for } i = 1, \ldots, n+1\}.$$

Given $x = (x_1, \ldots, x_{n+1}) \in \Omega_\uparrow^{n+1}$, we define a mapping ω_x from $\Gamma_\uparrow^n$ into Γ_S as follows: for every $(A_1, \ldots, A_n) \in \Gamma_\uparrow^n$, let $\omega_x(A_1, \ldots, A_n) = B$ for some $B \in \Gamma_S$ if there are $B_1, \ldots, B_n \in \Gamma_S$ such that $B_1 \subseteq \cdots \subseteq B_n$, $A_i\sim^*B_i$ for $i = 1, \ldots n$, and

$$x_1 \bigcirc_{B_1} x_2 \cdots x_n \bigcirc_{B_n} x_{n+1} \sim x_1 \bigcirc_B x_{n+1}.$$

By A1, A4, A6, and Lemma 1(3), ω_x is well defined on $\Gamma_{\uparrow}^n$.

We shall denote $\omega_x^k(AB) = \omega_x(A_1, \ldots, A_n)$ when $1 \le k < n$, $A_i = A$ for $i = 1, \ldots, k$, and $A_i = B$ for $i = k+1, \ldots, n$. We say that $1 \le k < n$ is *left-inessential* if for all $A, B, C \in \Gamma_S$, $\omega_x^k(AC) \sim^* \omega_x^k(BC)$ whenever $A \preceq^* B \preceq^* C$, and *right-inessential* if for all $A, B, C \in \Gamma_S$, $\omega_x^k(AB) \sim^* \omega_x^k(AC)$ whenever $A \preceq^* B \preceq^* C$. When k is not left (right)-inessential, we say that k is *left (right)-essential.* When k is left and right essential, we say that k is *essential.* When there is an essential $1 \le k < n$, we say that ω_x is essential. It follows from A1 and A4 that for $1 \le k < n$,

$$\begin{aligned} k \text{ is left-essential} &\iff x_1 \prec x_{k+1} \preceq x_{n+1}, \\ k \text{ is right-essential} &\iff x_1 \preceq x_{k+1} \prec x_{n+1}, \\ k \text{ is essential} &\iff x_1 \prec x_{k+1} \prec x_{n+1}. \end{aligned}$$

Given $x \in \Omega_{\uparrow}^{n+1}$ and an essential $1 \le k < n$, a standard sequence w.r.t. $\preceq^*$ is defined as a set of events, $\{A_i : i \in M\}$, for which there exist $A, B \in \Gamma_S$ such that not$(A \sim^* B)$, and either $\{A, B\} \preceq^* \{A_i\}$ and $\omega_x^k(AA_i) \sim^* \omega_x^k(BA_{i+1})$ for all $i, i+1 \in M$, or $\{A_i\} \preceq^* \{A, B\}$ and $\omega_x^k(A_iA) \sim^* \omega_x^k(A_{i+1}B)$ for all $i, i+1 \in M$. We say that a standard sequence w.r.t. $\preceq^*$, $\{A_i : i \in M\}$, is *strictly bounded* if $A \prec^* A_i \prec^* B$ for all $i \in M$ and some $A, B \in \Gamma_S$.

Given $x \in \Omega_{\uparrow}^{n+1}$ with $n \ge 2$, the triple $\langle \preceq^*, \omega_x, \Gamma_{\uparrow}^n \rangle$ is said to be a *weak multi-symmetric structure* if it satisfies the following six axioms, which are understood as applying to all $A, B, C, D, A_i, B_i \in \Gamma_S$ for $i = 1, \ldots, n$ and all $k = 1, \ldots, n-1$,

B1. *$\prec^*$ is a weak order.*

B2. *If k is left-essential and $\{A, B\} \preceq^* C$, then*

$$A \preceq^* B \iff \omega_x^k(AC) \preceq^* \omega_x^k(BC);$$

if k is right-essential and $C \preceq^* \{A, B\}$, then

$$A \preceq^* B \iff \omega_x^k(CA) \preceq^* \omega_x^k(CB).$$

B3. *If $\{A, B\} \preceq^* C$ and $\omega_x^k(AC) \preceq^* D \preceq^* \omega_x^k(BC)$, then $D \sim^* \omega_x^k(EC)$ for some $E \in \Gamma_S$ with $E \preceq^* C$; if $C \preceq^* \{A, B\}$ and $\omega_x^k(CA) \preceq^* D \preceq^* \omega_x^k(CB)$, then $D \sim^* \omega_x^k(CE)$ for some $E \in \Gamma_S$ with $C \preceq^* E$.*

B4. *Every strictly bounded standard sequence w.r.t. $\preceq^*$ is finite.*

B5. *If $A_1 \preceq^* \cdots \preceq^* A_n$, $B_1 \preceq^* \cdots \preceq^* B_n$, and $A_i \preceq^* B_i$ for $i = 1, \ldots, n$, then*

$$\omega_x(A_1, \ldots, A_n) \preceq^* \omega_x(B_1, \ldots, B_n).$$

B6. *If $A_1 \preceq^* \cdots \preceq^* A_n$, $B_1 \preceq^* \cdots \preceq^* B_n$, and $A_i \preceq^* B_i$ for $i = 1, \ldots, n$, then*

$$\omega_x^k(\omega_x(A_1, \ldots, A_n)\omega_x(B_1, \ldots, B_n)) \sim^* \omega_x(\omega_x^k(A_1B_1), \ldots, \omega_x^k(A_nB_n)).$$

Then we have the following proposition. The proof appears at the end of the section.

Proposition 1 *Suppose that $n \geq 2$, $x = (x_1, \ldots, x_{n+1}) \in \Omega_\uparrow^{n+1}$, and ω_x is essential. Then $\langle \preceq^*, \omega_x, \Gamma_\uparrow^n \rangle$ is a weak multi-symmetric structure.*

A numerical representation of the weak multi-symmetric structure, which is stated in the following proposition, yields a unique locally convex and dense capacity π on Γ_S. The proof is deferred to the end of the section. Note that when Ω is finite, the proposition provides the CEU representation of Theorem 1.

Proposition 2 *For all $x = (x_1, \ldots, x_{n+1}) \in \Omega_\uparrow^{n+1}$ and all $n \geq 2$, if ω_x is essential, then there exist unique real numbers $\lambda_i(x) \geq 0$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i(x) = 1$ and $\lambda_j(x)\lambda_k(x) > 0$ for some distinct $1 \leq j \leq n$ and $1 \leq k \leq n$, and a unique locally convex and dense capacity π on Γ_S agreeing with $\preceq^*$ such that for all $A_1, \ldots, A_n \in \Gamma_S$,*

$$\pi(\omega_x(A_1, \ldots, A_n)) = \sum_{i=1}^n \lambda_i(x)\pi(A_i) \text{ whenever } A_1 \preceq^* \cdots \preceq^* A_n.$$

Proof of Proposition 1. Suppose that $n \geq 2$, $x = (x_1, \ldots, x_{n+1}) \in \Omega_\uparrow^{n+1}$, and ω_x is essential. Then $x_1 \prec x_\ell \prec x_{n+1}$ for some $1 \leq \ell \leq n$. B1 follows from the definition of $\preceq^*$ and A1. Let $a = x_1$ and $b = x_{n+1}$, so $a \prec b$. Throughout the proof, we shall fix x and k. Thus without any confusion, for all $A, B \in \Gamma_S$, we shall write AB in place of $\omega_x^k(AB)$. Then note by definition that $a \bigcirc_{AB} b \sim (a \bigcirc_A x_{k+1}) \bigcirc_B b$.

To show B2, we show the first part of the claim. A similar analysis applies to get the second part. Suppose that k is left-essential and $\{A, B\} \preceq^* C$. Then $a \prec x_{k+1} \preceq b$. By Lemma 1(2), there are $A', B' \in \Gamma_S$ such that $A' \sim^* A$, $B' \sim^* B$, $A' \subseteq C$, and $B' \subseteq C$. Hence, we obtain that

$$\begin{aligned}
A \preceq^* B &\iff A' \preceq^* B' \\
&\iff a \bigcirc_{B'} x_{k+1} \preceq a \bigcirc_{A'} x_{k+1} \\
&\iff (a \bigcirc_{B'} x_{k+1}) \bigcirc_C b \preceq (a \bigcirc_{A'} x_{k+1}) \bigcirc_C b \quad \text{(by Lemma 1(4))} \\
&\iff a \bigcirc_{B'C} b \preceq a \bigcirc_{A'C} b \\
&\iff A'C \preceq^* B'C \\
&\iff AC \preceq^* BC.
\end{aligned}$$

To show B3, suppose that $\{A, B\} \preceq^* C$ and $AC \preceq^* D \preceq^* BC$. Then by B1 and B2, $A \preceq^* B$. We note by Lemma 1(3), A1, and A4 that there are $A', B' \in \Gamma_S$ such that $A' \subseteq B' \subseteq C$, $A \sim^* A'$, $B \sim^* B'$, $a \bigcirc_{AC} b \sim (a \bigcirc_{A'} x_{k+1}) \bigcirc_C b$, and $a \bigcirc_{BC} b \sim (a \bigcirc_{B'} x_{k+1}) \bigcirc_C b$. Then we have

$$\begin{aligned}
AC \preceq^* D \preceq^* BC &\iff a \bigcirc_{BC} b \preceq a \bigcirc_D b \preceq a \bigcirc_{AC} b \\
&\iff (a \bigcirc_{B'} x_{k+1}) \bigcirc_C b \preceq a \bigcirc_D b \preceq (a \bigcirc_{A'} x_{k+1}) \bigcirc_C b.
\end{aligned}$$

A6 implies that for some $F \in \Gamma_S$,

$$a \bigcirc_D b \sim ((a \bigcirc_{B'} x_{k+1}) \bigcirc_C b) \bigcirc_F ((a \bigcirc_{A'} x_{k+1}) \bigcirc_C b).$$

The right hand side is rearranged to give $(a \bigcirc_{A' \cup (B' \cap F)} x_{k+1}) \bigcirc_C b$. Thus letting $E = A' \cup (B' \cap F)$, we have that $D \sim^* EC$ and $E \preceq^* C$. The second part of B2 similarly follows.

To show B4, suppose that k is essential. Let $\{A_i : i \in M\}$ be a standard sequence w.r.t. $\preceq^*$ such that for some $A, B \in \Gamma_S$ with not$(A \sim^* B)$, $\{A, B\} \preceq^* \{A_i\}$ and $AA_i \sim^* BA_{i+1}$ for all $i, i+1 \in M$. Assume that $A \prec^* B$. When $B \prec^* A$, the proof is similar. By B1 and B3, $A_{i+1} \prec^* A_i$ for all $i, i+1 \in M$. Since $\emptyset \prec^* \{A_i\} \prec^* S$, the standard sequence w.r.t. $\preceq^*$ is strictly bounded. We are to show that $\{A_i\}$ is finite.

It follows from Lemma 1(3) that there are events A', B' and a decreasing sequence of events $\{A'_i : i \in M\}$ such that $A' \sim^* A, B' \sim^* B, A'_i \sim^* A_i$ and $A_i \supset A_{i+1} \supset B' \supset A'$ for all $i, i+1 \in M$. By the definition of ω_x, $AA_i \sim^* A'A'_i$ and $BA_{i+1} \sim^* B'A'_{i+1}$. Hence, by A7, $\{A_i\}$ must be finite. When $\{A_i\} \preceq^* \{A, B\}$ and $A_iA \sim^* A_{i+1}B$ for all $i, i+1 \in M$, finiteness of $\{A_i\}$ similarly follows.

To show B5 and B6, suppose that $A_1 \preceq^* \cdots \preceq^* A_n$, $B_1 \preceq^* \cdots \preceq^* B_n$, and $A_i \preceq^* B_i$ for $i = 1, \ldots, n$. Then by Lemma 1(3), there are events $A'_1, \ldots, A'_n, B'_1, \ldots, B'_n$ such that $A'_1 \subseteq \cdots \subseteq A'_n, B'_1 \subseteq \cdots \subseteq B'_n, A_i \sim^* A'_i, B_i \sim^* B'_i$, and $A'_i \subseteq B'_i$ for $i = 1, \ldots, n$. By A4,

$$x_1 \bigcirc_{B'_1} x_2 \cdots x_n \bigcirc_{B'_n} x_{n+1} \preceq x_1 \bigcirc_{A'_1} x_2 \cdots x_n \bigcirc_{A'_n} x_{n+1}.$$

Hence B5 readily follows from A1 and the definition of ω_x.

Next we show B6. Since ω_x is essential, let $a \prec c \prec b$ for some $c \in \{x_1, \ldots, x_n\}$. Then it follows from A4 and A6 that there are events $A'', B'', C'', C'_1, \ldots, C'_n$ such that $A'' \subseteq B'', C'_1 \subseteq \cdots \subseteq C'_n$,

$$\begin{aligned} x_1 \bigcirc_{A'_1} x_2 \cdots x_n \bigcirc_{A'_n} x_{n+1} &\sim x_1 \bigcirc_{A''} x_{n+1}, \\ x_1 \bigcirc_{B'_1} x_2 \cdots x_n \bigcirc_{B'_n} x_{n+1} &\sim x_1 \bigcirc_{B''} x_{n+1}, \end{aligned}$$

and for $i = 1, \ldots, n$,

$$\begin{aligned} (a \bigcirc_{A'_i} c) \bigcirc_{B'_i} b &\sim a \bigcirc_{C'_i} b, \\ (a \bigcirc_{A''} c) \bigcirc_{B''} b &\sim a \bigcirc_{C''} b. \end{aligned}$$

By A2, we obtain that

$$x_1 \bigcirc_{C'_1} x_2 \cdots x_n \bigcirc_{C'_n} x_{n+1} \sim x_1 \bigcirc_{C''} x_{n+1}.$$

Hence B6 holds. □

Proof of Proposition 2. Suppose that $n \geq 2$, $x = (x_1, \ldots, x_{n+1}) \in \Omega_\uparrow^{n+1}$, and ω_x is essential. Since, by Proposition 1, $\langle \preceq^*, \omega_x, \Gamma_\uparrow^n \rangle$ is a weak multi-symmetric structure, and ω_x is idempotent, i.e., $\omega_x(A, \ldots, A) \sim^* A$ for all $A \in \Gamma_S$, it follows from Theorem 1 in Nakamura (1992) that there are real numbers $\lambda_i(x) \geq 0$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i(x) = 1$ and $\lambda_j(x)\lambda_k(x) > 0$ for some distinct $1 \leq j \leq n$ and $1 \leq k \leq n$, and a real valued function π_x on Γ_S such that for all $A, B, A_1, \ldots, A_n \in$

Γ_S,

$$A \preceq^* B \iff \pi_x(A) \le \pi_x(B),$$
$$\pi_x(\omega_x(A_1, \ldots, A_n)) = \sum_{i=1}^{n} \lambda_i(x)\pi_x(A_i) \text{ whenever } A_1 \preceq^* \cdots \preceq^* A_n.$$

Moreover, $\lambda_i(x)$ for $i = 1, \ldots, n$ are unique, and π_x is unique up to a positive linear transformation.

For $x \in \Omega_\uparrow^{n+1}$ and $y \in \Omega_\uparrow^{m+1}$ with ω_x and ω_y essential, let $1 \le k \le n-1$ and $1 \le j \le m-1$ be essential. Then it follows from A2 and the definitions of ω_x and ω_y that for all $A, B, C, D \in \Gamma_S$ with $A \preceq^* \{B, C\} \preceq^* D$,

$$\omega_y^k(\omega_x^j(AB)\omega_x^j(CD)) \sim^* \omega_x^j(\omega_y^k(AC)\omega_y^k(BD)),$$

which is the weak isometry condition defined in Nakamura (1992). Thus Proposition 1 in Nakamura (1992) implies that $\pi_x = \pi_y$. Let $\pi = \pi_x = \pi_y$. Since π is unique up to a positive linear transformation, we take $\pi(\emptyset) = 0$ and $\pi(S) = 1$, so that π is a unique capacity agreeing with $\preceq^*$, and satisfies that for all $A_1, \ldots, A_n \in \Gamma_S$,

$$\pi(\omega_x(A_1, \ldots, A_n)) = \sum_{i=1}^{n} \lambda_i(x)\pi(A_i) \text{ whenever } A_1 \preceq^* \cdots, \preceq^* A_n.$$

By A5, there is at least one essential ω_x. It follows from B2 and the idempotency of ω_x that $\preceq^*$ is dense, i.e., if $A \prec^* B$ for $A, B \in \Gamma_S$, then $A \prec^* C \prec^* B$ for some $C \in \Gamma_S$. Hence density of π easily follows from the representation of $\pi(\omega_x(A_1, \ldots, A_n))$ in the preceding paragraph. Local convexity of π follows from Lemma 1(2). □

6 Finite Additivity

This section assumes that a binary relation $\preceq^*$ on Γ_S is primitive but not a derived notion as in Section 5. We show a qualitative probability structure that implies the existence of a unique locally convex and finitely additive probability measure π^* on Γ_S agreeing with $\preceq^*$. Our structure is slightly weaker than Savage's qualitative probability structure, since continuity is not required. It will be shown in the next section that the hypotheses of Theorems 2 and 3 satisfy our qualitative probablity structure when $\preceq^*$ is interpreted as a derived notion from $\preceq$.

Proposition 3 *Suppose that $\preceq^*$ on Γ_S satisfies the following seven conditions, which are understood as applying to all $A, B, C \in \Gamma_S$.*

C1. $\emptyset \preceq^* A$.

C2. $\emptyset \prec^* S$.

C3. $\prec^*$ *is a weak order.*

C4. *If* $(A \cup B) \cap C = \emptyset$, *then* $A \prec^* B \Longleftrightarrow A \cup C \prec^* B \cup C$.

C5. *If* $A \subseteq B$ *and* $A \preceq^* C \preceq^* B$, *then* $C \sim^* D$ *for some* $D \in \Gamma_S$ *with* $A \subseteq D \subseteq B$.

C6. *If* $(A \cup B) \cap C = \emptyset$ *and not*$(A \sim^* B)$, *and if a set* $\{A_i : A_i \subseteq C, i \in M\}$ *satisfies that* $A \cup A_i \sim^* B \cup A_{i+1}$ *for all* $i, i+1 \in M$, *then the set is finite.*

C7. $\emptyset \prec^* E \prec^* S$ *for some* $E \in \Gamma_S$.

Then there is a unique locally convex and finitely additive probability measure π^* *on* Γ_S *agreeing with* $\preceq^*$.

The proof is deferred later in the section. Axioms C1–C4, by which $\prec^*$ is called a qualitative probability relation, are necessary for the existence of π^*. It is well known that they are not sufficient. A *partition* of an event A, denoted $\sigma(A)$, is a finite set of mutually disjoint nonempty events whose union equals A. Savage (1954) added the following condition to axioms C1–C4, which applies to all $A, B \in \Gamma_S$.

C5*. *If* $A \prec^* B$, *then there is a partition* $\sigma(S)$ *for which* $A \cup C \prec^* B$ *for all* $C \in \sigma(S)$.

Then he showed that C1–C4 and $C5^*$ are necessary and sufficient for the existence of a unique locally convex, continuous, and finitely additive probability measure π^* on Γ_S agreeing with $\preceq^*$. In place of $C5^*$, we introduce a restricted solvability condition C5 and an Archimedean condition C6. Since C5 is a restatement of local convexity in terms of $\preceq^*$, it is necessary for the representation. C6 is also necessary. Although C5 is not necessary, we require it to avoid the unusual case of $0-1$ measures.

The following example shows that C1–C7 collectively do not require density (and also continuity). Let $S = \{s_1, \ldots, s_n\}$ and π^* be a probability measure on Γ_S that assigns equal probabilites to every state, i.e., $\pi^*(\{s_i\}) = \frac{1}{n}$ for all i. Define a binary relation $\preceq^*$ on Γ_S as follows: for all $A, B \in \Gamma_S$, $A \preceq^* B \Longleftrightarrow \pi^*(A) \leq \pi^*(B)$. Then it easily follows that $\preceq^*$ satisfies C1–C7.

Before providing the proof of Proposition 3, we need the following lemma.

Lemma 2 *Suppose that C1–C4 hold. Then we have*

(1) *If* $A \subseteq B$, *then* $A \preceq^* B$.
(2) *If* $A \cap C = B \cap D = \emptyset, C \sim^* D$, *and* $A \sim^* B$, *then* $A \cup C \sim^* B \cup D$.
(3) *If* $A \cap D = B \cap E = B \cap F = C \cap D = A \cap F = C \cap E = \emptyset$, *and if* $A \cup D \sim^* B \cup E$ *and* $B \cup F \sim^* C \cup D$, *then* $A \cup F \sim^* C \cup E$.

Proof. (1) and (2) follow from Fishburn (1970, Chapter 14).

(3) Suppose that the hypotheses of (3) hold. Then we have

$$\begin{aligned} A \cup D &= (A \setminus B) \cup (A \cap B) \cup (D \setminus E) \cup (D \cap E), \\ B \cup E &= (B \setminus A) \cup (A \cap B) \cup (E \setminus D) \cup (D \cap E). \end{aligned}$$

Thus by C4,

$$\begin{aligned} A \cup D \sim^* B \cup E &\iff [(A \setminus B) \cup (D \setminus E)] \cup [(A \cap B) \cup (D \cap E)] \\ &\quad \sim^* [(B \setminus A) \cup (E \setminus D)] \cup [(A \cap B) \cup (D \cap E)] \\ &\iff (A \setminus B) \cup (D \setminus E) \sim^* (B \setminus A) \cup (E \setminus D). \end{aligned}$$

Similarly we obtain

$$B \cup F \sim^* C \cup D \iff (B \setminus C) \cup (F \setminus D) \sim^* (C \setminus B) \cup (D \setminus F).$$

Since $[(A\setminus B)\cup(D\setminus E)]\cap[(B\setminus C)\cup(F\setminus D)] = [(B\setminus A)\cup(E\setminus D)]\cap[(C\setminus B)\cup(D\setminus F)] = \emptyset$, it follows from (2) that

$$[(A \setminus B) \cup (D \setminus E)]\cup[(B \setminus C) \cup (F \setminus D)] \sim^* [(B \setminus A) \cup (E \setminus D)]\cup[(C \setminus B) \cup (D \setminus F)].$$

This is rearranged to give

$$\begin{aligned} &[(A \setminus C) \cup (F \setminus E)] \cup [(B \setminus (A \cup C)) \cup ((A \cap C) \setminus B)] \\ &= \ [(C \setminus A) \cup (E \setminus F)] \cup [(B \setminus (A \cup C)) \cup ((A \cap C) \setminus B)]. \end{aligned}$$

Hence by C4,

$$\begin{aligned} &(A \setminus C) \cup (F \setminus E) \sim^* (C \setminus A) \cup (E \setminus F) \\ &\iff [(A \setminus C) \cup (F \setminus E)] \cup [(A \cap C) \cup (F \cap E)] \\ &\quad \sim^* [(C \setminus A) \cup (E \setminus F)] \cup [(A \cap C) \cup (F \cap E)] \\ &\iff A \cup F \sim^* C \cup E. \end{aligned}$$

This completes the proof. □

Proof of Proposition 3. Suppose that C1–C7 hold. Given an event $A \in \Gamma_S$, define a relative algebra as follows:

$$\Gamma_A = \{B \in \Gamma_S : B = A \cap C \text{ for some } C \in \Gamma_S\}.$$

We say that an event A is *essential* if $\emptyset \prec^* A \prec^* S$. If A is essential, then by C4, its complement is also essential. Thus it follows from C7 that there exists at least one partition of S that includes two essential events.

Suppose that $n \geq 2$ and there is a partition $\sigma(S) = \{E_1, \ldots, E_n\}$ for which at least two events in $\sigma(S)$ are essential. Then we define a binary relation $\prec^*_\sigma$ on $\Gamma_{E_1} \times \cdots \times \Gamma_{E_n}$ as follows: for all $A_i, B_i \in \Gamma_{E_i}$ and $i = 1, \ldots, n$,

$$(A_1, \ldots, A_n) \prec^*_\sigma (B_1, \ldots, B_n) \iff \bigcup_{i=1}^{n} A_i \prec^* \bigcup_{i=1}^{n} B_i.$$

Let $\preceq^*_\sigma$ and $\sim^*_\sigma$ be defined in the usual way.

It readily follows from C1-C7 that $(\prec^*_\sigma, \Gamma_{E_1} \times \cdots \times \Gamma_{E_n})$ is an additive conjoint structure (see Krantz, et al., 1971, Chapter 6), since the following six conditions hold for an essential event E, which are understood as applying to all $A, B, C \in \Gamma_E$ and all $A', B', C' \in \Gamma_{E^c}$.

D1. *$\prec^*$ is a weak order.*

D2. *if $A \cup A' \preceq^* B \cup A'$ then $A \cup B' \preceq^* B \cup B'$;*
if $A \cup A' \preceq^* A \cup B'$ then $B \cup A' \preceq^* B \cup B'$.

D3. *if $A \cup A' \sim^* B \cup B'$ and $B \cup C' \sim^* C \cup A'$, then $A \cup C' \sim^* C \cup B'$.*

D4. *if $A \cup A' \preceq^* C \cup C' \preceq^* B \cup A'$, then $C \cup C' \sim^* D \cup A'$ for some $D \in \Gamma_E$;*
if $A \cup A' \preceq^* C \cup C' \preceq^* A \cup B'$, then $C \cup C' \sim^* A \cup D'$ for some $D' \in \Gamma_{E^c}$.

D5. *if not($A' \sim^* B'$) and $A_i \cup A' \sim^* A_{i+1} \cup B'$ for $A_i, A_{i+1} \in \Gamma_E$ and all $i \in M$, then the set $\{A_i\}$ is finite, and if not($A \sim^* B$) and $A \cup A'_i \sim^* B \cup A'_{i+1}$ for $A'_i, A'_{i+1} \in \Gamma_{E^c}$ and all $i \in M$, then the set $\{A'_i\}$ is finite.*

D6. *$\emptyset \prec^* E$ and $\emptyset \prec^* E^c$.*

D1–D3, D5, and D6 follow from respectively C3, C4, Lemma 2(3), C6, and C7. To see that D4 holds, suppose that $A \cup A' \preceq^* C \cup C' \preceq^* B \cup A'$. Then by Lemma 2(1), $A' \preceq^* A \cup A'$, so by C3, $A' \preceq^* C \cup C' \preceq^* B \cup A'$. Thus it follows from C5 that $C \cup C' \sim^* D \cup A'$ for some $D \in \Gamma_E$.

Since $(\prec^*_\sigma, \Gamma_{E_1} \times \cdots \times \Gamma_{E_n})$ is an additive conjoint structure, there are real valued functions, $\psi^\sigma_{E_i}$ on Γ_{E_i} for $i = 1, \ldots, n$, such that for all $A_i, B_i \in \Gamma_{E_i}$ and $i = 1, \ldots, n$,

$$\bigcup_{i=1}^{n} A_i \prec^* \bigcup_{i=1}^{n} B_i \Longleftrightarrow \sum_{i=1}^{n} \psi^\sigma_{E_i}(A_i) < \sum_{i=1}^{n} \psi^\sigma_{E_i}(B_i).$$

Let $\psi^\sigma_{E_i}(\emptyset) = 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \psi^\sigma_{E_i}(E_i) = 1$. Therefore, it follows from the uniqueness of additive conjoint measurement that $\psi^\sigma_{E_i}$ for $i = 1, \ldots, n$ are uniquely determined. We note that $0 < \psi^\sigma_{E_i} < 1$ if E_i is essential, and $\psi^\sigma_{E_i} = 0$ otherwise.

Let $\sigma(S) = \{E, E^c\}$ for an essential E. Then we show that ϕ^σ_E on Γ_E is additive, i.e., for all $A, B \in \Gamma_E$ with $A \cap B = \emptyset$, $\psi^\sigma_E(A \cup B) = \psi^\sigma_E(A) + \psi^\sigma_E(B)$. If $B \sim^* \emptyset$, then $\psi^\sigma_E(B) = 0$ and by Lemma 2(2), $A \cup B \sim^* A$. Thus the desired result obtains. Therefore, we assume that A and B are essential. Let $\sigma'(S) = \{A, B, E \backslash (A \cup B), E^c\}$. By the uniqueness of additive conjoint measurement, we have

$$\begin{aligned}
\psi^\sigma_E(A \cup B) &= \psi^{\sigma'}_A(A) + \psi^{\sigma'}_B(B), \\
\psi^\sigma_E(A) &= \psi^{\sigma'}_A(A), \\
\psi^\sigma_E(B) &= \psi^{\sigma'}_B(B).
\end{aligned}$$

Therefore, additivity of ψ^σ_E follows.

Take any essential event E as assured by C7. Then it follows from the preceding paragraph that ψ^σ_E and $\psi^\sigma_{E^c}$ are additive, where $\sigma(S) = \{E, E^c\}$. Thus we define a finitely additive probability measure π^* on Γ_S as follows: for all $A \in \Gamma_S$,

$$\pi^*(A) = \phi_E(A \cup E) + \phi_{E^c}(A \cap E^c).$$

Then local convexity of π^* follows from C5, and π^* agrees with $\preceq^*$. It remains to show that the definition is consistent, i.e., if E_i for $i = 1, 2$ are essential, then for all $A \in \Gamma_S$,

$$\psi^{\sigma_1}_{E_1}(A \cap E_1) + \psi^{\sigma_1}_{E_1^c}(A \cap E_1^c) = \psi^{\sigma_2}_{E_2}(A \cap E_2) + \psi^{\sigma_2}_{E_2^c}(A \cap E_2^c),$$

where $\sigma_i(S) = \{E_i, E_i^c\}$ for $i = 1, 2$. By additivity, we have that for $i = 1, 2$,

$$\begin{aligned}
\psi^{\sigma_1}_{E_1}(A \cap E_1) + \psi^{\sigma_1}_{E_1^c}(A \cap E_1^c) &= \psi^{\sigma_1}_{E_1}(A \cap E_1 \cap E_2) + \psi^{\sigma_1}_{E_1}(A \cap E_1 \cap E_2^c) \\
&\quad + \psi^{\sigma_1}_{E_1^c}(A \cap E_1^c \cap E_2) + \psi^{\sigma_1}_{E_1^c}(A \cap E_1^c \cap E_2^c), \\
\psi^{\sigma_2}_{E_2}(A \cap E_2) + \psi^{\sigma_2}_{E_2^c}(A \cap E_2^c) &= \psi^{\sigma_2}_{E_2}(A \cap E_1 \cap E_2) + \psi^{\sigma_2}_{E_2}(A \cap E_1^c \cap E_2) \\
&\quad + \psi^{\sigma_2}_{E_2^c}(A \cap E_1 \cap E_2^c) + \psi^{\sigma_2}_{E_2^c}(A \cap E_1^c \cap E_2^c).
\end{aligned}$$

Let $\sigma'(S) = \{E_1 \cap E_2, E_1 \cap E_2^c, E_1^c \cap E_2, E_1^c \cap E_2^c\}$. Since $\sigma'(S)$ includes at least two essential events, it follows from the uniqueness of additive conjoint measurement that

$$\begin{aligned}
\psi^{\sigma'}_{E_1 \cap E_2}(A \cap E_1 \cap E_2) &= \psi^{\sigma_1}_{E_1}(A \cap E_1 \cap E_2) &= \psi^{\sigma_2}_{E_2}(A \cap E_1 \cap E_2), \\
\psi^{\sigma'}_{E_1 \cap E_2^c}(A \cap E_1 \cap E_2^c) &= \psi^{\sigma_1}_{E_1}(A \cap E_1 \cap E_2^c) &= \psi^{\sigma_2}_{E_2^c}(A \cap E_1 \cap E_2^c), \\
\psi^{\sigma'}_{E_1^c \cap E_2}(A \cap E_1^c \cap E_2) &= \psi^{\sigma_1}_{E_1^c}(A \cap E_1^c \cap E_2) &= \psi^{\sigma_2}_{E_2}(A \cap E_1^c \cap E_2), \\
\psi^{\sigma'}_{E_1^c \cap E_2^c}(A \cap E_1^c \cap E_2^c) &= \psi^{\sigma_1}_{E_1^c}(A \cap E_1^c \cap E_2^c) &= \psi^{\sigma_2}_{E_2^c}(A \cap E_1^c \cap E_2^c).
\end{aligned}$$

Hence the desired result follows. □

7 Proofs of Theorems 1, 2, and 3

Throughout the section we assume that Assumption 1 holds. This section provides the proofs of Theorems 1, 2, and 3.

Proof of Theorem 1. Suppose that axioms A1–A7 hold. Since the representation of the theorem readily follows from Proposition 2 when Ω is finite, we shall assume that Ω is infinite. Let π on Γ_S, $\lambda_i(x)$ for all $x \in \Omega^{n+1}_{\uparrow}$, all integers $n > 2$, and all $i = 1, \ldots, n$ be obtained in Proposition 2. We are to show that there is a real valued function u on Ω such that for all $f, g \in \mathcal{F}^s$,

$$f \preceq g \iff \int_S u(f(s))d\pi(s) \leq \int_S u(g(s))d\pi(s),$$

where

$$\int_S u(f(s))d\pi(s) = \sum_{i=1}^{n} \pi(A_i)(u(x_i) - u(x_{i+1})) + u(x_{n+1}),$$

when $f = x_1 \bigcirc_{A_1} x_2 \cdots x_n \bigcirc_{A_n} x_{n+1}$.

Given $a, b \in \Omega$ with $a \prec c \prec b$ for some $c \in \Omega$, let

$$\Omega_{ab} = \{x \in \Omega : a \preceq x \preceq b\}.$$

For all $x \in \Omega_{ab}$ with $x = (a, x, b)$, define a real valued function λ_{ab} on Ω_{ab} as follows:

$$\lambda_{ab}(x) = \begin{cases} \lambda_1(x) & \text{if } a \prec x \prec b, \\ 0 & \text{if } x \sim a, \\ 1 & \text{if } x \sim b. \end{cases}$$

Suppose that $x_1 \sim a$, $x_{n+1} \sim b$, and $x = (x_1, \ldots, x_{n+1}) \in \Omega_{\uparrow}^{n+1}$ for $n > 1$. Then we show that for all $k = 1, \ldots, n$,

$$\lambda_k(x) = \lambda_{ab}(x_{k+1}) - \lambda_{ab}(x_k) \geq 0.$$

Since, by Proposition 2, $\pi(\omega_x(A_1, \ldots, A_n)) = \sum_{i=1}^{n} \lambda_i(x)\pi(A_i)$ for all $(A_1, \ldots, A_n) \in \Gamma_{\uparrow}^n$, we obtain that for $(A, B) \in \Gamma_{\uparrow}^2$ and $k = 1, \ldots, n-1$,

$$\pi(\omega_x^k(AB)) = \left(\sum_{i=1}^{k} \lambda_i(x)\right)\pi(A) + \left(1 - \sum_{i=1}^{k} \lambda_i(x)\right)\pi(B).$$

On the other hand, we have

$$\pi(\omega_x^k(AB)) = \lambda_{ab}(x_{k+1})\pi(A) + (1 - \lambda_{ab}(x_{k+1}))\pi(B).$$

To see this, given $1 \leq k < n$, let $y = (a, x_{k+1}, b) \in \Omega_{\uparrow}^3$. We note by the definitions of ω_x and ω_y that for $(A, B) \in \Gamma_{\uparrow}^2$, $a \bigcirc_{\omega_x^k(AB)} b \sim a \bigcirc_{\omega_x^k(AB)} b$. Thus $\omega_x^k(AB) {\sim^*} \omega_y(AB)$. Therefore, by Proposition 2, the desired result obtains.

It follows from two expression of $\pi(\omega_x^k(AB))$ in the preceding paragraph that for $k = 1, \ldots, n-1$,

$$\lambda_{ab}(x_{k+1}) = \sum_{i=1}^{k} \lambda_i(x).$$

Solving those $n-1$ equations w.r.t. $\lambda_i(x)$, $i = 1, \ldots, n-1$ yields that for all $k = 1, \ldots, n$, $\lambda_k(x) = \lambda_{ab}(x_{k+1}) - \lambda_{ab}(x_k) \geq 0$.

Assign any real numbers $u_{ab}(a)$ and $u_{ab}(b)$ with $u_{ab}(a) < u_{ab}(b)$ to a and b. Then define u_{ab} on Ω_{ab} as follows: for all $x \in \Omega_{ab}$,

$$u_{ab}(x) = \lambda_{ab}(x)(u_{ab}(b) - u_{ab}(a)) + u_{ab}(a).$$

It easily follows that u_{ab} on Ω_{ab} is unique up to a positive linear transformation. Let $x = (x_1, \ldots, x_{n+1}) \in \Omega_{\uparrow}^{n+1}$ and $y = (y_1, \ldots, y_{m+1}) \in \Omega_{\uparrow}^{m+1}$ with $a \sim x_1 \sim y_1$ and $b \sim x_{n+1} \sim y_{m+1}$. Then for all $(A_1, \ldots, A_n) \in \Gamma_{\uparrow}^n$ and all $(B_1, \ldots, B_m) \in \Gamma_{\uparrow}^m$, we obtain

$$\begin{aligned} & x_1 \bigcirc_{A_1} x_2 \cdots x_n \bigcirc_{A_n} x_{n+1} \preceq y_1 \bigcirc_{B_1} y_2 \cdots y_m \bigcirc_{B_m} y_{m+1} \\ \iff\ & a \bigcirc_{\omega_x(A_1,\ldots,A_n)} b \preceq a \bigcirc_{\omega_y(B_1,\ldots,B_m)} b \\ \iff\ & \omega_y(B_1, \ldots, B_m) \preceq^* \omega_x(A_1, \ldots, A_n) \\ \iff\ & \pi(\omega_y(B_1, \ldots, B_m)) \leq \pi(\omega_x(A_1, \ldots, A_n)) \end{aligned}$$

$$\begin{aligned}
&\iff \sum_{i=1}^{m}\lambda_i(y)\pi(B_i) \le \sum_{i=1}^{n}\lambda_i(x)\pi(A_i)\\
&\iff \sum_{i=1}^{m}(\lambda_{ab}(y_{i+1}) - \lambda_{ab}(y_i))\pi(B_i) \le \sum_{i=1}^{n}(\lambda_{ab}(x_{i+1}) - \lambda_{ab}(x_i))\pi(A_i)\\
&\iff \sum_{i=1}^{n}(u_{ab}(x_i) - u_{ab}(x_{i+1}))\pi(A_i) \le \sum_{i=1}^{m}(u_{ab}(y_{i+1}) - u_{ab}(y_i))\pi(B_i).
\end{aligned}$$

Note that every $f \in \mathcal{F}_{ab}$ can be represented as follows: $f = x_1 \bigcirc_{A_1} x_2 \cdots x_n \bigcirc_{A_n} x_{n+1}$ for some positive number n, some $(A_1, \dots, A_n) \in \Gamma_{\uparrow}^{n}$ and some $(x_1, \dots, x_{n+1}) \in \Omega_{\uparrow}^{n+1}$ with $x_1 \sim a$ and $x_{n+1} \sim b$. Hence it readily follows that for all $f, g \in \mathcal{F}_{ab}$,

$$f \preceq g \iff \int_S u_{ab}(f(s))d\pi(s) \le \int_S u_{ab}(g(s))d\pi(s).$$

Suppose that $\Omega_{ab} \subset \Omega_{cd}$. We show that for some $\alpha > 0$ and β, $u_{cd}(x) = \alpha u_{ab}(x) + \beta$. Let $x = (c, a, x, b, d)$ and $y = (a, x, b)$. It follows from the preceding paragraph that

$$\begin{aligned}
&c \bigcirc_{\emptyset} a \bigcirc_A x \bigcirc_B b \bigcirc_S d \preceq c \bigcirc_{\emptyset} a \bigcirc_C x \bigcirc_D b \bigcirc_S d\\
&\quad\iff (u_{cd}(a) - u_{cd}(x))\pi(A) + (u_{cd}(x) - u_{cd}(b))\pi(B)\\
&\quad\qquad \le (u_{cd}(a) - u_{cd}(x))\pi(C) + (u_{cd}(x) - u_{cd}(b))\pi(D).\\
&a \bigcirc_A x \bigcirc_B b \preceq a \bigcirc_C x \bigcirc_D b\\
&\quad\iff (u_{ab}(a) - u_{ab}(x))\pi(A) + (u_{ab}(x) - u_{ab}(b))\pi(B)\\
&\quad\qquad \le (u_{ab}(a) - u_{ab}(x))\pi(C) + (u_{ab}(x) - u_{ab}(b))\pi(D).
\end{aligned}$$

Since $c \bigcirc_{\emptyset} a \bigcirc_A x \bigcirc_B b \bigcirc_S d = a \bigcirc_A x \bigcirc_B b$ and $c \bigcirc_{\emptyset} a \bigcirc_C x \bigcirc_D b \bigcirc_S d = a \bigcirc_C x \bigcirc_D b$, the uniqueness of additive representation gives

$$\frac{u_{cd}(a) - u_{cd}(x)}{u_{cd}(a) - u_{cd}(b)} = \frac{u_{ab}(a) - u_{ab}(x)}{u_{ab}(a) - u_{ab}(b)}.$$

Hence $u_{cd}(x) = \alpha u_{ab}(x) + \beta$, where $\alpha = (u_{cd}(a) - u_{cd}(b))/(u_{ab}(a) - u_{ab}(b))$ and $\beta = u_{ab}(a)u_{cd}(b) - u_{ab}(b)u_{cd}(a)$.

It follows from the preceding paragraphs that under appropriate positive linear transformations of u_{ab} for all $a, b \in \Omega$ with $a \prec c \prec b$ for some $c \in X$, we can construct a real valued function u on X such that for all $f, g \in \mathcal{F}^s$,

$$f \preceq g \iff \int_S u(f(s))d\pi(s) \le \int_S u(g(s))d\pi(s).$$

Moreover, u is unique up to a positive linear transformation. □

Proof of Theorem 2. Suppose that axioms $A1$, $A(2)$, $A2$, $A3$, $A4^*$, $A5$, $A6$, and $A7^*$ hold. First in Claim 1 below, we show that there is a unique locally convex and finitely additive probability measure π^* on Γ_S agreeing with $\preceq^*$. Next, we show that $A4$ and $A7$ hold, so that Theorem 1 obtains. Thus $(\preceq, \mathcal{F}^s)$ has a CEU representation (u, π). Finally, the density of π^* is proved in Claim 3. Therefore, there exists a unique strictly increasing function ϕ on $R(\pi)$ such that $\pi(A) = \phi(\pi^*(A))$ for all $A \in \Gamma_S$. Suppose that $0 < \alpha < 1$ and $\alpha = \pi(A)$ for no $A \in \Gamma_S$. Since π and π^* are dense,

$$\sup_{\pi(A)<\alpha} \phi(\pi(A)) = \inf_{\pi(A)>\alpha} \phi(\pi(A)).$$

Letting $\phi(\alpha) = \sup_{\pi(A)<\alpha} \phi(\pi(A))$, ϕ must be continuous on $[0,1]$. Hence the conclusion of the theorem obtains.

Claim 1 *There exists a unique locally convex and finitely additive probability measure π^* on Γ_S agreeing with $\preceq^*$.*

Proof. It suffices to show that the qualitative probability relation $\preceq^*$ satisfies C1–C7 in Proposition 3. We note by A5 that $a \prec b \prec c$ for some $a, b, c \in \Omega$. C1 and C2 follow from A3, and C3 follows from A1.

C4. Suppose that $(A \cup B) \cap C = \emptyset$. Then by $A(2)$, we have

$$\begin{aligned} A \prec^* B &\iff a \bigcirc_B b \prec a \bigcirc_A b \\ &\iff (a \bigcirc_B b) \bigcirc_{A\cup B} (a \bigcirc_C b) \prec (a \bigcirc_A b) \bigcirc_{A \cup B} (a \bigcirc_C b) \\ &\iff a \bigcirc_{B\cup C} b \prec x \bigcirc_{A\cup C} b \\ &\iff A \cup C \prec^* B \cup C. \end{aligned}$$

C5. Suppose that $A \subseteq B$ and $A \preceq^* C \preceq^* B$. Then we have $a \bigcirc_B b \preceq a \bigcirc_C b \preceq a \bigcirc_A b$. By A6, there is an event F such that $a \bigcirc_C b \sim (a \bigcirc_B b) \bigcirc_F (a \bigcirc_A b)$. Thus $a \bigcirc_C b \sim a \bigcirc_E b$, so $C \sim^* E$, where $E = A \cup (F \cap B)$ and $A \subseteq E \subseteq B$.

C6. Suppose that $(A \cup B) \cap C = \emptyset$, $A_i \subseteq C$, and $A \cup A_i \sim^* B \cup A_{i+1}$ for all $i, i+1 \in M$. Then we have

$$a \bigcirc_{A \cup A_i} b = (a \bigcirc_A a) \bigcirc_{A\cup A_i} b \sim a \bigcirc_{B \cup A_{i+1}} b = (a \bigcirc_B a) \bigcirc_{B \cup A_{i+1}} b,$$

so that finiteness of $\{A_i\}$ follows from $A7^*$.

C7. Since $a \prec b \prec c$, A6 implies $b \sim a \bigcirc_E c$ for some event E. Thus by A1, $a \bigcirc_S c = a \prec a \bigcirc_E c \prec c = a \bigcirc_\emptyset c$, so that $\emptyset \prec^* E \prec^* S$.

Hence by Proposition 3, there is a unique locally convex and finitely additive probability measure π^* on Γ_S agreeing with $\preceq^*$. □

A7 follows from $A7^*$. To see that A4 holds, we need the following claim.

Claim 2 *For $A, B \in \Gamma_S$, $x_1, \ldots, x_n \in \Omega$, and $g \in \mathcal{F}^s$, let $\sigma(A) = \{A_1, \ldots, A_n\}$, $\sigma(B) = \{B_1, \ldots, B_n\}$, $A \cap B = A_1 = B_1$, and $g(A_i) = g(B_i) = x_i$ for $i = 1, \ldots, n$. If $x_1 \preceq \cdots \preceq x_n$ and $B_i \preceq^* A_i$ for $i = 1, \ldots, n$, then for all $f \in \mathcal{F}^s$, $(x_1 \bigcirc_A g) \bigcirc_{A\cup B} f \preceq (x_1 \bigcirc_B g) \bigcirc_{A \cup B} f$. If, in addition, $B_k \prec^* A_k$ for some k, then $(x_1 \bigcirc_A g) \bigcirc_{A \cup B} f \prec (x_1 \bigcirc_B g) \bigcirc_{A\cup B} f$.*

Proof. Suppose that the hypotheses of the claim hold. Let $f_1 = (x_1 \bigcirc_A g) \bigcirc_{A\cup B} f$. Then we recursively define acts, $f_2, \ldots, f_n$, as follows: for $i = 2, \ldots, n$,

$$f_i = (x_1 \bigcirc_{B_i} x_i) \bigcirc_{B_i \cup A_i} f_{i-1}.$$

Note $f_n = (x_1 \bigcirc_B g) \bigcirc_{A\cup B} f$. Since $x_1 \bigcirc_{A_i} x_i \preceq x_1 \bigcirc_{B_i} x_i$ for $i = 2, \ldots, n$, $A(2)$ implies

$$(x_1 \bigcirc_{A_i} x_i) \bigcirc_{A_i \cup B_i} f_{i-1} \preceq (x_1 \bigcirc_{B_i} x_i) \bigcirc_{B_i \cup A_i} f_{i-1},$$

so $f_{i-1} \preceq f_i$. Thus by A1, $f_1 \preceq f_n$. If, in addition, $B_k \prec^* A_k$ for some k, then $f_{k-1} \prec f_k$. Thus by A1, $f_1 \prec f_n$. This completes the proof of the claim. □

Suppose that $f(S) = g(S) = \{x_1, \ldots, x_n\}$, $x_1 \preceq \cdots \preceq x_n$, and for $i = 1, \ldots, n$, $g^{-1}(\{x_1, \ldots, x_i\}) \preceq^* f^{-1}(\{x_1, \ldots, x_i\})$. We are to show that $f \preceq g$, and that $f \prec g$ whenever $g^{-1}(\{x_1, \ldots, x_i\}) \prec^* f^{-1}(\{x_1, \ldots, x_i\})$ for some i.

Let $f_1 = f$. For $k = 1, \ldots, n-1$, we recursively define acts, g_k and f_{k+1}, as follows. Let $\sigma(g^{-1}(x_k)) = \{B_k^k, B_k^{k+1}, \ldots, B_k^n\}$, where $B_k^i = g^{-1}(x_k) \cap f_k^{-1}(x_i)$ for $i = k, \ldots, n$. Since by Claim 1, π^* on Γ_S agrees with $\preceq^*$ and is locally convex and additve, there exists a partition $\sigma(f_k^{-1}(x_k)) = \{A_k^k, A_k^{k+1} \cup A_k^0, A_k^{k+2}, \ldots, A_k^n\}$ such that $A_k^k = B_k^k$ and $A_k^i \sim^* B_k^i$ for $i = k+1, \ldots, n$, where

$$A_k^0 = f_k^{-1}(x_k) \setminus \bigcup_{i=k}^{n} A_k^i.$$

Then we define an act $g_k \in \mathcal{F}^s$ by

$$g_k(A_k^0) = \{x_{k+1}\} \text{ and } g_k(A_k^i) = g_k(B_k^i) = \{x_i\} \text{ for } i = k, \ldots, n.$$

Given g_k, let

$$f_{k+1} = (x_k \bigcirc_{g^{-1}(x_k)} g_k) \bigcirc_{f^{-1}(x_k) \cup g^{-1}(x_k)} f_k.$$

Since $f_k = (x_k \bigcirc_{f_k^{-1}(x_k)} g_k) \bigcirc_{f_k^{-1}(x_k) \cup g_k^{-1}(x_k)} f_k$ and $B_k^{k+1} \preceq^* A_k^{k+1} \cup A_k^0$, Claim 2 implies that $f_k \preceq f_{k+1}$. Hence $f = f_1 \preceq \cdots \preceq f_n = g$, so that the first part of A4 holds.

Let ℓ be the smallest number in $\{1, \ldots, n-1\}$ such that $g^{-1}(\{a_1, \ldots, x_\ell\}) \prec f^{-1}(\{x_1, \ldots, x_\ell\})$. Then it is easy to see that $\pi^*(A_\ell^0) > 0$, so that $B_\ell^{\ell+1} \prec^* A_\ell^{\ell+1} \cup A_\ell^0$. Hence by Claim 2, $f_\ell \prec f_{\ell+1}$, so $f \prec g$. Therefore, the second part of A4 holds.

We note at this stage that there exist a locally convex and dense capacity π on Γ_S and a locally convex and finitely additive probability measure π^* on Γ_S, both agreeing with $\preceq^*$.

Claim 3 *π^* is dense.*

Proof. Let $A_0 = S$ and $a \prec b \prec c$. Define a sequence $\{A_i\}$ of events as follows: for $i = 0, 1, 2, \ldots$,

$$b \bigcirc_{A_i} c \sim a \bigcirc_{A_{i+1}} c.$$

We show that such a sequence exists and that $\emptyset \prec^* A_{i+1} \prec^* A_i \prec^* S$ for all integers $i > 0$. Since $a \prec b \bigcirc_{A_0} c \prec c$, A6 implies that $b \bigcirc_{A_0} c \sim a \bigcirc_{A_1} c$ for some $A_1 \in \Gamma_S$. Thus $a \prec a \bigcirc_{A_1} c \prec c$, so by definition, $\emptyset \prec^* A_1 \prec^* S$.

Suppose that $A_1, \ldots, A_k$ exist for some $k \geq 1$ and $\emptyset \prec^* A_k \prec^* \cdots \prec^* A_1 \prec^* S$. We are to show that A_{k+1} exists and $\emptyset \prec^* A_{k+1} \prec^* A_k$. Since $b \prec b \bigcirc_{A_k} c \prec c$, we have that $a \prec b \bigcirc_{A_k} c \prec c$. By A6, there exists an $A_{k+1} \in \Gamma_S$ such that $b \bigcirc_{A_k} c \sim a \bigcirc_{A_{k+1}} c$. Thus $(a \bigcirc_{\emptyset} b) \bigcirc_{A_k} c \sim a \bigcirc_{A_{k+1}} c$. On the other hand, since $a \bigcirc_{A_k} b \prec a \bigcirc_{\emptyset} b$, it follows form $A(2)$ that $a \bigcirc_{A_k} c = (a \bigcirc_{A_k} b) \bigcirc_{A_k} c \prec (a \bigcirc_{\emptyset} b) \bigcirc_{A_k} c$. Hence $a \bigcirc_{A_k} c \prec a \bigcirc_{A_{k+1}} c$, so by definition, $A_{k+1} \prec^* A_k$. Since $a \bigcirc_{A_{k+1}} c \prec c$, we have $\emptyset \prec^* A_{k+1}$.

In the sequel, we show that 0 is an accumulation point of $R(\pi^*)$, i.e., given a sequence $\{A_i\}$ constructed in the preceding paragraph,

$$\lim_{i \to \infty} \pi^*(A_i) = 0.$$

Hence it follows from additivity and local convexity of π^* that π^* must be dense.

Suppose on the contrary that 0 is not an accumulation point of $R(\pi^*)$, i.e.,

$$1 > \lim_{i \to \infty} \pi^*(A_i) = \alpha > 0.$$

Given $0 < \epsilon < \alpha$, there is a $k \geq 1$ such that

$$\alpha < \pi^*(A_{k+1}) < \pi^*(A_k) < \alpha + \epsilon,$$

so that $0 < \pi^*(A_k) - \pi^*(A_{k+1}) < \epsilon$. Therefore, local convexity and additivity of π^* imply that there is an event $A \in \Gamma_S$ such that $0 < \pi^*(A) < \alpha$. Thus $\emptyset \prec^* A \prec^* A_i$ for all i.

Since $\emptyset \prec^* A$, we have $\pi(A) > 0$. By Theorem 1 and the construction of $\{A_i\}$,

$$\lim_{i \to \infty} \pi(A_i) = 0.$$

Therefore, there is an $\ell \geq 1$ sucht that $\pi(A_\ell) < \pi(A)$, so $A_\ell \prec^* A$. This is a contradiction. Hence 0 must be an accumulation point of $R(\pi^*)$. □

Proof of Theorem 3. Suppose that axioms $A1$, $A(3)$, $A2$, $A3$, $A4^*$, $A5$, $A6$, $A7^*$ hold. Since $A(3)$ implies $A(2)$, Theorem 2 implies that $(\mathcal{F}^s, \preceq)$ has an RDEU representation (u, π^*, ϕ). We are to show that ϕ is a linear function on $I = [0,1]$, i.e., $\phi(\alpha) = \gamma\alpha + \nu$ for all $\alpha \in I$ and some reals, $\gamma > 0$ and ν.

Throughout the proof we shall fix $a, b, c \in \Omega$ with $a \prec b \prec c$ as assured by $A5$. With no loss of generality we assume that $u(a) = 0$ and $u(c) = 1$. Let $\lambda = u(b)$, so $0 < \lambda < 1$.

We define a mapping μ from $I \times I$ into I such that for all $(\alpha, \beta) \in I \times I$,

$$\phi(\mu(\alpha, \beta)) = \lambda\phi(\alpha) + (1 - \lambda)\phi(\beta).$$

Since ϕ is continuous and strictly increasing on I, μ is uniquely well defined. We note that ϕ is a cardinal scale, i.e., unique up to a positive linear transformation. Let $J \subseteq I$. Then we say that μ on $J \times J$ is *augmented invariant* if for all $\alpha, \beta, \delta \in J$,

$$\mu(\alpha + \delta, \beta + \delta) = \mu(\alpha, \beta) + \delta,$$

whenever $\alpha + \delta$ and $\beta + \delta$ are in J. Pfanzagl (1959, Theorem 3) proves that if μ on $I \times I$ is augmented invariant, then ϕ is either a linear function or an exponential function.

Next we show that μ on $I \times I$ is augmented invariant. Since ϕ is continuous, it suffices to show that μ on $J \times J$ is augmented invariant for some dense subset J of I, i.e., $I = \overline{J}$. For $A, B \in \Gamma_S$ with $A \subseteq B$, $a \preceq (a \bigcirc_A b) \bigcirc_B c \preceq c$. Thus by $A6$, $(a \bigcirc_A b) \bigcirc_B c \sim a \bigcirc_C c$ for some $C \in \Gamma_S$. By Theorem 2,

$$\lambda\phi(\pi^*(A)) + (1-\lambda)\phi(\pi^*(B)) = \phi(\pi^*(C)).$$

Therefore, $\mu(\pi^*(A), \pi^*(B)) = \pi^*(C)$.

Throughout the rest of the proof, let $f = (a \bigcirc_A b) \bigcirc_B c$ and $g = a \bigcirc_C c$, so $f \sim g$. Since $|f(S) \cup g(S)| = 3$, it follows from $A(3)$ that, for all $D, E \in \Gamma_S$, $f \bigcirc_D (a \bigcirc_E c) \sim g \bigcirc_D (a \bigcirc_E c)$. Taking any D and E to satisfy that $B \subseteq D$ and $D \cap E = \emptyset$, we obtain

$$(a \bigcirc_{A \cup E} b) \bigcirc_{B \cup E} c \sim (a \bigcirc_{C \cup E} b) \bigcirc_{C \cup E} c.$$

Therefore, by Theorem 2,

$$\lambda\phi(\pi^*(A) + \pi^*(E)) + (1-\lambda)\phi(\pi^*(B) + \pi^*(E)) = \phi(\pi^*(C) + \pi^*(E)),$$

so that we have

$$\begin{aligned} \mu(\pi^*(A) + \pi^*(E), \pi^*(B) + \pi^*(E)) &= \pi^*(C) + \pi^*(E) \\ &= \mu(\pi^*(A), \pi^*(B)) + \pi^*(E). \end{aligned}$$

Thus μ on $R(\pi^*) \times R(\pi^*)$ is augmented invariant. Since $\overline{R(\pi^*)} = I$, μ on $I \times I$ is augmented invariant.

Last we show that ϕ must be a linear function. Suppose on the contrary that ϕ is an exponential function, i.e., for some real numbers γ, ϵ, and ν with $\gamma\epsilon > 0$,

$$\phi(\alpha) = \gamma e^{\epsilon\alpha} + \nu.$$

Since $f \sim g$, it follows from Theorem 2 that

$$\lambda e^{\epsilon\pi^*(A)} + (1-\lambda)e^{\epsilon\pi^*(B)} = e^{\epsilon\pi^*(C)},$$

which is rearranged to give

$$\lambda(e^{\epsilon\pi^*(A)} - e^{\epsilon\pi^*(C)}) = (1-\lambda)(e^{\epsilon\pi^*(C)} - e^{\epsilon\pi^*(B)}).$$

Taking any D and E to satisfy that $B \subseteq D$, $D \cap E = \emptyset$, and $\pi^*(E) \neq 0$. we obtain by $A(3)$ that $f \bigcirc_D (b \bigcirc_E c) \sim g \bigcirc_D (b \bigcirc_E c)$, which is equivalent to

$$(a \bigcirc_A b) \bigcirc_{B \cup E} c \sim (a \bigcirc_C b) \bigcirc_{C \cup E} c.$$

Therefore, by Theorem 2,

$$\lambda e^{\epsilon\pi^*(A)} + (1-\lambda)e^{\epsilon(\pi^*(B)+\pi^*(E))} = \lambda e^{\epsilon\pi^*(C)} + (1-\lambda)e^{\epsilon(\pi^*(C)+\pi^*(E))},$$

which is rearranged to give

$$\lambda(e^{\epsilon\pi^*(A)} - e^{\epsilon\pi^*(C)}) = (1-\lambda)e^{\epsilon\pi^*(E)}(e^{\epsilon\pi^*(C)} - e^{\epsilon\pi^*(B)}).$$

It follows from the last equation of the preceding paragraph that $e^{\epsilon\pi^*(E)} = 1$, so that $\epsilon = 0$. This is a contradiction. Hence ϕ must be a linear function. □

8 Proofs of Theorems 4 and 5

This section shows the proofs of Theorems 4 and 5. Since the representations (2) and (3) in each theorem readily follow from the representation (1) in those theorems, and Theorems 2 and 3, we shall prove only the representations (1) in Theorems 4 and 5. Throughout the section, we shall assume that Assumption 1, axioms $A8$ and $A9$, and the hypotheses of Theorem 1 hold with $A3$ replaced by $A3^*$.

Let $\mathcal{F}^*$ denote the set of all measurable acts that are bounded in preferences, i.e.,

$$\mathcal{F}^* = \{f \in \mathcal{F} : a \preceq f \preceq b \text{ for some } a, b \in \Omega\}.$$

By $A3^*$, $\mathcal{F}^b \subseteq \mathcal{F}^*$. Let u and π be resepectively a real valued function on Ω and a capacity on Γ_S obtained in Theorem 1.

For $f \in \mathcal{F}^*$, there are $a, b \in \Omega$ such that $a \preceq f \preceq b$ and $a \prec b$. Let

$$\begin{aligned} R_f^+(a,b) &= \{\pi(A) : a \bigcirc_A b \prec f\}, \\ R_f^-(a,b) &= \{\pi(A) : f \prec a \bigcirc_A b\}. \end{aligned}$$

Since π is dense, we have $\inf R_f^+(a,b) = \sup R_f^-(a,b)$. Let $\alpha_f = \inf R_f^+(a,b)$. Then we define

$$U(f) = \alpha_f u(a) + (1-\alpha_f)u(b).$$

We note that the definition of $U(f)$ does not depend on the choice of a and b.

We show that for all $f, g \in \mathcal{F}^*$, $f \prec g \iff U(f) < U(g)$. With no loss of generality we assume that $a \preceq f \preceq b$ and $a \preceq g \preceq b$. If $f \prec g$, then by A8, $f \prec a \bigcirc_A b \prec g$ for some $A \in \Gamma_S$, so that $\alpha_g < \alpha_f$. Hence $U(f) < U(g)$. If $U(f) < U(g)$, then by definition, $\alpha_g < \alpha_f$, so $\alpha_g < \pi(A) < \alpha_f$ for some $A \in \Gamma_S$. Thus $f \prec a \bigcirc_A b \prec g$, so $f \prec g$.

Given u on Ω, $f \in \mathcal{F}$, and π on Γ_S, define

$$E(u,f,\pi) = \int_0^{+\infty} (1 - \pi(\{s : u(f(s)) \leq \tau\}))d\tau - \int_{-\infty}^0 \pi(\{s : u(f(s)) \leq \tau\})d\tau.$$

We note that $E(u, f, \pi)$ is well defined when $f \in \mathcal{F}^*$.

Proof of Theorem 4. We are to show that $U(f) = E(u, f, \pi)$ for all $f \in \mathcal{F}^*$. First we show the following claim.

Claim 4 *Suppose that $f \in \mathcal{F}^b$, $g \in \mathcal{F}^s$, $c_0 < c_1 < \cdots < c_n$, and $\sigma(S) = \{A_1, \ldots, A_n\}$.*

(1) *If $c_{i-1} \leq u(f(s)) < c_i$ for all $s \in A_i$ and $i = 1, \ldots, n$, and $g \prec f$, then*

$$E(u, g, \pi) < \sum_{k=1}^{n} (\pi(\bigcup_{i=1}^{k} A_i) - \pi(\bigcup_{i=1}^{k-1} A_i))c_i.$$

(2) *If $c_{i-1} < u(f(s)) \leq c_i$ for all $s \in A_i$ and $i = 1, \ldots, n$, and $f \prec g$, then*

$$\sum_{k=1}^{n} (\pi(\bigcup_{i=1}^{k} A_i) - \pi(\bigcup_{i=1}^{k-1} A_i))c_{i-1} < E(u, g, \pi).$$

Proof. We show (1). The proof of (2) is similar. Suppose that the hypotheses of the claim hold. For $i = 1, \ldots, n$, let

$$X_i = \{x \in \Omega : c_{i-1} \leq u(x) < c_i\},$$

so $A_i = f^{-1}(X_i)$. Given $x_i \in X_i$ for $i = 1, \ldots, n$, let $h(s) = x_i$ if $s \in A_i$. If there is no h such that $g \preceq h$, then by A9, $f \preceq g$. This is a contradiction. Hence $g \preceq h$ for some h. Therefore,

$$E(u, g, \pi) \leq E(u, h, \pi) < \sum_{k=1}^{n} (\pi(\bigcup_{i=1}^{k} A_i) - \pi(\bigcup_{i=1}^{k-1} A_i))c_i.$$

This completes the proof of the claim. □

Since $f \in \mathcal{F}^*$, let $a \preceq f \preceq b$ for some $a, b \in \Omega$. With no loss of generality, assume that $u(a) = 0$ and $u(b) = 1$. Devide S into n cumulative events as follows: for $i = 1, 2, \ldots, n$,

$$A_i = \{s \in S : u(f(s)) \leq \frac{i}{n}\}.$$

By the definition of expectation, we have

$$\sum_{i=1}^{n} \frac{i-1}{n}(\pi(A_i) - \pi(A_{i-1})) \leq E(u, f, \pi) \leq \sum_{i=1}^{n} \frac{i}{n}(\pi(A_i) - \pi(A_{i-1})).$$

It follows from Claim 4 that for any $\epsilon > 0$,

$$\sup_{\alpha \in R_f^+(a,b)} (1 - \alpha) \leq \sum_{i=1}^{n} \frac{i+\epsilon}{n}(\pi(A_i) - \pi(A_{i-1})),$$

$$\sum_{i=1}^{n} \frac{i-1-\epsilon}{n}(\pi(A_i) - \pi(A_{i-1})) \leq \inf_{\alpha \in R_f^-(a,b)} (1 - \alpha).$$

Therefore, we have

$$\sum_{i=1}^{n} \frac{i-1-\epsilon}{n}(\pi(A_i) - \pi(A_{i-1})) \leq U(f) \leq \sum_{i=1}^{n} \frac{i+\epsilon}{n}(\pi(A_i) - \pi(A_{i-1})).$$

Letting n get large, we obtain that $U(f) = E(u, f, \pi)$. This completes the proof of the theorem. □

Proof of Theorem 5. Suppose that axiom A10 holds. We have two cases to examine: $f \in \mathcal{F}^* \setminus \mathcal{F}^b$; $f \notin \mathcal{F}^*$.

Case 1(all $f \in \mathcal{F}^* \setminus \mathcal{F}^b$). Suppose that $f \in \mathcal{F}^*$. We are to show that $U(f) = E(u, f, \pi)$ for all $f \in \mathcal{F}^* \setminus \mathcal{F}^b$. Let $a \preceq f \preceq b$ and $a \prec b$ for $a, b \in \Omega$.

By $A3^*$, $f^x \preceq f \preceq f_x$ for all $x \in \Omega$, so that $\sup_{x \in \Omega} U(f^x) \leq U(f) \leq \inf_{x \in \Omega} U(f_x)$. Suppose first that f is bounded below, i.e., $d \preceq f(s)$ for all $s \in S$ and some $d \in \Omega$. When f is bounded above, the proof is similar. Note that $E(u, f, \pi)$ is well defined. Since f^x is bounded, $U(f^x) = E(u, f^x, \pi)$. By the definition of $E(u, f, \pi)$, $\sup_{x \in \Omega} E(u, f^x, \pi) = E(u, f, \pi)$. Therefore, $E(u, f, \pi) \leq U(f)$. It remains to show that the equality holds. Assume that $E(u, f, \pi) < U(f)$. Since $U(f) = \alpha_f u(a) + (1 - \alpha_f)u(b)$ for some $0 \leq \alpha_f \leq 1$, it follows from the density of π that there is an $A \in \Gamma_S$ such that $E(u, f, \pi) < \pi(A)u(a) + (1 - \pi(A))u(b) < \alpha_f u(a) + (1 - \alpha_f)u(b)$. Thus $a \bigcirc_A b \prec f$. By $A10$, $a \bigcirc_A b \prec f^c$ for some $c \in \Omega$. However, we have

$$E(u, f^c, \pi) \leq E(u, f, \pi) < \pi(A)u(a) + (1 - \pi(A))u(b),$$

so $f^c \prec a \bigcirc_A b$. This is a contradiction. Hence $U(f) = E(u, f, \pi)$.

Suppose next that f is unbounded below and above. If $E(u, f, \pi)$ is well defined, a similar analysis of the preceding paragraph provides that $U(f) = E(u, f, \pi)$. Thus it suffices to verify that $E(u, f, \pi)$ is well defined. Suppose that $E(u, f, \pi)$ is undefined, so that

$$\int_0^{+\infty} (1 - \pi(\{s : u(f(s)) \leq \tau\}))d\tau = \int_{-\infty}^{0} \pi(\{s : u(f(s)) \leq \tau\})d\tau = +\infty.$$

We note that $E(u, g, \pi) = -\infty$ if g is bounded above. Since f is unbounded, there is a $c \in \Omega$ such that $c \prec a$. Thus by A1, $c \prec f$. By $A10$, $c \prec f^d$ for some $d \in \Omega$. Since f^d is bounded above, $U(c) < U(f^d) = E(u, f^d, \pi) = -\infty$. This is a contradiction. Hence $E(u, f, \pi)$ is well defined.

Case 2 (all $f \in \mathcal{F} \setminus \mathcal{F}^*$). A similar proof of Case 1 gives that for all $f \in \mathcal{F} \setminus \mathcal{F}^*$, $E(u, f, \pi)$ is well defined. We are to show that if for all $f \in \mathcal{F} \setminus \mathcal{F}^*$, we define $U(f) = E(u, f, \pi)$, then for all $f, g \in \mathcal{F}$, $f \preceq g$ iff $U(f) \leq U(g)$. It suffices to verify the following claim.

Claim 5 (1) *If either $x \prec \{f, g\}$ for all $x \in \Omega$, or $\{f, g\} \prec x$ for all $x \in \Omega$, then $f \sim g$.*
(2) *If $x \prec f$ for all $x \in \Omega$, then $E(u, f, \pi) = \sup_{x \in \Omega} u(x)$.*
(3) *If $f \prec x$ for all $x \in \Omega$, then $E(u, f, \pi) = \inf_{x \in \Omega} u(x)$.*

Proof. (1) Suppose that $x \prec \{f,g\}$ for all $x \in \Omega$. When $\{f,g\} \prec x$ for all $x \in \Omega$, the proof is similar. First we show that $f \sim g$. Assume $f \prec g$. Then by $A10$, $f \prec g^a$ for some $a \in \Omega$. By $A3^*$, $g^a \preceq a$, so by $A1$, $f \prec a$. This is a contradiction, so $f \preceq g$. Similarly, we must have $g \preceq f$. Hence $f \sim g$.

(2) Suppose that $x \prec f$ for all $x \in \Omega$. Assume that $E(u,f,\pi) < \sup_{x\in\Omega} u(x)$. Then there is an $a \in \Omega$ such that $E(u,f,\pi) < u(a)$. Since $a \prec f$, $A10$ implies that $a \prec f^b$ for some $b \in \Omega$. Since $a \prec f^b \preceq b$ by $A3^*$, Case 1 implies that $u(a) < E(u,f^b,\pi)$. Since $E(u,f,\pi)$ is well defined, $E(u,f^b,\pi) \leq E(u,f,\pi)$. Thus $u(a) < E(u,f,\pi)$, a contradiction. Hence $\sup_{x\in\Omega} u(x) \leq E(u,f,\pi)$.

It remains to show that the equality holds. Assume that $\sup_{x\in\Omega} u(x) < E(u,f,\pi)$. By the definition of $E(u,f,\pi)$, $\sup_{x\in\Omega} E(u,f^x,\pi) = E(u,f,\pi)$. Thus $\sup_{x\in\Omega} u(x) < E(u,f^a,\pi)$ for some $a \in \Omega$. However, we have $E(u,f^a,\pi) \leq u(a)$. This is a contradiction. Hence $E(u,f,\pi) = \sup_{x\in\Omega} u(x)$.

(3) Similar to (2). □

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