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Mere and Specific Knowledge of the Existence of a Nash Equilibrium

by

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Abstract

The mere knowledge of the existence of a Nash equilibrium for a game is not sufficient for a player to play the game. Instead, he needs the specific knowledge of what a Nash equilibrium is. The standard existence theorem ensures the former if the real number axioms are known to the players with complete logical abilities, but not necessarily the latter. In this paper, we consider the implications of this distinction, and explore conditions on the language and axioms for the latter to be obtained. We formulate classical game theory in an ordered field language, and review the epistemic axiomatization of final decisions, which leads to the common knowledge of a Nash equilibrium. This additional knowledge structure enables us to distinguish between the mere and specific knowledge of the existence of a Nash equilibrium. Then we show that in this language, any axioms additional to the ordered field axioms are not sufficient for our purpose for games with more than two players. This gives the undecidability result for many games. The introduction of radicals solves undecidability for some games, but not for all games. Then we extend the language and axioms in two ways so that all real algebraic numbers have names, and show that each these extensions is sufficient for the specific knowledge of the existence of a Nash equilibrium for any finite game.

1. Introduction

It has been often regarded as an acceptable view among game theorists and economists that if the players have "complete" logical (mathematical) abilities, each player could find his strategy to be played whenever such a strategy exists. Instead, it has been a

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problem in the literature of game theory what such a strategy is. Nevertheless, we would find this view to be too naive by recalling the literature of foundations of mathematics: Gödel's incompleteness theorem is often referred as an example to warn us of such a naive view, but the distinction between classical mathematics and constructionist mathematics is more to the point. This distinction is relevant specifically to game theory, since game theory should take actual choices as its objects in addition to pure deductive considerations of decision making. The distinction may be relevant in many phases of game theoretical decision making.[1] In this paper, we consider the distinction between the mere knowledge of the existence of a Nash equilibrium (e.g. by the standard fixed point argument) and the specific knowledge of where a Nash equilibrium is.

For the explicit consideration of the above distinction, we need some framework where the logical abilities of the players as well as some mathematics sufficient for classical game theory can be explicitly described. Kaneko-Nagashima [12] and [13] have developed such a framework, called *game logic*. The game logic framework has several variants depending upon choices of assumptions on the knowledge of the players. In this paper, we adopt one variant, GL_ω, where the logical-introspective abilities of the players are assumed to be common knowledge as well as the description of the logical ability of the investigator is given. Specifically, it is an infinitary predicate extension of propositional (modal) epistemic logic, KD4, with the knowledge operators of n players. It is infinitary, i.e., allows infinitary conjunctions and disjunctions, as to describe common knowledge explicitly as a conjunctive formula,[2] and is a predicate logic to allow the treatment of some real number theory for classical game theory. The players and investigator have logical abilities in the sense of classical mathematics but not in the constructionist sense. The constructionist aspect emerges in the interactions between the knowledge and inferences of the players and those of the investigator. To state this aspect, we need to discuss an epistemic consideration of *ex ante* decision making in a game.

In GL_ω, Kaneko-Nagashima [10] and [12] gave an epistemic axiomatization of individual *ex ante* decision making in a game, and showed that their axioms lead to the

[1] This has been discussed by some authors, e.g., Rabin [22]. He gave a 3-stage 2-person 0-sum game with countable action spaces where all descriptions are effectively computable but the a strategy induced by backward induction is not effectively computable. See also Tashiro [26]. In classical game theory in the sense of von Neumann [31] and Nash [20], however, this view has not been discussed, with the exception of the work on algorithms for finding Nash equilibria.

[2] We formulate the common knowledge $C(A)$ of a formula A, to be an infinitary conjunction, instead of introducing the operator symbol $C_0(\cdot)$. Therefore we employ the infinitary predicate extension GL_ω of KD4. There is a finitary approach to logics with common knowledge (cf., Halpern-Moses [3] and Lismont-Mongin [17]). In fact, Kaneko [5] proved that the (KD4-variant of) common knowledge logic of [3] and [17] can be faithfully embedded into a propositional fragment of GL_ω (under some restriction of the Barcan axiom in GL_ω).

common knowledge of a Nash equilibrium,

$$C(\mathrm{Nash}_g(\overrightarrow{a})), \tag{1.1}$$

for a *solvable* game g in Nash's [20] sense, i.e., the equilibria are interchangeable. That is, a mixed strategy profile $\overrightarrow{a} = (\overrightarrow{a}_1, ..., \overrightarrow{a}_n)$ consists of decisions predicted by a player if and only if it is common knowledge that $\overrightarrow{a}$ is a Nash equilibrium. For an unsolvable game, the solution is the common knowledge of a *subsolution* in Nash's sense (Kaneko [6]). Also, it is also argued in Kaneko [7] that for a game which has some *dominant* strategies, the common knowledge is not necessarily involved. We focus, however, on the solution of the form of (1.1), since this case is central in the problem of this paper.

Once the solution is determined as (1.1), the playability is formulated as

$$\exists \overrightarrow{x} C(\mathrm{Nash}_g(\overrightarrow{x})) \tag{1.2}$$

instead of

$$C(\exists \overrightarrow{x} \mathrm{Nash}_g(\overrightarrow{x})). \tag{1.3}$$

The former states that there exists a strategy profile $\overrightarrow{x}$ such that it is common knowledge that $\overrightarrow{x}$ is a Nash equilibrium, and the latter simply states the common knowledge of the existence of a Nash equilibrium. The latter could be obtained if the "complete" logical (mathematical) abilities as well as appropriate real number axioms are common knowledge. To play a game, however, the former is required, while it may not be derived from the common knowledge of real number axioms.

Here we meet the constructionist aspect. The distinction between (1.2) and (1.3) is substantiated by the *term-existence theorem* proved in Kaneko-Nagashima [13], rather than just giving different interpretations to them. It requires that to have (1.2), the existence of such a strategy profile should be obtained by constructing it with permissible computational units together with some operations, for example, zero $\mathbf{0}$, one $\mathbf{1}$, and the four arithmetic operations $+, -, \cdot, /$. Thus, we do not directly adopt constructionist mathematics, but our distinction emerge as a consequence.[3]

Mathematical and game theoretical axioms are assumed in addition to the logical axioms of game logic GL_ω. Throughout this paper, we adopt at least the ordered field language having $\mathbf{0}, \mathbf{1}$ as well as the four arithmetic operations $+, -, \cdot, /$, and the *ordered field axioms* Φ_{of}. The ordered field axioms Φ_{of} may not be enough to prove the existence

[3]There are various constructionist schools: they have developed various constructive mathematical systems, cf., Beeson [1] and van Dalen [28]. In game logic GL_ω, classical logic is assumed for each player as well as the investigator. The constructive aspect emerges only as a consequence of the interactions of the knowledge and inferences of the players and that of the investigator. Also, the constructiveness in this paper may be accepted by these constructive mathematics schools, since we do not go to more than the real closed field theory.

of a Nash equilibrium. For general existence, we need some additional continuity axiom. One choice of such an axiom is the *real-closedness axiom* (schema), and we obtain the *real closed field axioms* Φ_{rcf} as the union of Φ_{of} and the real closedness axiom. The real closed field theory enjoys the special property: it is complete, which is known as Tarski's *completeness theorem*. Using Tarski's completeness theorem and Nash's [20] existence theorem, we have

$$\Phi_{\mathrm{rcf}} \vdash_0 \exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}), \tag{1.4}$$

i.e., the existence of a Nash equilibrium is derived as a theorem from the real closed fields axioms Φ_{rcf}, where $\vdash_0$ denotes the provability relation of classical logic. It follows from (1.4) that in game logic GL_ω,

$$C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(\exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x})), \tag{1.5}$$

where $\vdash_\omega$ is the provability relation of GL_ω. That is, when all the axioms Φ_{rcf} are common knowledge, the existence of a Nash equilibrium is also common knowledge.

On the other hand, $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ may not be provable even if the real closed field axioms Φ_{rcf} are common knowledge. Kaneko-Nagashima [13] and [13] showed that in some 3-person game g with a *unique* Nash equilibrium,

$$\text{neither } C(\Phi_{\mathrm{rcf}}) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x})) \text{ nor } C(\Phi_{\mathrm{rcf}}) \vdash_\omega \neg\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x})). \tag{1.6}$$

The first states that the specific existence $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ is not obtained, and the second that the negation of this existence is not obtained either. Hence any player cannot have a definite, positive or negative, conclusion, and continues looking for an equilibrium forever. In fact, the essential part of (1.6) is the first statement in the sense that the second can never hold and involves no deep considerations. Hence we will focus on the first in the remaining of this introduction.

The term-existence theorem of [13] states that the assertion, $C(\Phi_{\mathrm{rcf}}) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$, is equivalent to

$$C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})) \text{ for some profile } \overrightarrow{t} \text{ of closed terms.} \tag{1.7}$$

Here the *closed terms* are expressions constructed by basic constant symbols in addition to $0, 1$ and the four arithmetic operations $+, -, \cdot, /$ in the formalized language together with appropriate axioms. The closed terms constitute *permissible computational units* in our formalized theory. We can categorize the existence proofs into two classes: general proofs allowing abstract axioms and constructive proofs based only on the permissible computational units and four arithmetic operations. By (1.7), we need to have a constructive proof of a Nash equilibrium for the specific existence $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$.

In the *pure* ordered field language allowing only $0, 1$ and $+, -, \cdot, /$, the closed terms express only rational numbers. Hence if a Nash equilibrium involves irrational numbers, (1.7) would not hold. For 3-person games, irrational numbers are typically involved – one example is given by Kaneko-Nagashima [13] where $\sqrt{51}$ is involved in the equilibrium strategies. However, if $\sqrt{51}$ is added to the language as a permissible computational unit, then the undecidability result, (1.6), is removed. Thus (1.6) depends upon the choice of a language, in other words, the constructive proofs are dependent upon the choice of permissible computational units. It is known that the abstract proofs are independent of computational units in the above sense.

The fact that the addition of a symbol removes undecidability (1.6) does not necessarily mar its point, since the players living in the fixed language cannot find the necessity of introducing new computational units. Nevertheless, we can ask the problems of how robust the above undecidability is and how we can avoid it. Specifically, the problem is how and how much such computational units should be introduced together with appropriate mathematical axioms in order to have $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ for some and/or all games. In this paper, we explore these problems in details.

In Section 2, we formulate game logic GL_ω and present basic theorems necessary in this paper. In Section 3, we review briefly the epistemic axiomatization of individual *ex ante* decision making. Section 4 is the first main section. There we start wit the pure ordered field language. Then we show that (the common knowledge of) any axioms additional to the ordered field axioms Φ_{of} are superfluous for obtaining $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$. It means that in this choice of a language, the ordered field axioms Φ_{of} should be sufficient for $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ if it is ever provable. It follows from Lemke-Howson [16] that for any 2-person game g, $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ is obtained from the common knowledge of the ordered field axioms. This does not work, however, for a game with more than two players. The reason for it is, as stated above, that a Nash equilibrium may involve irrational number probabilities for games with more than two players.

Undecidability (1.6) is removed by adding a symbol together with an appropriate axiom to describe $\sqrt{51}$ to our theory. Then we would ask whether or not the introduction of radical expressions is sufficient to remove the undecidability for any game. In fact, this question is answered negatively by the Abel-Galois theorem on the unsolvability of a polynomial equation of degree 5 together with Bubelis' [2] result. This consideration does not directly lead to our final conclusion, but is important in the consideration of the distinction of the abstract and constructive existence proofs. In Section 5, we will reflect on the entire problem of decision making involving a real number theory.

In Section 6, we extend the language so that all real algebraic numbers are expressed by constant symbols. Then we show that $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ is provable from the common knowledge of the real closed field axioms as well as of additional axioms determining real algebraic numbers. In this extension, we prove that $\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ holds for any finite game. In this extension, each player can compute a Nash equilibrium and generate the probabilities prescribed by the equilibrium. Hence in this extension, every game is playable as far as it is solvable. We will discuss also applications of this extension to other game theoretical concepts.

In Section 7, we give another extension of the ordered field theory, which is weaker than the extension in Section 6 and is sufficient for our specific purpose of the specific knowledge of the existence of a Nash equilibrium. This extension can be regarded as purely constructive, but may not be sufficient for some other game theoretical concepts.

A final remark should be given on our mathematical method. We will discuss our game theoretical problems in a formalized language as well as will use sometimes model theoretic methods and nonformalized mathematics. Our game theoretical objects are described in the formalized mathematics, but the formalized mathematics is not a formalization of game theoretical objects in nonformalized mathematics. In this sense, model theoretic methods as well as nonformalized mathematics are understood as metamathematics for investigations in game logic GL_ω.

2. Logics GL_0 and GL_ω

The language of an ordered field theory suffices for classical game theory, but some extension is needed to discuss epistemic aspects. First, we give finitary and infinitary languages. Then we give base logic GL_0 which is classical predicate logic with no epistemic axioms, and we formulate game logic GL_ω which is an infinitary predicate extension of propositional epistemic KD4 and in which we discuss our epistemic axiomatization of final decisions. We provide some relevant results without proofs.

2.1. Languages $\mathcal{P}^f(\mathcal{L})$ and $\mathcal{P}^\omega(\mathcal{L})$

The following is the list $\mathcal{L}_{of}$ of basic symbols:

Logic Symbols:

Free variables: $a_0, a_1, ...$; *Bound variables*: $x_0, x_1, ...$;

Logical connectives: $\neg$ (*not*), $\supset$ (*implies*), $\bigwedge$ (*and*), $\bigvee$ (*or*), $\forall$ (*for all*), $\exists$ (*exists*);

Parentheses: (,);

Ordered Field Language Symbols:

Constants: 0, 1; *Binary functions*: $+$, $-$, $\cdot$, $/$;

Binary predicates: $\geq$, $=$;

Game-Epistemic Symbols:

$\sum_{i=1}^{n} \ell_i$ *-ary predicates*: $D_1, ..., D_n$;

Knowledge operators: $K_1, ..., K_n$.

We denote the above list of symbols by $\mathcal{L}_{of}$. We denote, by $\mathcal{L}$, a list of symbols obtained from $\mathcal{L}_{of}$ by adding some constant and function symbols. In Sections 5, 6 and 7, we will give specific extensions of $\mathcal{L}_{of}$.

The indices $1, ..., n$ are the names of players. Each player i has ℓ_i pure strategies ($\ell_i \geq 1$). The intent of $D_i(\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$ is that player i predicts that each player j chooses a mixed strategy $\overrightarrow{t}_j = (t_{j1}, ..., t_{j\ell_j})$ as his final decision, where $\overrightarrow{t}_j = (t_{j1}, ..., t_{j\ell_j})$ is a ℓ_j-probability vector. Mathematically, they will defined presently. By the expression $K_i(A)$, we mean that player i knows formula A.

In this paper, the concept of terms – function expressions – plays a crucial role as *computational units* as mentioned in Section 1. We define the *terms* by the standard finite induction:

(T0): each free variable a is a term and all constants are also terms;

(T1): if f is a k–ary function symbol and if $t_1, ..., t_k$ are terms, then $f(t_1, ..., t_k)$ is also a term.[4]

A term having no free variable is said to be a *closed term*. The space of terms depends upon the list of primitive symbols. Particularly, we mean, by *the terms in* $\mathcal{L}_{of}$, the terms generated by T0 and T1 based on constant $0, 1$ and four function symbols $+, -, \cdot, /$.

For any terms, t_1, t_2 and vectors of terms, $\overrightarrow{t}_1 = (t_{11}, ..., t_{1\ell_1}), ..., \overrightarrow{t}_n = (t_{n1}, ..., t_{n\ell_n})$, the expressions $t_1 \geq t_2, t_1 = t_2$ and $D_i(\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$ $(i = 1, ..., n)$ are called *atomic formulae.*

Let $\mathcal{P}^0(\mathcal{L})$ be the set of all formulae generated by the standard finitary inductive definition with respect to $\neg, \supset, \forall, \exists$ and $K_1, ..., K_n$ from the atomic formulae, i.e.,

(F0) any atomic formula is in $\mathcal{P}^0(\mathcal{L})$;

(F1) if A and B are in $\mathcal{P}^0(\mathcal{L})$, so are $\neg A, A \supset B, \forall x A(x), \exists x A(x)$ and $K_i(A)$;

where $A(x)$ is obtained from A by substituting x for all the occurrences of some variable a in A. Suppose that $\mathcal{P}^k(\mathcal{L})$ is already defined ($k = 0, 1, ...$). We call a nonempty countable subset Φ of $\mathcal{P}^k(\mathcal{L})$ an *allowable set* iff it contains a finite number of free variables. For an allowable set Φ, the expressions $(\bigwedge \Phi)$ and $(\bigvee \Phi)$ are considered here. From the union $\mathcal{P}^k(\mathcal{L}) \cup \{(\bigwedge \Phi), (\bigvee \Phi) : \Phi$ is an allowable set in $\mathcal{P}^k(\mathcal{L})\}$, we obtain the space $\mathcal{P}^{k+1}$ of formulae by the standard finitary inductive definition with respect to $\neg, \supset, \forall, \exists$ and $K_1, ..., K_n$. We denote $\bigcup_{k<\omega} \mathcal{P}^k(\mathcal{L})$ by $\mathcal{P}^\omega(\mathcal{L})$.[5,6] A formula having

[4] and (T2): the terms generated by finite numbers of applications of T0 and T1 are only terms. In the following inductive definitions, we skip this third statement.

[5] Note that $\bigwedge \Phi$ and $\bigvee \Phi$ may not be in $\mathcal{P}^\omega(\mathcal{L})$ for some countable subsets Φ of $\mathcal{P}^\omega(\mathcal{L})$. For our purpose, however, this does not matter and the space $\mathcal{P}^\omega(\mathcal{L})$ is large enough.

[6] This space is already uncountable. Some smaller, countable, space of formulae suffices for our purpose. For example, a countable and constructive space of formulae is provided in Kaneko-Nagashima [10]. We adopt the above space for presentational simplicity.

no free variable is called *closed*. We abbreviate $\bigwedge\{A,B\}, \bigvee\{A,B\}, \bigwedge\{A_1,...,A_m\}$ as $A\bigwedge B, A\bigvee B$, $\bigwedge_{k=1}^{m} A_k$, etc.

We say that a formula A is *finitary* iff it contains no infinitary conjunctions and no infinitary disjunctions. We denote the set of all finitary formulae including no D_i and K_i $(i = 1,...n)$ by $\mathcal{P}^f(\mathcal{L})$. The space $\mathcal{P}^f(\mathcal{L})$ is closed with respect to $\neg, \supset, \forall, \exists$ and finitary $\bigwedge, \bigvee$. When $\mathcal{L}$ is specified to be $\mathcal{L}_{of}$, the set of formulae $\mathcal{P}^f(\mathcal{L}_{of})$ is called the *pure ordered field language.*

The primary reason for the infinitary language is to express common knowledge explicitly as a conjunctive formula. The common knowledge of a formula A is defined as follows: For any $m \geq 0$, we denote the set $\{K_{i_1}K_{i_2}...K_{i_m}$: each K_{i_t} is one of $K_1,...,K_n$ and $i_t \neq i_{t+1}$ for all $t = 1,...,m-1\}$ by $\mathsf{K}(m)$. When $m = 0$, $K_{i_1}K_{i_2}...K_{i_m}$ is interpreted as the null symbol. We define the common knowledge of A by

$$\bigwedge\{K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\},$$

which we denote by $C(A)$. If A is in $\mathcal{P}^m(\mathcal{L})$, then $C(A)$ is in $\mathcal{P}^{m+1}(\mathcal{L})$. Hence the space $\mathcal{P}^\omega(\mathcal{L})$ is closed with respect to the operation $C(\cdot)$.

2.2. Classical Logic GL_0

We define logic GL_0 by the following seven axiom schemata and five inference rules. In GL_0, all formulae occurring in these axioms and inferences are assumed to be in $\mathcal{P}^f(\mathcal{L})$. For any formulae A, B, C, finite set of formulae Φ, and term t,

(L1): $A \supset (B \supset A)$;

(L2): $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

(L3): $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;

(L4): $\bigwedge \Phi \supset A$, where $A \in \Phi$;

(L5): $A \supset \bigvee \Phi$, where $A \in \Phi$;

(L6): $\forall x A(x) \supset A(t)$;

(L7): $A(t) \supset \exists x A(x)$;

$$\frac{A \supset B \quad A}{B} \text{ (MP)}$$

$$\frac{\{A \supset B : B \in \Phi\}}{A \supset \bigwedge \Phi} (\bigwedge\text{-Rule}) \qquad \frac{\{A \supset B : A \in \Phi\}}{\bigvee \Phi \supset B} (\bigvee\text{-Rule})$$

$$\frac{A \supset B(a)}{A \supset \forall x B(x)} (\forall\text{-Rule}) \qquad \frac{A(a) \supset B}{\exists x A(x) \supset B} (\exists\text{-Rule}),$$

where the free variable a must not occur in $A \supset \forall x B(x)$ of $\forall$-Rule and $\exists x A(x) \supset B$ of $\exists$-Rule.

A *proof* in GL_0 is a finite tree with the following properties:

(i): a formula in $\mathcal{P}^f(\mathcal{L})$ is associated with each node, and the formula associated with each leaf is an instance of L1 – L7;

(ii) adjoining nodes together with their associated formulae form an instance of the above inferences.

For a formula A in $\mathcal{P}^f(\mathcal{L})$, we denote by $\vdash_0 A$ iff there is a proof in GL_0 such that A is associated with the root of the proof. For any subset Γ of $\mathcal{P}^f(\mathcal{L})$, we write $\Gamma \vdash_0 A$ iff $\vdash_0 \bigwedge \Phi \supset A$ for some finite subset Φ of Γ. We will use some abbreviations, e.g., $\Gamma \cup \Theta \vdash_0 A$ and $\Gamma \cup \{B\} \vdash_0 A$ are denoted by $\Gamma, \Theta \vdash_0 A$ and $\Gamma, B \vdash_0 A$.

Logic GL_0 is classical predicate logic, and it is *sound and complete* with respect to the standard interpretations. That is, all *valid* formulae in this sense are provable, and *vice versa* (cf., Mendelson [18], Chapter 2). We will use, later on, the soundness theorem: if A is provable in a theory $(\mathcal{P}^f(\mathcal{L}), \Phi)$, then A is *true* in any model of $(\mathcal{P}^f(\mathcal{L}), \Phi)$.

2.3. Game Logic GL_ω: Players' Logical and Introspective Abilities

In game logic GL_ω, we allow logical axioms L1 – L7 and inference rules MP – $\exists$-Rule in the language $\mathcal{P}^\omega(\mathcal{L})$, and add the following axioms and inference rule in $\mathcal{P}^\omega(\mathcal{L})$: for any formulae A, B, allowable set Φ and $i = 1, ..., n$;

(MP$_i$): $K_i(A \supset B) \bigwedge K_i(A) \supset K_i(B)$;

($\bot_i$): $\neg K_i(\neg A \bigwedge A)$;

(PI$_i$): $K_i(A) \supset K_i K_i(A)$;

($\bigwedge$-B$_i$): $\bigwedge K_i(\Phi) \supset K_i(\bigwedge \Phi)$, where $K_i(\Phi) = \{K_i(A) : A \in \Phi\}$;

and

(Necessitation): $$\frac{A}{K_i(A)}.$$

In GL_ω, a *proof* is defined to be a countable tree satisfying:

(i): every path in the tree is finite;

(ii): a formula in $\mathcal{P}^\omega(\mathcal{L})$ is associated with each node, and the formula associated with each leaf is an instance of L1 – L7 and the additional four axioms MP$_i$ to $\bigwedge$-B in $\mathcal{P}^\omega(\mathcal{L})$;

(iii) adjoining nodes together with their associated formulae form an instance of MP – $\exists$-Rule and Nec.

We write $\vdash_\omega A$ iff there is a proof in GL_ω such that A is associated with the root of the proof, and also write $\Gamma \vdash_\omega A$ iff $\vdash_\omega \bigwedge \Phi \supset A$ for some finite subset Φ of Γ.

Since logical axioms MP_i, $\bot_i$ and PI_i are called K, D and 4 in the literature of modal logic, GL_ω is an infinitary predicate extension together with axiom $\bigwedge$-B of propositional epistemic logic KD4.[7,8]

Logic GL_0 may be regarded as a description of the logical ability of the *investigator.* In logic GL_ω, Nec and MP_i gives each player the infinitary extension of the complete logical ability of the investigator, and $\bot_i$ requires the knowledge of each player based his basic knowledge to be consistent in GL_ω. Axiom PI_i, called the *Positive Introspection,* gives each player the introspective ability in the sense that he knows A, he knows that he knows A. Moreover, these are assumed to be common knowledge. Nevertheless, it is important to note that in the statement $\vdash_\omega \bigwedge \Phi \supset A$, the game theoretical axioms which are described in Φ are not necessarily common knowledge, while whole statement $\bigwedge \Phi \supset A$ is common knowledge. See Kaneko-Nagashima [13] for more details.

Axiom $\bigwedge$-B is called the *Barcan axiom.* When Φ is finite, this is derived from other axioms, but not for infinite Φ. For the development of our framework, $\bigwedge$-B will be used to derive the fixed-point property:

$$C(A) \supset K_iC(A) \quad \text{for } i = 1, ..., n. \tag{2.1}$$

That is, if A is common knowledge, then each player i knows that it is common knowledge, which plays an important role in the epistemic axiomatization of final decisions in Section 3.[9]

The following theorem is a corollary of Propositions 3.4 and 4.1 of Kaneko-Nagashima [13]. In th following, we denote the set $\{C(B) : B \in \Gamma\}$ by $C(\Gamma)$.

Theorem 2.1. Let Γ be a set of closed formulae in $\mathcal{P}^f(\mathcal{L})$, and A a closed formula in $\mathcal{P}^f(\mathcal{L})$. Then $C(\Gamma) \vdash_\omega C(A)$ if and only if $\Gamma \vdash_0 A$.

The next theorem was proved in Kaneko-Nagashima [13], which will play a crucial role in this paper. As discussed in Section 1, the theorem requires that existence should be obtained by constructing objects in terms of permissible computational units.

Theorem 2.2 (Term-Existence for GL_ω). Let Γ be a set of closed formulae in $\mathcal{P}^f(\mathcal{L})$, and $\exists x_1 ... \exists x_k A(x_1, ..., x_k)$ a closed formula in $\mathcal{P}^f(\mathcal{L})$. If $C(\Gamma) \vdash_\omega \exists x_1 ... \exists x_k C(A(x_1, ..., x_k))$, there are closed terms $t_1, ..., t_k$ such that $C(\Gamma) \vdash_\omega C(A(t_1, ..., t_k))$.

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[7]The logic obtained by the replacement of $\bot_i$ by T: $K_i(A) \supset A$ is called S4. In S4, the truth of knowledge can be defined relative to the investigator, while in KD4, knowledge is required only to be consistent within each player. In this sense, KD4 is a system of cognitive relativism, and knowledge in KD4 is often called belief in the literature of epistemic logic.

[8]Game logic GL_ω can be regarded as a modal logic variation of a fragment of infinitary logic $L_{\omega_1\omega}$ (cf., Karp [14] and Keisler [15]). As a space of formulae, $\mathcal{P}^\omega(\mathcal{L})$, is much smaller than the space of formulae in $L_{\omega_1\omega}$. Since our primary purpose of the infinitary extension is to express common knowledge explicitly as a formula, the present extension suffices for our purpose

[9]In Kaneko-Nagashima [12] and [13], the $\forall$-Barcan axiom, $\forall x K_i(A(x)) \supset K_i(\forall x A(x))$, is also assumed for GL_ω. The results we use in this paper are independent of $\forall$-Barcan.

It is important to notice that this is a general term-existence theorem for game logic GL_ω, instead of for a particular theory (i.e., a combination of a logic and a set of nonlogical axioms), while Theorems 6.3 and 7.2 to be given later are term-existence for particular theories. Theorem 2.2 gives a general requirement, while the others show the capabilities of those theories.[10]

Theorem 2.2 implies that the assertion $C(\Gamma) \vdash_\omega \exists x_1 ... \exists x_k C(A(x_1, ..., x_k))$ is potentially different from $C(\Gamma) \vdash_\omega C(\exists x_1 ... \exists x_k A(x_1, ..., x_k))$. On the other hand, if we put the negation symbol $\neg$ to these conclusions, they become equivalent (see Kaneko-Nagashima [12]):

Lemma 2.3. Let Γ be a set of closed formulae in $\mathcal{P}^{\mathrm{f}}(\mathcal{L})$, and $\exists x_1 ... \exists x_k A(x_1, ..., x_k)$ a closed formula in $\mathcal{P}^{\mathrm{f}}(\mathcal{L})$. Then $C(\Gamma) \vdash_\omega \neg \exists x_1 ... \exists x_k C(A(x_1, ..., x_k))$ if and only if $C(\Gamma) \vdash_\omega C(\neg \exists x_1 ... \exists x_k A(x_1, ..., x_k))$.

2.4. Axioms for Ordered Fields and for Real Closed Fields

Here we give the axioms for ordered fields and also for real closed fields.

We start with the equality axioms: In the following axioms, x, y, z are bound variables $\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2$.

(Reflexivity): $\forall x\, (x = x)$;

(Substitutivity): $\forall x \forall y \forall z\, (x = z \supset f(x, y) = f(z, y))$;
$\forall x \forall y \forall z\, (y = z \supset f(x, y) = f(x, z))$;
$\forall x \forall y \forall z\, (x = z \supset (P(x, y) \supset P(z, y))$; and
$\forall x \forall y \forall z\, (y = z \supset (P(x, y) \supset P(x, z)))$,

where f is one of the function symbols $+, -, \cdot, /$ and P is either $=$ or $\geq$.

The following are the main part of the ordered field axioms:

(Commutativity): $\forall x \forall y (x + y = y + x)$; $\forall x \forall y (x \cdot y = y \cdot x)$;

(Associativity): $\forall x \forall y \forall z\, ((x + y) + z = x + (y + z))$; $\forall x \forall y \forall z\, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$;

(Inverse Operations): $\forall x \forall y\, (x + (y - x) = y)$; $\forall x \forall y\, (\neg(x = 0) \supset x(y/x) = y)$;

(Distributive Law): $\forall x \forall y \forall z\, (x \cdot (y + z) = x \cdot y + x \cdot z)$;

(Unit Elements): $\forall x (x + 0 = x)$; $\forall x (x \cdot 1 = x)$;

(Distinctive Elements): $\neg(0 = 1)$;

(0-Denominator): $\forall x (x/0 = 0)$;[11]

[10] Several forms of term-existence theorems have been known for some logics and some theories. For example, intuitionistic logic permits term-existence in some form, and some theories in intuitionisitc logic enjoys term-existence. See van Dalen [28].

[11] It suffices to put some fixed number in the right-hand side.

(Reflexivity): $\forall x(x \geq x)$;
(Antisymmetry): $\forall x \forall y ((x \geq y) \bigwedge (y \geq x) \supset x = y)$;
(Transitivity): $\forall x \forall y \forall z ((x \geq y) \bigwedge (y \geq z) \supset x \geq z)$;
(Totality): $\forall x \forall y ((x \geq y) \bigvee (y \geq x))$;
(Structure Preserving): $\forall x \forall y \forall z (x \geq z \supset x + z \geq y + z)$;
$\forall x \forall y \forall z ((x \geq y) \bigwedge (z \geq \mathbf{0}) \supset x \cdot z \geq y \cdot z)$.

We denote the set of above axioms by Φ_{of}. The pair $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}}), \Phi_{\mathrm{of}})$ is called the *pure ordered field theory.*[12]

Under the ordered field axioms, integers and rational numbers are expressed as terms. We define *numerals* as follows: $[0]$ is $\mathbf{0}$, $[m+1]$ is $[m] + \mathbf{1}$ for any nonnegative integer m, and $[m]$ is $\mathbf{0} - [-m]$ for a negative integer m. For a rational number $q = k/m$ (k/m are irreducible and $m > 1$), we define $[q]$ to be $[k]/[m]$. Numerals are closed terms.

The following lemma is crucial in this paper, which will be proved in the Appendix.

Lemma 2.4.(1): For any closed term t in $\mathcal{L}_{\mathrm{of}}$, there is a rational numeral $[k]/[m]$ such that $\Phi_{\mathrm{of}} \vdash_0 t = [m]/[k]$.

(2): either $\Phi_{\mathrm{of}} \vdash_0 t_1 \geq t_2$ or $\Phi_{\mathrm{of}} \vdash_0 \neg(t_1 \geq t_2)$ for any closed terms t_1 and t_2 in $\mathcal{L}_{\mathrm{of}}$.

This lemma implies that the set of closed terms in the pure ordered field theory is regarded as corresponding to the set of rational numbers in the non-formalized mathematics. We should note that if r_1, r_2 are the rational numbers corresponding to closed terms t_1, t_2 in $\mathcal{L}_{\mathrm{of}}$, then

$$\Phi_{\mathrm{of}} \vdash_0 t_1 \geq t_2 \text{ if and only if } r_1 \geq r_2, \tag{2.2}$$

where the right-hand side is evaluated in the ordered field Q of rational numbers.

We denote $\neg(t_1 \geq t_2) \bigwedge (t_2 \geq t_1)$ by $t_2 > t_1$. Since $\Phi_{\mathrm{of}} \vdash_0 (t_1 \geq t_2) \bigvee (t_2 \geq t_1)$ by Totality, Lemma 2.4.(2) is restated as $\Phi_{\mathrm{of}} \vdash_0 t_1 > t_2$, $\Phi_{\mathrm{of}} \vdash_0 t_2 > t_1$ or $\Phi_{\mathrm{of}} \vdash_0 t_1 = t_2$.

Finally, we add the following axioms to have the real closed field theory which is one possible choice of a real number theory:

(Real-Closedness): $\forall x \exists y (x \geq \mathbf{0} \supset (y^2 = x))$; and
for any odd natural number m,
$\forall y_{m-1} ... \forall y_0 \exists x (x^m + y_{m-1} x^{m-1} + ... + y_1 x + y_0 = \mathbf{0})$.

We denote the union of Φ_{of} and the set of the formulae of Real-Closedness by Φ_{rcf}. Note that Φ_{rcf} is an infinite set, while Φ_{of} is a finite set. The theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}}), \Phi_{\mathrm{rcf}})$ is called the *real closed field theory.* An important theorem on the real closed field theory is known as Tarski's theorem (cf., Rabin [23] and van den Dries [29] for recent survey):

[12] We use, without references, elementary propositions, such as $\neg(\mathbf{0} \geq \mathbf{1})$, derivable from these axioms.

Tarski's Completeness Theorem. For any closed A in $\mathcal{P}^{f}(\mathcal{L}_{of})$, either $\Phi_{rcf} \vdash_0 A$ or $\Phi_{rcf} \vdash_0 \neg A$.

Here, we should mention some model theoretic facts, which will be used later. Let Q the set of all *rational* numbers, A the set of all *real algebraic* numbers, and R the set of all *real* numbers. Then $(\mathsf{Q}; +, -, \times, / ; \geq), (\mathsf{A}; +, -, \times, /; \geq)$ and $(\mathsf{R}; +, -, \times, / ; \geq)$ are all ordered fields. Here we use the *same* symbols $+, -, \times, /$, $\geq$ to denote their *interpretations*, but would not cause any confusions. The first is a subfield of the second, and the second is a subfield of the third. By interpreting 0 and 1 to be 0 and 1, $(\mathsf{Q}; +, -, \times, / ; \geq; 0, 1)$, $(\mathsf{A}; +, -, \times, /; \geq)$ and $(\mathsf{R}; +, -, \times, /; \geq)$ are models of the pure ordered field theory $(\mathcal{P}^{f}(\mathcal{L}_{of}), \Phi_{of})$. Also, $(\mathsf{A}; +, -, \times, / ; \geq; 0, 1)$ and $(\mathsf{R}; +, -, \times, /; \geq)$ are models of the real closed field theory $(\mathcal{P}^{f}(\mathcal{L}_{of}), \Phi_{rcf})$ (cf.,van der Waerden[30], Section70).[13]

3. Games and Individual Decision Making

In this section, we provide a formalization of classical game theory in our language, and review the epistemic axiomatization of final decisions given in Kaneko-Nagashima [10] and Kaneko [6].

3.1. Finite Games and Nash Equilibrium

Consider an *n-person finite* game g. Recall that each player has ℓ_i pure strategies. The *payoff function* of player i is a function from the set of *pure strategy profiles* to the set of rational numbers. That is, for each pure strategy profile $(s_1, ..., s_n)$, the payoff to player i is given as a rational number $\mathrm{g}_i(s_1, ..., s_n)$.

Now we describe these payoff functions in our formalized language: the payoff to player i from each strategy profile $(s_1, ..., s_n)$ is given as the rational numeral $[\mathrm{g}_i(s_1, ..., s_n)]$. Pure strategies $s_1, ..., s_n$ are not directly formalized in this paper. The game g itself is not formalized either in this paper but is treated as a part of the description of Nash equilibrium. In fact, this treatment is rather for simplicity. To discuss the knowledge on the game, it would be better to formalize the game explicitly, but is a side problem in this paper. It can be done without much difficulty, cf., Kaneko [6].

A *mixed strategy* for player i is formulated as an attribute of an ℓ_i-vector of terms $\overrightarrow{t}_i = (t_{i1}, ..., t_{i\ell_i})$ satisfying the following formula:

$$(\sum_{k=1}^{\ell_i} t_{ik} = 1) \wedge (\bigwedge\{t_{ik} \geq 0 : k = 1, ..., \ell_i\}), \tag{3.1}$$

[13] In these models, equality = is interpreted as the standard identity, i.e., these are *normal* models. Hence we do not include = in these models.

which we denote by $\mathrm{St}_i(\overrightarrow{t}_i)$.[14] Next, the payoff to player i from a *mixed strategy profile* $\overrightarrow{t} = (\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$ is given as the expected payoff with respect to the probability distribution over the pure strategy combinations $(s_1, ..., s_n)$ induced by $\overrightarrow{t}$:

$$\sum_{k_1} \cdots \sum_{k_n} t_{1k_1} \cdot \ \dots \ \cdot t_{nk_n} \cdot [\mathrm{g}_i(s_{k_1}, ..., s_{k_n})], \tag{3.2}$$

which we denote by $g_i(\overrightarrow{t})$. Note that this $g_i(\overrightarrow{t})$ is a term. In the following, we denote $(\overrightarrow{t}_1, ..., \overrightarrow{t}_{i-1}, \overrightarrow{t}_{i+1}, ..., \overrightarrow{t}_n)$ by $\overrightarrow{t}_{-i}$, and $(\overrightarrow{t}_i, \overrightarrow{t}_{-i})$ means $\overrightarrow{t}$ itself.

A *Nash equilibrium* is defined to be a mixed strategy profile $\overrightarrow{t} = (\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$ satisfying the following formula:

$$\bigwedge\{\mathrm{St}_i(\overrightarrow{t}_i) \wedge \forall \overrightarrow{x}_i \left(\mathrm{St}_i(\overrightarrow{x}_i) \supset g_i(\overrightarrow{t}) \geq g_i(\overrightarrow{x}_i, \overrightarrow{t}_{-i})\right) : i = 1, ..., n\}, \tag{3.3}$$

where $\forall \overrightarrow{x}_i A(\overrightarrow{x}_i)$ means $\forall x_{i1} ... \forall x_{i\ell} A(x_{i1}, ..., x_{i\ell})$ and, later on, $\exists \overrightarrow{x}_i A(\overrightarrow{x}_i)$ will be used to denote $\exists x_{i1} ... \exists x_{i\ell} A(x_{i1}, ..., x_{i\ell})$. We denote the formula of (3.3) by $\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})$. Note that the formula $\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})$ is relative to a specific game g.

The game of Table 3.1 is the "Prisoner's dilemma" and has a unique Nash equilibrium $((0,1),(0,1))$ in the formal language.

	N	C
N	5, 5	1, 6
C	6, 1	3, 3

Table 3.1

	B	M
B	2, 1	0, 0
M	0, 0	1, 2

Table 3.2

The game of Table 3.2, called "the Battle of Sexes", has three equilibria, $(B, B), (M, M)$ and $((2/3, 1/3), (1/3, 2/3))$, and their formal counterparts are $((1,0),(1,0))$, $((0,1),(0,1))$ and $(([2/3],[1/3]),([1/3],[2/3]))$.

The following lemma is a well-known criterion for a Nash equilibrium, which eliminates the universal quantifiers in the formula $\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{a})$. We denote the ℓ_i-vector with 0's except 1 in the k-*th* entry by u_k.

Lemma 3.1. $\Phi_{\mathrm{of}} \vdash_0 \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{a}) \equiv \bigwedge_{i=1}^{n} \left(\mathrm{St}_i(\overrightarrow{a}_i) \wedge \left(\bigwedge_{k \leq \ell} g_i(\overrightarrow{a}) \geq g_i(\mathrm{u}_k, \overrightarrow{a}_{-i})\right)\right)$.

Proof. It suffices to prove that $\Phi_{\mathrm{of}} \vdash_0 \bigwedge_{k \leq \ell_i} g_i(\overrightarrow{a}) \geq g_i(\mathrm{u}_k, \overrightarrow{a}_{-i}) \supset$

[14] $\sum_{k=1}^{\ell} t_{ik}$ is an abbreviation of $(...((a_{i1}+a_{i2})+a_{i3})+...+a_{i\ell_i})...)$. Under the ordered field axioms, the order of summation does not matter, but we do not assume these axioms in the epistemic axiomatization. We use also other abbreviations without noting.

$\forall \overrightarrow{x}_i \left(\mathrm{St}_i(\overrightarrow{x}_i) \supset g_i(\overrightarrow{a}) \geq g_i(\overrightarrow{x}_i, \overrightarrow{a}_{-i})\right)$ for each i. We show that for any terms $t, t_1, ..., t_\ell$

$$\Phi_{\mathrm{of}} \vdash_0 \bigwedge_{k \leq \ell_i} t \geq t_k \supset \left(\sum_{k=1}^{\ell_i} b_j = 1 \wedge (\bigwedge_{k \leq \ell_i} b_k \geq 0) \supset \sum_{k \leq \ell_i} b_k \cdot t_k \leq t\right), \tag{3.4}$$

where $b_1, ..., b_\ell$ are free variables not occurring in $t, t_1, ..., t_\ell$. This together with $\forall$-Rule implies $\Phi_{\mathrm{of}} \vdash_0 \bigwedge_{k \leq \ell_i} t \geq t_k \supset \forall \overrightarrow{x}_i \left(\mathrm{St}_i(\overrightarrow{x}_i) \supset \sum_{k \leq \ell_i} x_{ik} \cdot t_k \leq t\right)$. From this, we obtain what we have to prove, by substituting $g_i(\overrightarrow{a})$ and $g_i(\mathrm{u}_k, \overrightarrow{a}_{-i})$ for t and t_k $(k = 1, ..., \ell)$.

Assertion (3.4) is equivalent to

$$\Phi_{\mathrm{of}} \vdash_0 (\bigwedge_{k \leq \ell_i} t \geq t_k) \wedge \sum_{k=1}^{\ell_i} b_j = 1 \wedge (\bigwedge_{k \leq \ell_i} b_k \geq 0) \supset \sum_{k \leq \ell_i} b_k \cdot t_k \leq t. \tag{3.5}$$

This is proved as follows: First, $\Phi_{\mathrm{of}} \vdash_0 \bigwedge_{k \leq \ell_i} (t \geq t_k \wedge b_k \geq 0) \supset \sum_{k \leq \ell_i} b_k \cdot t_k \leq \sum_{k \leq \ell_i} b_k \cdot t$, and second, $\Phi_{\mathrm{of}} \vdash_0 \sum_{k=1}^{\ell_i} b_j = 1 \supset \sum_{k \leq \ell_i} b_k \cdot t = t$. These imply (3.5). □

The above lemma states that the Nash equilibrium property can be verified by checking a finite number of equalities and inequalities. By Lemma 2.4, when we plug closed terms in $\mathcal{L}_{\mathrm{of}}$ to free variables in the right formula of Lemma 3.1, the right formula is decidable. Hence $\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})$ is decidable, too.

Lemma 3.2. Suppose that $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}), \Phi)$ satisfies

$$\Phi \vdash_0 t_1 \geq t_2 \text{ or } \Phi \vdash_0 \neg(t_1 \geq t_2) \text{ for any closed terms } t_1, t_2 \text{ in } \mathcal{L}. \tag{3.6}$$

Then $\Phi \vdash_0 \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})$ or $\Phi \vdash_0 \neg\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})$ for any profiles $\overrightarrow{t}$ of closed terms in $\mathcal{L}$.

Proof. Let $(\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$ be a profile of closed terms. It follows from (3.6) that (i) $\Phi \vdash_0 \mathrm{St}_i(t_i)$ or $\Phi \vdash_0 \neg\mathrm{St}_i(t_i)$ for each t_i; and (ii) $\Phi \vdash_0 g_i(\overrightarrow{t}) \geq g_i(\mathrm{u}_k, \overrightarrow{t}_{-i})$ or $\Phi \vdash_0 \neg\left(g_i(\overrightarrow{t}) \geq g_i(\mathrm{u}_k, \overrightarrow{t}_{-i})\right)$ for each $k \leq \ell_i$. If all of these hold without negation $\neg$, then $\Phi \vdash_0 \bigwedge_{i \leq n} \left(\mathrm{St}_i(\overrightarrow{t}_i) \wedge \left(\bigwedge_{k \leq \ell_i} g_i(\overrightarrow{t}) \geq g_i(\mathrm{u}_k, \overrightarrow{t}_{-i})\right)\right)$; and otherwise, we have $\Phi \vdash_0 \neg \bigwedge_{i \leq n} \left(\mathrm{St}_i(\overrightarrow{t}_i) \wedge \left(\bigwedge_{k \leq \ell} g_i(\overrightarrow{t}) \geq g_i(\mathrm{u}_k, \overrightarrow{t}_{-i})\right)\right)$. Thus it follows from Lemma 3.1 that $\Phi \vdash_0 \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})$ or $\Phi \vdash_0 \neg\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t})$. □

The standard existence proof of a Nash equilibrium for any finite game g with mixed strategies relies upon Brouwer's fixed point theorem (von Neumann [32] and Nash [20]). Since $(\mathrm{R}; +, -, \cdot, /; \geq; 0, 1)$, is a model of $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}}), \Phi_{\mathrm{rcf}})$, the existence of a Nash equilibrium, $\exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x})$, is *true* in this model. Hence it follows from the soundness theorem for $(\mathcal{P}^f(\mathcal{L}_{of}), \Phi_{rcf})$ that it is not the case that $\Phi_{rcf} \vdash_0 \neg\exists \vec{x}\,\mathrm{Nash}_g(\vec{x})$. Thus $\Phi_{rcf} \vdash_0 \exists \vec{x}\,\mathrm{Nash}_g(\vec{x})$ by Tarski's completeness theorem, and then $C(\Phi_{rcf}) \vdash_\omega C(\exists \vec{x}\,\mathrm{Nash}_g(\vec{x}))$ by Theorem 2.1.

Theorem 3.3. Let g be any n-person finite game. Then

(1): $\Phi_{rcf} \vdash_0 \exists \vec{x}\,\mathrm{Nash}_g(\vec{x})$;

(2): $C(\Phi_{rcf}) \vdash_\omega C(\exists \vec{x}\,\mathrm{Nash}_g(\vec{x}))$.

Thus, in logic GL_ω, the existence of a Nash equilibrium is common knowledge when the real closed field axioms are common knowledge. Nevertheless, this is different from $C(\Phi_{rcf}) \vdash_\omega \exists \vec{x}\, C(\mathrm{Nash}_g(\vec{x}))$, which is required to be a necessary condition by the epistemic axiomatization.

3.2. Infinite Regress of the Knowledge of the Final Decision Axioms and its Solution

In a game g, each player deliberates his and the others' strategy choices and may reach a prediction on their final decisions. The expression $D_i(\vec{a})$ intends to mean that player i predicts the strategy profile $\vec{a} = (\vec{a}_1, ..., \vec{a}_n)$ to be chosen by the players. We characterize these predictions $D_i(\vec{a})$ by the following five axioms: for $i = 1, ..., n$,

(D0$_i^o$): $\forall \vec{x}\left(D_i(\vec{x}) \supset \bigwedge_j \mathrm{St}_j(\vec{x}_j)\right)$;

(D1$_i^o$): $\forall \vec{x}\left(D_i(\vec{x}) \supset \forall \vec{y}_i\left(\mathrm{St}_i(\vec{y}_i) \supset g_i(\vec{x}) \geq g_i(\vec{y}_i, \vec{x}_{-i})\right)\right)$;

(D2$_i^o$): $\forall \vec{x}\left(D_i(\vec{x}) \supset \bigwedge_j D_j(\vec{x})\right)$;

(D3$_i^o$): $\forall \vec{x}\left(D_i(\vec{x}) \supset K_i(D_i(\vec{x}))\right)$;

(D4$_i^o$): $\forall \vec{x} \forall \vec{y} \bigwedge_j \left(D_i(\vec{x}) \wedge D_i(\vec{y}) \supset D_i(\vec{x}_j, \vec{y}_{-j})\right)$.

We denote D0$\wedge ... \wedge$D4 by D$_i$(0–4). Each axiom is described as follows.

D0$_i^o$: (*Strategies*): Predictions are mixed strategies;

D1$_i^o$: (*Best Response to Predicted Decisions*): When player i predicts final decisions $\vec{x}_1, ..., \vec{x}_n$ for the players, his own decision $\vec{x}_i$ maximizes his payoff against his prediction $\vec{x}_{-i}$, that is, $\vec{x}_i$ is a best response to $\vec{x}_{-i}$.

D2$_i^o$: (*Identical Predictions*): The other players reach the same predictions as player i's.

D3$_i^o$: (*Knowledge of Predictions*): Player i knows his own predictions.

D4$_i^o$: (*Interchangeability*): Player i's predictions are interchangeable, which is a requirement for independent decision making.

We assume that player i himself knows these axioms as his *behavioral postulate*. Thus, the axiom for him is $(\mathrm{D0}_i^o \wedge \cdots \wedge \mathrm{D4}_i^o) \wedge K_i(\mathrm{D0}_i^o \wedge \cdots \wedge \mathrm{D4}_i^o)$, which we denote by $\mathrm{D}_i(0\text{–}4)$. We denote $\bigwedge_i \mathrm{D}_i(0\text{–}4)$ by D(0–4).

In fact, Axiom D(0–4) is far from being sufficient to determine $D_1(\cdot), ..., D_n(\cdot)$. Each $\mathrm{D}_i(0\text{–}4)$ requires player i to know the axioms, $\bigwedge_{j \neq i} \mathrm{D}_j(0\text{–}4)$, for the other players, since $\mathrm{D2}_i^o$ includes the other $D_j(\cdot), j \neq i$, that is, without knowing these axioms, $\mathrm{D2}_i^o$ could not make sense for player i. Thus, we assume that all the players know D(0–4), i.e., $\bigwedge_i K_i(\mathrm{D}(0\text{–}4))$. However, this addition does not solve the insufficiency: under this addition, we have

$$\mathrm{D}(0\text{–}4), \bigwedge_i K_i(\mathrm{D}(0\text{–}4)) \vdash_\omega D_i(\overrightarrow{a}) \supset K_j K_k(D_i(\overrightarrow{a})). \tag{3.7}$$

This requires the imaginary player k of the mind of player j to know the behavioral postulate, $\mathrm{D}_i(0\text{–}4)$, for player i, for otherwise, consequence (3.7) would not make sense for k. This suggests to add another formula $\bigwedge_j \bigwedge_i K_j K_i(\mathrm{D}(0\text{–}4))$. Under this addition, however, we meet the same difficulty as that in (3.7) of the depth of one more degree, and need to go to the next step. This process leads to an infinite regress of adding the knowledge of D(0–4) of any finite depth. The infinite regress is described as

$$\{K(\mathrm{D}(0\text{–}4)) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}. \tag{3.8}$$

The conjunction of this set is the common knowledge, $C(\mathrm{D}(0\text{–}4))$, of D(0–4). We adopt this infinite regress as an axiom for $D_i(\cdot), i = 1, ..., n$. Then:

Lemma 3.4. $C(\mathrm{D}(0\text{–}4)) \vdash_\omega \forall \overrightarrow{x} \left(D_i(\overrightarrow{x}) \supset C(\mathrm{Nash_g}(\overrightarrow{x}))\right)$.[15]

In fact, the formula, $C(\mathrm{Nash_g}(\overrightarrow{a}))$, can be regarded as the solution of $C(\mathrm{D}(0\text{–}4))$ for the class of solvable games: A game g is said to satisfy *interchangeability* (in the sense of Nash [20]) iff the following holds:

$$\forall \overrightarrow{x} \forall \overrightarrow{y} \bigwedge_j \left(\mathrm{Nash_g}(\overrightarrow{x}) \wedge \mathrm{Nash_g}(\overrightarrow{y}) \supset \mathrm{Nash_g}(\overrightarrow{x}_j, \overrightarrow{y}_{-j})\right). \tag{3.9}$$

This is satisfied by the game of Table 3.1 but not by that of Table 3.2. We denote this formula by $\mathrm{INT_g}$. Of course, this is satisfied by a game g with a unique equilibrium.

Then we can prove that $C(\mathrm{Nash_g}(\overrightarrow{a}))$ satisfies $C(\mathrm{D}(1\text{–}4))$ under the common knowledge of $\mathrm{INT_g}$, that is, if every occurrence of $D_i(\overrightarrow{x})$ in $C(\mathrm{D}(0\text{–}4))$ is replaced by $C(\mathrm{Nash_g}(\overrightarrow{x}))$ for $i = 1, ..., n$, which is denoted by $C(\mathrm{D}(0\text{–}4))[C(\mathrm{Nash_g})]$, then:

Lemma 3.5. $C(\mathrm{INT_g}) \vdash_\omega C(\mathrm{D}(0\text{–}4))[C(\mathrm{Nash_g})]$.[16]

[15] Axiom $\mathrm{D4}_i^o, i = 1, ..., n$ are not necessary in this lemma. For such details, see Kaneko [6].

[16] In fact, $\bigwedge$-B_i is used for this lemmam but not for Lemma 3.4. Also, $\mathrm{D4}_i^o$ is used here, but not for Lemma 3.4.

Thus, $C(\text{Nash}(a))$ is a solution of $C(\text{D}(0\text{–}4))$, and Lemma 3.1 states that it is the deductively weakest. Hence $C(\text{Nash}_{\text{g}}(\overrightarrow{a}))$ can be regarded as what $C(\text{D}(0\text{–}4))$ determines. To formulate this claim explicitly, we introduce one more axiom schema:

(WD): $C(\text{D}(0\text{–}4))[\mathcal{A}] \supset \bigwedge_i \forall \overrightarrow{x}(A_i(\overrightarrow{x}) \supset D_i(\overrightarrow{x}))$,

where $\mathcal{A}$ is a family $\{A_i(\overrightarrow{a}) : i = 1, ..., n\}$ of formulae, and $C(\text{D}(0\text{–}4))[\mathcal{A}]$ is obtained from $C(\text{D}(0\text{–}4))$ by replacing all occurrences of $D_i(\overrightarrow{x})$ by $A_i(\overrightarrow{x})$ for all $i = 1, ..., n$. Lemmas 3.4 and 3.5 together with WD imply the following theorem.

Theorem 3.6. $C(\text{D}(0\text{–}4)), C(\text{INT}_{\text{g}}), \text{WD} \vdash_\omega \forall \overrightarrow{x} \left(D_i(\overrightarrow{x}) \equiv C(\text{Nash}_{\text{g}}(\overrightarrow{x}))\right)$.

It should be remarked that since we defined only implicitly a game as a part of Nash equilibrium, it is unclear whether game g is known to the players. However, it would be fair to say that since the players are not given the knowledge of basic axioms in Φ_{of} or Φ_{rcf}, they do not know the structure of the game. When they know these axioms, they are well informed of the game. In this sense, Theorem 3.6 is an abstract characterization of prediction $D_i(\cdot)$.

When the real closed field axioms Φ_{rcf} are assumed to be common knowledge, the game included in $\text{Nash}_{\text{g}}(\cdot)$ is also common knowledge. In this case, when a game g satisfies (3.9), we have

$$C(\text{D}(0\text{–}4)), C(\Phi_{\text{rcf}}), \text{WD} \vdash_\omega \forall \overrightarrow{x}(D_i(\overrightarrow{x}) \equiv C(\text{Nash}_{\text{g}}(\overrightarrow{x}))),$$

since $C(\Phi_{\text{rcf}}) \vdash_\omega C(\text{INT}_{\text{g}})$.

In order for player i to play the game g, we need to have $\exists \overrightarrow{x} D_i(\overrightarrow{x})$. By the above theorem, this is equivalent to having $\exists \overrightarrow{x} C(\text{Nash}_{\text{g}}(\overrightarrow{x}))$. Of course, since $C(\text{D}(0\text{–}4)), C(\text{INT}_{\text{g}})$ and WD are axioms for the considerations of $D_1, ..., D_n$, we need to assume some additional axioms including the ordered field axioms Φ_{of}. In the following, we consider the question of what Γ including no $D_1, ..., D_n$ can ensure

$$\Gamma \vdash_\omega \exists \overrightarrow{x} C(\text{Nash}_{\text{g}}(\overrightarrow{x})). \tag{3.10}$$

Suppose $\Gamma \vdash_\omega C(\text{INT}_{\text{g}})$. If (3.10) holds, then we have $C(\text{D}(0\text{–}4)), \Gamma, \text{WD} \vdash_\omega \exists \overrightarrow{x} D_i(\overrightarrow{x})$, which gives an affirmative answer to our playability problem. On the other hand, if (3.10) does not hold, it is not the case that $C(\text{D}(0\text{–}4)), \Gamma, \text{WD} \vdash_\omega \exists \overrightarrow{x} D_i(\overrightarrow{x})$, which is shown in Kaneko-Nagashima [12]. Hence it suffices to consider whether or not (3.10) holds.

When g has an equilibrium in pure strategies, it follows from Lemmas 2.4, 3.1 and 3.2 that $\Phi_{\text{of}} \vdash_0 \text{Nash}_{\text{g}}(\overrightarrow{r})$ for some profile $\overrightarrow{r}$ of terms consisting of $\mathbf{0}, \mathbf{1}$. Hence $C(\Phi_{\text{of}}) \vdash_\omega \exists \overrightarrow{x} C(\text{Nash}_{\text{g}}(\overrightarrow{x}))$ by Theorem 2.1 and $\exists$-Rule. Hence $C(\Phi_{\text{of}})$ suffices for Γ in

(3.10). In the following, we will focus on the existence of a Nash equilibrium in mixed strategies.

For an unsolvable game g, Theorem 3.6 fails to hold, but Lemma 3.4 does hold. Hence (3.10) is a necessary condition to have a final decision: unless (3.10) holds, it would be a conclusion that the game is unplayable. Theorem 3.6 is modified for an unsolvable game so that Nash equilibrium is replaced by a subsolution (a maximal subset of equilibria satisfying interchangeability) in the sense of Nash [20]. Kaneko [6] discussed this modification for finite games with pure strategies. For a game with mixed strategies, we meet some difficulties, for example, the subsolution concept may involves the second-order language. However, we can avoid these difficulties by restricting games so that each Nash equilibrium is locally unique and assuming some strong theory such as those given in Sections 6 and 7. In such a formulation, we would have the same conclusions (with some apparent modifications) with those for solvable games.

4. Conditions for the Specific Existence of a Nash Equilibrium

In this section, we give a general criterion for the specific knowledge of the existence of a Nash equilibrium. We start with the following result.

Theorem 4.A. Let Φ be a set of closed formulae in $\mathcal{P}^{f}(\mathcal{L})$ satisfying decidability (3.6). Let Γ be a consistent set of closed formulae in $\mathcal{P}^{f}(\mathcal{L})$ with $\Gamma \vdash_0 A$ for any $A \in \Phi$, and let g any finite game. Then the following three statements are equivalent:

(1): $C(\Gamma) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$;

(2): $C(\Phi) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$;

(3): there is a profile $\overrightarrow{t}$ of closed terms such that $\Phi \vdash_0 \mathrm{Nash_g}(\overrightarrow{t})$.

Proof. (2) $\Rightarrow$ (1) is straightforward, and (3) $\Rightarrow$ (2) is obtained by Theorem 2.1 and $\exists$-Rule. We prove (1) $\Rightarrow$ (3). Suppose $C(\Gamma) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$. It follows from Theorem 2.2 (Term-Existence Theorem) that $C(\Gamma) \vdash_\omega C(\mathrm{Nash_g}(\overrightarrow{t}))$ for some profile $\overrightarrow{t}$ of closed terms. Then $\Gamma \vdash_0 \mathrm{Nash_g}(\overrightarrow{t})$ by Theorem 2.1. By (3.6) and Lemma 3.2, $\Phi \vdash_0 \mathrm{Nash_g}(\overrightarrow{t})$ or $\Phi \vdash_0 \neg\mathrm{Nash_g}(\overrightarrow{t})$. If $\Phi \vdash_0 \neg\mathrm{Nash_g}(\overrightarrow{t})$, then $\Gamma \vdash_0 \neg\mathrm{Nash_g}(\overrightarrow{t})$ because $\Gamma \vdash_0 A$ for all A in Φ_{of}, and then $\Gamma \vdash_0 \neg\mathrm{Nash_g}(\overrightarrow{t}) \wedge \mathrm{Nash_g}(\overrightarrow{t})$, which is a contradiction to the consistency of Γ. Thus $\Phi \vdash_0 \mathrm{Nash_g}(\overrightarrow{t})$. □

The direction (1) $\Rightarrow$ (2) states that if the specific existence, $\exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$, is obtained from the common knowledge of some set Γ of axioms stronger than Φ, then we could actually obtain it from the common knowledge of Φ. This mean that to obtain $\exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$, the common knowledge of any axioms additional to the given axioms

Φ having decidability (3.6) are superfluous. Hence a deductively weakest set of axioms having (3.6) determines whether $\exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$ can be obtained.

When the list $\mathcal{L}$ is specified to be $\mathcal{L}_{\mathrm{of}}$, the ordered field axioms Φ_{of} satisfies (3.6) by Lemma 2.4. In this case, Φ_{rcf} does not help us obtain $\exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$, though $C(\exists \overrightarrow{x} \mathrm{Nash_g}(\overrightarrow{x}))$ is obtained from $C(\Phi_{\mathrm{rcf}})$. When $\mathcal{L}$ is $\mathcal{L}_{\mathrm{of}}$, we can strengthen Theorem 4.A as follows:

Theorem 4.B. Let $\mathcal{L}$ be $\mathcal{L}_{\mathrm{of}}$, and let Φ be Φ_{of}. Then each of the assertions (1), (2) and (3) of Theorem 4.A is equivalent to each of the following:

(4): $C(\Phi_{\mathrm{of}}) \vdash_\omega C(\exists \overrightarrow{x} \mathrm{Nash_g}(\overrightarrow{x}))$;

(5): $\Phi_{\mathrm{of}} \vdash_0 \exists \overrightarrow{x} \mathrm{Nash_g}(\overrightarrow{x})$.

Proof. (3) $\Rightarrow$ (5) is straightforward, and the equivalence (4) $\Leftrightarrow$ (5) follows from Theorem 2.1. We prove (5) $\Rightarrow$ (3): Suppose $\Phi_{\mathrm{of}} \vdash_0 \exists \overrightarrow{x} \mathrm{Nash_g}(\overrightarrow{x})$. Then $\exists \overrightarrow{x} \mathrm{Nash_g}(\overrightarrow{x})$ is true in the model $(Q; +, -, \times, / \,; \geq; 0, 1)$ by the soundness theorem. This means that there is a profile $(\overrightarrow{r}_1, ..., \overrightarrow{r}_n)$ of rational numbers such that $\mathrm{Nash_g}(\overrightarrow{a}_1, ..., \overrightarrow{a}_n)$ is true for an assignment σ in the model $(Q; +, -, \times, / \,; \geq; 0, 1)$ where $\sigma(a_{ik}) = r_{ik}$ for $k = 1, ..., \ell_i, i = 1, ..., n$. Let $\overrightarrow{\mathbf{r}}_1, ..., \overrightarrow{\mathbf{r}}_n$ be the vectors of the rational numerals corresponding to $\overrightarrow{r}_1, ..., \overrightarrow{r}_n$. Then $\mathrm{Nash_g}(\overrightarrow{\mathbf{r}}_1, ..., \overrightarrow{\mathbf{r}}_n)$ is true in $(Q; +, -, \times, / \,; \geq; 0, 1)$. This together with the soundness theorem implies $\Phi_{\mathrm{of}} \vdash_0 \mathrm{Nash_g}(\overrightarrow{\mathbf{r}}_1, ..., \overrightarrow{\mathbf{r}}_n)$, since $\Phi_{\mathrm{of}} \vdash_0 \mathrm{Nash_g}(\overrightarrow{\mathbf{r}}_1, ..., \overrightarrow{\mathbf{r}}_n)$ or $\Phi_{\mathrm{of}} \vdash_0 \neg \mathrm{Nash_g}(\overrightarrow{\mathbf{r}}_1, ..., \overrightarrow{\mathbf{r}}_n)$ by Lemmas 2.4, 3.1 and 3.2. □

Under the assumptions of Theorem 4.B, the assertion (3) is further equivalent to, by Lemma 2.4.(1),

(3_{r}): there is a profile $\overrightarrow{\mathbf{r}}$ of rational numerals such that $\Phi_{\mathrm{of}} \vdash_0 \mathrm{Nash_g}(\overrightarrow{\mathbf{r}})$.

It is a consequence from Theorem 4.B that although $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(\exists \overrightarrow{x} \mathrm{Nash_g}(\overrightarrow{x}))$ by Theorem 3.2 for any game g, $C(\Phi_{\mathrm{rcf}}) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$ would be obtained only if $\Phi_{\mathrm{of}} \vdash_0 \exists \overrightarrow{x} \mathrm{Nash_g}(\overrightarrow{x})$. Thus (2) holds if and only if the existence is obtained from the ordered field axioms Φ_{of} in the standard sense. Hence unless $\exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$ is provable from $C(\Phi_{\mathrm{of}})$, any strengthening in the axioms does not help to obtain $\exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$.

In the pure ordered field theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}}), \Phi_{\mathrm{of}})$, only the four arithmetic operations together with equality-inequality comparisons are allowed. Therefore the existence looks difficult to obtain. In the literature of game theory, we already know a positive answer for some class of games and a negative answer for some other class of games.

For 2-person finite games, Lemke-Howson [16] gave a finite algorithm to calculate a Nash equilibrium, based on the ordered field axioms.[17] A counterpart in the formalized mathematics is as follows:

[17] Rosenmüller [24] and Wilson [27] gave "algorithms" to calculate Nash equilibria for any finite game

Theorem 4.1 (Lemke-Howson [16]). Let g be a 2-person finite game. There is a profile $\overrightarrow{r}$ of rational numerals such that $\Phi_{of} \vdash_0 \mathrm{Nash}_g(\overrightarrow{r})$.

This together with Theorem 4.A implies that each of the assertions (1) through (5) holds for any 2-person game, in addition to their equivalence, in $(\mathcal{P}^{\omega}(\mathcal{L}_{of}), C(\Phi_{of}))$.

When a 2-person game g is zero-sum, it is proved to be also solvable in $(\mathcal{P}^{f}(\mathcal{L}_{of}), \Phi_{of})$, i.e., $\Phi_{of} \vdash_0 \mathrm{INT}_g$, which implies $C(\Phi_{of}) \vdash_{\omega} C(\mathrm{INT}_g)$. Hence every zero-sum 2-person zero-sum game passes the playability test fully, that is, $C(\mathrm{D}(0\text{–}4)), \mathrm{WD}, C(\Phi_{of}) \vdash_{\omega} \exists \overrightarrow{x} D_i(\overrightarrow{x})$.

For a game with more than two players, we have often a negative answer to the above existence question even if the game has a unique Nash equilibrium when $\mathcal{L}$ is $\mathcal{L}_{of}$. A Nash equilibrium may involve irrational number probability weights for a game with more than two players. If this is true for a game g, all the assertions of Theorem 4.A fail to hold. Furthermore, the negation of such an existence claim is also unprovable when $\mathcal{L}$ is $\mathcal{L}_{of}$. This is the undecidability result presented in Kaneko-Nagashima [12], which is now discussed below.

Consider the following 3-person game:

γ_1	β_1	β_2
α_1	0,0,1	1,0,0
α_2	1,1,0	2,0,8

Table 4.1

γ_2	β_2	β_2
α_1	2,0,9	0,1,1
α_2	0,1,1	1,0,0

Table 4.2

The tables mean that when the players choose pure strategies, say, $\alpha_1, \beta_2, \gamma_2$, the right upper vector (0,1,1) of Table 4.2 gives payoffs to the players. This game has no Nash equilibrium in pure strategies, but has a unique Nash equilibrium $((p, 1-p), (q, 1-q), (r, 1-r))$ in mixed strategies, where

$$p = (30 - 2\sqrt{51})/29, \ q = (2\sqrt{51} - 6)/21 \text{ and } r = (9 - \sqrt{51})/12. \qquad (4.1)$$

The probability weights in equilibrium are irrational numbers. Therefore those probabilities are not represented as closed terms in $\mathcal{L}_{of}$ by Lemma 2.4. Therefore it follows from Theorem 4.A that when $\mathcal{L}$ is $\mathcal{L}_{of}$, it is not the case that $C(\Gamma) \vdash_{\omega} \exists \overrightarrow{x} C(\mathrm{Nash}_g(\overrightarrow{x}))$, even though Γ is very strong.

By Lemma 2.3, $C(\Gamma) \vdash_{\omega} \neg\exists \overrightarrow{x} C(\mathrm{Nash}_g(\overrightarrow{x}))$ is equivalent to $C(\Gamma) \vdash_{\omega} C(\neg\exists \overrightarrow{x} \mathrm{Nash}_g(\overrightarrow{x}))$. When Γ is nonepistemic, this is further equivalent to $\Gamma \vdash_0 \neg\exists \overrightarrow{x} \mathrm{Nash}_g(\overrightarrow{x})$. If a set Γ of

as extensions of the Lemke-Hosen algorithm. However, these algorithms require more than the four arithemic operations and does not work for an ordered field, which is shown by the game of Tables 4.1 and 4.2. They need to trace on algebraic curves, which step does not have an algorithm. Hence theirs are not regarded as algorithms.

nonepistemic closed formulae stronger than Φ_{of} does not satisfy $\Gamma \vdash_0 \neg\exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x})$, then we have the following theorem.

Theorem 4.2 (Kaneko-Nagashima [12]). Let g be the 3-person game given by Tables 4.1 and 4.2, and let Γ be a set of closed formulae in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}})$ which satisfies $\Gamma \vdash_0 A$ for all $A \in \Phi_{\mathrm{of}}$ but not $\Gamma \vdash_0 \neg\exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x})$. Then neither $C(\Gamma) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$ nor $C(\Gamma) \vdash_\omega \neg\exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$.

5. Extensions, and Computations to Play a Game

Some readers may be puzzled by the undecidability assertion that a player cannot reach the Nash equilibrium which is specifically given by (4.1). The reason for the above undecidability result is that radical expressions are not allowed in our formal language. Once radical expressions are introduced, undecidability is removed for that specific 3-person game. However, this will not solve undecidability completely. In this section, we discuss the undecidability result when we extend the theory by adding some constant or function symbols. Then we consider the meaning of computations in such an extension, and provide the desiderata which should be satisfied by an extension.

5.1. Radical Expressions and the Abel–Galois Theorem

We add the following constant symbol to the list $\mathcal{L}_{\mathrm{of}}$ of symbols:

Constant: $\sqrt{51}$.

We denote the new list of symbols by $\mathcal{L}_{\mathrm{of}\sqrt{51}}$. We need the following axiom to determine $\sqrt{51}$ to be the square root of 51:

$(\sqrt{51})$: $(\sqrt{51} \geq 0) \bigwedge (\sqrt{51} \cdot \sqrt{51} = [51])$.

We denote $\Phi_{\mathrm{of}} \cup \{(\sqrt{51})\}$ by $\Phi_{\mathrm{of}\sqrt{51}}$.

In this extended theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}\sqrt{51}}), \Phi_{\mathrm{of}\sqrt{51}})$, the irrational numbers of (4.1) are expressible by closed terms. Denote the terms representing these irrational numbers by p, q, r, e.g., p is $([30] - [2]\sqrt{51}))/[29]$. Then we can prove that for the 3-person game g of Tables 4.1 and 4.2,

$$\Phi_{\mathrm{of}\sqrt{51}} \vdash_0 \mathrm{Nash}_{\mathrm{g}}((\mathrm{p}, 1-\mathrm{p}), (\mathrm{q}, 1-\mathrm{q}), (\mathrm{r}, 1-\mathrm{r})).$$

This together with Theorem 2.1 and $\exists$-Rule implies $C(\Phi_{\mathrm{of}\sqrt{51}}) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$. Thus the game g is no longer an example for undecidability in the present extension.

The above extension removes undecidability for only games where equilibria are expressed in terms of rational numerals with $\sqrt{51}$. However, Theorem 4.A is applied also to this $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}\sqrt{51}}), \Phi_{\mathrm{of}\sqrt{51}})$, since (3.6) holds in $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}\sqrt{51}}), \Phi_{\mathrm{of}\sqrt{51}})$, which can be proved by modifying the proof of Lemma 2.4.(2). Hence if a game involves another radical, then the above extension does not remove the new undecidability. Nevertheless, we can remove such undecidability by adding a constant to describe the new radical to the theory $(\mathcal{P}^f(\mathcal{L}_{\mathrm{of}\sqrt{51}}), \Phi_{\mathrm{of}\sqrt{51}})$ in a similar manner. Now we have a question of whether some extension of the theory $(\mathcal{P}^f(\mathcal{L}_{\mathrm{of}\sqrt{51}}), \Phi_{\mathrm{of}})$ by the addition of radicals removes undecidability for any finite game.

In fact, we can find an answer to this question in the literatures of game theory and of algebra. A theorem of Bubelis [2] states that for any real algebraic number $\alpha \in [0, 1]$, there is a 3-person game such that its unique equilibrium has α as the probability weight for some pure strategy. The Abel-Galois theorem states that a polynomial equation of degree five or more with rational coefficients is not necessarily solved by the four arithmetic operations together with radicals. For example, $x^5 - 4x + 2 = 0$ has three real roots, which cannot be solved by the four operations and radicals (cf., van der Waerden [30], Section 60). For any of these roots, Bubelis' theorem provides a 3-person game whose unique equilibrium involves that root (the game is relatively small; players 1, 2 and 3 have two, six and six pure strategies, respectively). Consequently, undecidability cannot be removed for such 3-person games by any extension of our theory by adding radicals.

The addition of radical expressions does not suffice for the complete resolution of undecidability. Bubelis' [2] theorem states that every real algebraic number is involved in classical game theory. Conversely, The argument for Theorem 3.3 states that the real algebraic numbers are sufficient for the existence problem of a Nash equilibrium. The Abel-Galois theorem simply states that the four arithmetic operations and radicals do not express all the real algebraic numbers. Now our problem is to find a method of expressing all the real algebraic numbers by closed terms. In Sections 6 and 7, we give two procedures.

5.2. Terms as Computation Units, Transformation into Decimal Expansions, and Generating Irrational Probabilities

Before going to the consideration of expressing all real algebraic numbers by closed terms, we consider computations involved in playing a game.

To play a game, each player needs not only to reach a specific Nash equilibrium but also to generate the probabilities prescribed by the equilibrium whenever proper probabilities are involved. This statement may include several kinds of computations and a probability generator. As Theorem 2.2 (Term-Existence Theorem) states, each player needs to compute a Nash equilibrium in closed terms. In this computation, closed terms form computation units, which depend upon the choice of a language. For example, in the pure ordered field theory $(\mathcal{P}^f(\mathcal{L}_{\mathrm{of}}), \Phi_{\mathrm{of}})$, rational numbers are only legitimate computation units, and in the extended theory $(\mathcal{P}^f(\mathcal{L}_{\mathrm{of}\sqrt{51}}), \Phi_{\mathrm{of}\sqrt{51}})$, expressions in terms of the four arithmetic operations including radicals are allowed. Once those computation units are specified, some algorithm for searching a Nash equilibrium is defined. For some games, a Nash equilibrium is found in such computation units, but sometimes not. When a Nash equilibrium is found in closed terms, we have the question of how a player plays such a strategy expressed by closed terms.

Let us return to a theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}), \Phi)$ with property (3.6). This decidability of $t_1 \geq t_2$ plays two important roles in our consideration. The first is: for any profiles of closed terms $(\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$,

$$\Phi \vdash_0 \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t}) \text{ or } \Phi \vdash_0 \neg\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t}) \text{ for any profiles } \overrightarrow{t} \text{ of closed terms.}$$

Suppose that a Nash equilibrium exists in closed terms in the theory $(\mathcal{P}^{\mathrm{f}}, \Phi)$. If the closed terms are enumerable – we will construct theories in Sections 6 and 7 whose closed terms are enumerable –, then the profiles $(\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$ of closed terms are also enumerable. Hence the step-by-step verification of each candidate $(\overrightarrow{t}_1, ..., \overrightarrow{t}_n)$ to be a Nash equilibrium until a Nash equilibrium is found forms an algorithm (though this procedure is very inefficient from the practical point of view). Hence (3.6) ensures the existence of an algorithm if there exists a Nash equilibrium in closed terms.

Now suppose that a player reaches a Nash equilibrium in closed terms. The other problem of how he plays such a strategy expressed by closed terms. For a 2-person game, such closed terms are rational numerals. Hence a fair roulette (a choice of a ball from an urn) can generate the probability described by each term. However, for a game with more than two players, a Nash equilibrium involves irrational numbers. Now his problem is how such irrational number probabilities are actually generated. In fact, this is always possible if the irrational numbers are transformed into decimal (or binary) expansions.

Consider the problem of how the probability $p = (30 - 2\sqrt{51})/29$ is generated in the game of Section 4. Let the decimal expansion of p be $.d_1d_2...d_k...$ $(= .5419...)$. Prepare a fair roulette (an urn containing 10 balls) with outcomes $0, 1, ..., 9$. Player 1 chooses pure strategy α_1 or α_2 by spinning the roulette repeatedly according the following rule. Suppose that he goes to the k–th spin of the roulette. If the outcome of the k-th spin is smaller than d_k, then choose α_1; and if it is greater than d_k, then choose α_2. In these two cases, player 1 does not spin the roulette anymore, but if the outcome is d_k, then he goes to the $(k+1)$–th spin of the roulette. The probability of choosing pure strategy α_1 is exactly $p = d_1/10 + d_2/10^2 + d_3/10^3 + \cdots = (30 - 2\sqrt{51})/29$. The process halts and player 1 chooses α_1 or α_2 with probability 1.[18]

Although we separate the process of making the decimal expansion from the roulette spins, practically these should be processed simultaneously, that is, if the roulette spin

[18] This procedure was suggested by R. Aumann.

goes to the next round, then the player needs the next digit of the expansion. When an extended theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}), \Phi)$ satisfies (3.6), this decimal expansion can be obtained up to any given order. In $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}\sqrt{51}}), \Phi_{\mathrm{of}\sqrt{51}})$, the first digit d_1 is found by checking $\Phi_{\mathrm{of}\sqrt{51}} \vdash_0 [k]/[10] \leq \mathrm{p} < [k+1]/[10]$ for $k = 0, ..., 9$, and the second d_2 is found by $\Phi_{\mathrm{of}\sqrt{51}} \vdash_0 [d_1]/[10] + [k]/[100] \leq \mathrm{p} < [d_1]/[10] + [k+1]/[100]$ for $k = 0, ..., 9$, and so on. Hence (3.6) suffices for obtaining decimal expansions.

We summarize the above considerations as follows:

Computation of a Nash equilibrium in closed terms: In a given language with suitable axioms where legitimate computation units are specified as closed terms, a Nash equilibrium could be computed if a Nash equilibrium exists in closed terms.

Transformation into a decimal expansion: Once a player obtains a Nash equilibrium in closed terms, he needs to transform the probabilities involved in the equilibrium into decimal (or binary) expressions up to any given order.

Generating Probabilities: Finally, he needs to have some random mechanism to generate the probabilities prescribed by the Nash equilibrium computed.

The last process conditional upon the first two is external to the choice of a theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}), \Phi)$. To have the first two, we should construct a theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}),\Phi)$ so that it enjoys decidability (3.6), term-existence:

$$\begin{array}{c}\text{if } \Phi \vdash_0 \exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}), \\ \text{then } \Phi \vdash_0 \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{t}) \text{ for some profile } \overrightarrow{t} \text{ of closed terms,}\end{array} \tag{5.1}$$

as well as the existence of a Nash equilibrium itself. We will give such two theories in Sections 6 and 7.

6. The Real Closed Field Theory with Real Algebraic Numbers

It was discussed in Section 5 that a new extended theory should be able to express all the real algebraic numbers by closed terms as well as to guarantee the existence of a Nash equilibrium. Theorem 3.3 states that the existence of a Nash equilibrium is obtained in the real closed field theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}}), \Phi_{\mathrm{rcf}})$. Hence it would be natural to add to this theory some symbols to express all the real algebraic numbers. By this addition, we will construct the new theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{rcfr}})$ which satisfies decidability (3.6), term-existence (5.1) as well as has the existence of a Nash equilibrium. In fact, by using Tarski's completeness theorem, this theory is proved to be complete. This enables us to obtain a positive answer to parallel questions about refinements of Nash equilibrium, which will be remarked in the second subsection.

6.1. A Formalization of Real Algebraic Numbers

To describe a real algebraic number, we need to consider a polynomial equation with integer coefficients. Identifying a polynomial $b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0$ with a vector $(b_m, ..., b_1, b_0)$, we can regard the set of all polynomials with integral coefficients as the set $\bigcup_{0<m<\omega} P(m)$, where $P(m) = \{(b_m, ..., b_1, b_0) : b_m, ..., b_1, b_0$ are integers and $b_m \neq 0\}$.

We add the following constant symbols to the list $\mathcal{L}_{\mathrm{of}}$ of basic symbols:

Constant symbols: $\mathbf{e}_k^f$ for $k = 1, ..., m$, $f \in P(m)$ and $m = 1, 2, ...$

The new list is denoted by $\mathcal{L}_{\mathrm{ofr}}$. The symbol $\mathbf{e}_k^f$ is intended to express the k–th smallest real root of polynomial equation $f(x) = 0$ if it exists, and to be $\mathbf{0}$ otherwise. We call these symbols *algebraic numerals.*

In our formalized language, a polynomial equation can be described as

$$[b_m]a^m + ... + [b_1]a + [b_0] = \mathbf{0}, \tag{6.1}$$

where a is a free variable and $[b_m], ..., [b_1], [b_0]$ are numerals corresponding to integers $b_m, ..., b_1, b_0$, where $f = (b_m, ..., b_1, b_0) \in P(m)$. We denote the above equation of (6.1) by $\mathbf{f}(a) = \mathbf{0}$.

We denote, by $R_f^k(a)$, the following formula describing "$\mathbf{f}(x) = \mathbf{0}$ has at least $k-1$ real roots smaller than real root a":

$$\exists x_1 ... \exists x_{k-1} \left(\bigwedge_{t=1}^{k-1} (\mathbf{f}(x_t) = \mathbf{0}) \bigwedge (x_1 < ... < x_{k-1} < a) \right) \bigwedge (\mathbf{f}(a) = \mathbf{0}).$$

We denote, by $E_k^f(a)$, the following formula describing "a is the k-th smallest roof of $\mathbf{f}(x) = \mathbf{0}$":

$$R_k^f(a) \bigwedge \forall y \left(R_k^f(y) \supset y \geq a \right).$$

Hence if $f(x) = 0$ has at least k real roots, then $E_k^f(\mathbf{e}_k^f)$ describes "$\mathbf{e}_k^f$ is the k–th smallest root of $\mathbf{f}(x) = \mathbf{0}$", and otherwise, we determine $\mathbf{e}_k^f$ to be $\mathbf{0}$. We make this as an axiom:

$$\text{(RT): } \left(\exists x R_k^f(x) \supset E_k^f(\mathbf{e}_k^f)\right) \bigwedge \left(\neg \exists x R_k^f(x) \supset \mathbf{e}_k^f = \mathbf{0}\right),$$

where $f \in P(m)$, $k = 1, ..., m$ and $m = 1, ...$ We denote the union of Φ_{rcf} and the set of all instances of RT by Φ_{rcfr}. Since $\Phi_{\mathrm{rcf}} \vdash_0 \exists x R_k^f(x)$ or $\Phi_{\mathrm{rcf}} \vdash_0 \neg\exists x R_k^f(x)$ by Tarski's completeness theorem, either $\Phi_{\mathrm{rcfr}} \vdash_0 E_k^f(\mathbf{e}_k^f)$ or $\Phi_{\mathrm{rcfr}} \vdash_0 \mathbf{e}_k^f = \mathbf{0}$. Here $\mathbf{e}_k^f$ is completely determined. In the first case, this complete determination is guaranteed by the first of the following lemma. The second assertion of the lemma is the elimination of algebraic numerals.

Lemma 6.1.(1): $\Phi_{\mathrm{rcfr}}, \exists x R_k^f(x) \vdash_0 \forall x(E_k^f(x) \supset x = \mathrm{e}_k^f)$.
(2): For any closed formula A in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}})$, there is a closed formula B in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}})$ such that $\Phi_{\mathrm{rcfr}} \vdash_0 A \equiv B$.

Proof. (1): Observe $\Phi_{\mathrm{rcfr}}, \exists x R_k^f(x) \vdash_0 E_j^k(a) \supset E_j^k(a) \wedge R_j^k(\mathrm{e}_k^f)$ and $\vdash_0 E_j^k(a) \wedge R_j^k(\mathrm{e}_k^f) \supset a \leq \mathrm{e}_k^f$, which imply $\Phi_{\mathrm{rcfr}}, \exists x R_k^f(x) \vdash_0 E_j^k(a) \supset a \leq \mathrm{e}_k^f$. The converse holds, too, i.e., $\Phi_{\mathrm{rcfr}}, \exists x R_k^f(x) \vdash_0 E_j^k(a) \supset a \geq \mathrm{e}_k^f$. Hence $\Phi_{\mathrm{rcfr}}, \exists x E_k^f(x) \vdash_0 E_j^k(a) \supset a = \mathrm{e}_k^f$.

(2): Let $A(\mathrm{e}_k^f)$ include e_k^f. There are two cases: $\Phi_{\mathrm{rcf}} \vdash_0 \exists x R_k^f(x)$ and $\Phi_{\mathrm{rcf}} \vdash_0 \neg\exists x R_k^f(x)$. We prove that in each case, there is another formula A' without including e_k^f such that $\Phi_{\mathrm{rcfr}} \vdash_0 A(\mathrm{e}_k^f) \equiv A'$. Repeating this process, we eliminate all algebraic numerals in $A(\mathrm{e}_k^f)$, and then obtain B in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}})$ so that $\Phi_{\mathrm{rcfr}} \vdash_0 A(\mathrm{e}_k^f) \equiv B$.

Consider the case where $\Phi_{\mathrm{rcf}} \vdash_0 \neg\exists x R_k^f(x)$. Then $\Phi_{\mathrm{rcfr}} \vdash_0 \mathrm{e}_k^f = \mathbf{0}$ by RT. Hence $\Phi_{\mathrm{rcfr}} \vdash_0 A(\mathrm{e}_k^f) \equiv A(\mathbf{0})$, where $A(\mathbf{0})$ is obtained from $A(\mathrm{e}_k^f)$ by substituting $\mathbf{0}$ for all occurrences of e_k^f in $A(\mathrm{e}_k^f)$.

Now consider the case where $\Phi_{\mathrm{rcf}} \vdash_0 \exists x R_k^f(x)$. Then $\Phi_{\mathrm{rcfr}} \vdash_0 E_k^f(\mathrm{e}_k^f)$ by RT. It suffices to show $\Phi_{\mathrm{rcfr}} \vdash_0 \forall x \left(E_k^f(x) \supset A(x)\right) \equiv A(\mathrm{e}_k^f)$, where $A(x)$ is the expression obtained from $A(\mathrm{e}_k^f)$ by substituting x for all occurrences of e_k^f in $A(\mathrm{e}_k^f)$.

First, $\vdash_0 \forall x \left(E_k^f(x) \supset A(x)\right) \supset \left(E_k^f(\mathrm{e}_k^f) \supset A(\mathrm{e}_k^f)\right)$, equivalently, $\vdash_0 E_k^f(\mathrm{e}_k^f) \supset \left(\forall x \left(E_k^f(x) \supset B(x)\right) \supset B(\mathrm{e}_k^f)\right)$. Since $\Phi_{\mathrm{rcfr}} \vdash_0 E_k^f(\mathrm{e}_k^f)$, we have $\Phi_{\mathrm{rcfr}} \vdash_0 \forall x \left(E_k^f(x) \supset A(x)\right) \supset A(\mathrm{e}_k^f)$. Conversely, since $\Phi_{\mathrm{rcf}} \vdash_0 \exists x R_k^f(x)$, it follows from (1) that $\Phi_{\mathrm{rcfr}} \vdash_0 E_k^f(a) \supset \left(A(\mathrm{e}_k^f) \supset A(a)\right)$, equivalently, $\Phi_{\mathrm{rcfr}} \vdash_0 A(\mathrm{e}_k^f) \supset \left(E_k^f(a) \supset A(a)\right)$. Thus $\Phi_{\mathrm{rcfr}} \vdash_0 A(\mathrm{e}_k^f) \supset \forall x \left(E_k^f(x) \supset A(x)\right)$. □

Lemma 6.1.(2) implies that the provability of a closed formula in $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}),\Phi_{\mathrm{rcfr}})$ is reduced into $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}}),\Phi_{\mathrm{rcf}})$. This together with Tarski's completeness theorem implies the following.

Theorem 6.2 (Completeness of $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}),\Phi_{\mathrm{rcfr}})$). Let A be a closed formula in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}})$. Then $\Phi_{\mathrm{rcfr}} \vdash_0 A$ and $\Phi_{\mathrm{rcfr}} \vdash_0 \neg A$.

Proof. Let A be a closed formula in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}})$. By Lemma 6.1.(2), there is a closed formula B in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{of}})$ such that $\Phi_{\mathrm{rcfr}} \vdash_0 A \equiv B$. By Tarski's completeness theorem, $\Phi_{\mathrm{rcf}} \vdash_0 B$ or $\Phi_{\mathrm{rcf}} \vdash_0 \neg B$. Hence $\Phi_{\mathrm{rcfr}} \vdash_0 A$ or $\Phi_{\mathrm{rcfr}} \vdash_0 \neg A$. □

Decidability (3.6) is a small part of Theorem 6.2. The other key property for the specific existence is term-existence (5.1).

Theorem 6.3 (Term-Existence for $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}),\Phi_{\mathrm{rcfr}})$). Let $\exists x_1...\exists x_m A(x_1,...,x_m)$ be a closed formula in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}})$. Then if $\Phi_{\mathrm{rcfr}} \vdash_0 \exists x_1...\exists x_m A(x_1,...,x_m)$, there are algebraic numerals $\mathrm{e}_1,...,\mathrm{e}_m$ such that $\Phi_{\mathrm{rcfr}} \vdash_0 A(\mathrm{e}_1,...,\mathrm{e}_m)$.

Before proving this theorem, we construct a model of $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}),\Phi_{\mathrm{rcfr}})$. To discuss a model of $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}),\Phi_{\mathrm{rcfr}})$, we should specify the interpretations of algebraic numerals in addition to those of $+,-,\times,/$ and $\geq$. We denote the k–th root of a polynomial equation $f(x)=0$ by e^f_k ($f \in P(m), m=1,...$). Define the structure $\mathcal{A}=(\mathsf{A};+,-,\times,/\,;\geq;\psi)$ by the following interpretation ψ of constants:

$$\psi(0)=0, \psi(1)=1$$

$$\text{if } f(x)=0 \text{ has } \ell \text{ real roots, then } \psi(\mathrm{e}^f_k)= e^f_k \text{ for } k=1,...,\ell; \quad \text{and } \psi(\mathrm{e}^f_k)=0 \text{ for } k=\ell+1,...,m. \tag{6.2}$$

Then every instance of RT is true in $\mathcal{A}$. Therefore $\mathcal{A}=(\mathsf{A};+,-,\times,/;\geq;\psi)$ is a model of $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}),\Phi_{\mathrm{rcfr}})$.

Proof of Theorem 6.3. Suppose $\Phi_{\mathrm{rcfr}} \vdash_0 \exists x_1...\exists x_m A(x_1,...,x_m)$. Then $\exists x_1...\exists x_m A(x_1,...,x_m)$ is true in the model $\mathcal{A}$. Hence $A(a_1,...,a_m)$ is true in $\mathcal{A}$ for some free variables $a_1,...,a_m$ and some assignment σ which assigns a value in A to each free variable. Let $\sigma(a_1),...,\sigma(a_m)$ be $e_1,...,e_m$. For each e_s ($s=1,...,m$),there is a polynomial equation $f_s(x)=0$ with integer coefficients such that e_s is its k_s–th smallest root. Thus $\psi(\mathrm{e}^{k_s}_{f_s})=e_s$ for $s=1,...,m$. Then $A(\mathrm{e}^{k_1}_{f_1},...,\mathrm{e}^{k_m}_{f_m})$ is true in $\mathcal{A}$. By Theorem 6.2, we have $\Phi_{\mathrm{rcfr}} \vdash_0 A(\mathrm{e}^{k_1}_{f_1},...,\mathrm{e}^{k_m}_{f_m})$. □

Now we can state the game theoretical result.

Theorem 6.A. Let $\mathcal{L}$ be $\mathcal{L}_{\mathrm{ofr}}$. Let g be any finite game. Then the following hold:

(1): $C(\Phi_{\mathrm{rcfr}}) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$;

(2): there is a profile $\overrightarrow{\mathrm{e}}$ of algebraic numerals such that $\Phi_{\mathrm{rcfr}} \vdash_0 \mathrm{Nash_g}(\overrightarrow{\mathrm{e}})$.

Proof. By Theorem 3.2.(1), $\Phi_{\mathrm{rcf}} \vdash_0 \exists \overrightarrow{x}\mathrm{Nash_g}(\overrightarrow{x})$. By Theorem 6.3, there is a profile $\overrightarrow{\mathrm{e}}$ of algebraic numerals such that $\Phi_{\mathrm{rcfr}} \vdash_0 \mathrm{Nash_g}(\overrightarrow{\mathrm{e}})$, which is (1). Assertion (2) follows (1): By Theorem 2.1, $C(\Phi_{\mathrm{rcfr}}) \vdash_\omega C(\mathrm{Nash_g}(\overrightarrow{\mathrm{e}}))$. Then $C(\Phi_{\mathrm{rcfr}}) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash_g}(\overrightarrow{x}))$ by $\exists$-Rule. □

We can observe that in the present extension, the desiderata discussed in Subsection 5.2 are fulfilled. Although we add an infinite number of constants symbols, they are enumerable and so are the instances of Axiom RT. We can repeat the verification of whether or not $\Phi_{\mathrm{rcfr}} \vdash_0 \mathrm{Nash_g}(\overrightarrow{\mathrm{e}}_1,...,\overrightarrow{\mathrm{e}}_n)$ for each profile $(\overrightarrow{\mathrm{e}}_1,...,\overrightarrow{\mathrm{e}}_n)$, until it holds. This verification terminates by Theorem 6.A, and we will find $(\overrightarrow{\mathrm{e}}_1,...,\overrightarrow{\mathrm{e}}_n)$ so that $\Phi_{\mathrm{rcfr}} \vdash_\omega \mathrm{Nash_g}(\overrightarrow{\mathrm{e}}_1,...,\overrightarrow{\mathrm{e}}_n)$. Since $\Phi_{\mathrm{rcfr}} \vdash_0 t_1 \geq t_2$ or $\Phi_{\mathrm{rcfr}} \vdash_0 \neg(t_1 \geq t_2)$ for any closed terms t_1, t_2 by Theorem 6.2, every term in $\vec{e}_1, ..., \vec{e}_n$ are transformed into decimal expansions. Thus the probabilities described by algebraic numerals can be generated.

6.2. Possible Applications to Refinements

The extension presented in the above subsection is very powerful. Theorem 6.A could hold for many refinement concepts, for example, Selten's [25] perfect equilibrium and Nash's [19] and [21] bargaining solution. Here we give remarks on possible applications to such refinement concepts.

By adding more axioms, the epistemic axiomatization of "final decision predictions" would yield a refinement concept of Nash equilibrium. For example, in the case of perfect equilibrium in Selten [25], the final decision prediction $D_i(\vec{a})$ would become $C(\text{Selten}_g(\vec{a}))$ instead of $C(\text{Nash}_g(\vec{a}))$, where $\text{Selten}_g(\vec{a})$ is a perfect equilibrium.[19] Hence the relevant question becomes the provability of $\exists \vec{x} C(\text{Selten}_g(\vec{x}))$. Then Theorem 6.A holds for $\exists \vec{x} C(\text{Selten}_g(\vec{x}))$ in any finite game. In a similar manner, we can apply our argument to other refinement concepts.

Another, slightly different, application is Nash's [19] and [21] bargaining theory. The Nash bargaining solution with a fixed threat can be regarded as a Nash equilibrium with the other axioms (Independence of Irrelevant Alternatives, Invariance under Affine Transformation and Anonymity) for demand games (cf., Kaneko-Mao [9]). The epistemic axiomatization given in Section 3 can be modified to incorporate the other axioms, and the resulting solution concept $C(\text{Bar}_S(\vec{a}))$ for a polyhedral bargaining problem S. In the 2-person case, the pure ordered field theory $(\mathcal{P}^f(\mathcal{L}_{of}), \Phi_{of})$ suffices, which was shown by Kaneko [4]. For a bargaining game with more than two players, irrational numbers may be involved again in the Nash solution. However our theory $(\mathcal{P}^f(\mathcal{L}_{ofr}), \Phi_{rcfr})$ suffices for the Nash bargaining solution and Theorem 6.A holds for it. It is important to emphasize that this theory gives a unique bargaining outcome and hence it satisfies always the solvability condition.

Theorem 6.A holds, too, in the case of the Nash bargaining solution with variable threats. In the 2-person case, the pure ordered field theory $(\mathcal{P}^f(\mathcal{L}_{of}), \Phi_{of})$ suffices and it is also solvable. For a bargaining game with variable threats and more than two players, $(\mathcal{P}^f(\mathcal{L}_{ofr}), \Phi_{rcfr})$ is sufficient for Theorem 6.A. But the game may be unsolvable (cf., Kaneko-Mao [9]).

[19] Selten [25] himself used the definition in terms of a convergent sequence, and this involves a second-order language – it is not allowed in our first-order language. Nevertheless, it is possible to define perfect equilibrium in our first-order language.

7. Ordered Field Theory with the Sturm Axiom

The real closed field theory, $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{rcfr}})$, with real algebraic numbers fulfills the desiderata discussed in Section 5, and works for purposes other than the specific existence of a Nash equilibrium, as stated in Subsection 6.2. It includes, however, one slightly nonconstructive axiom – Real-Closedness. In fact, without this axiom, Axiom RT could not work to determine the real algebraic numbers. Hence we can ask the question of whether we have a purely constructive theory which consists only of the axioms for the four arithmetic operations and of ones determining the real algebraic numbers. In this section, we construct such a theory, $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{ofst}})$, which is properly weaker than $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{rcfr}})$, but it suffices for the specific existence of a Nash equilibrium, though it cannot be applied to some refinement concepts.

We will adopt the Sturm method to determine the real algebraic numbers instead of Axiom RT. In Subsection 6.1, we will prepare such algebraic notions in nonformalized mathematics, and then formulate our new theory, $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{ofst}})$, in Subsection 6.2.

7.1. Some Algebraic Notions

Given polynomial $f(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$ $(f \in P(m), m = 1, \ldots)$, we define the *Sturm sequence* $f_0(x) = f(x), f_1(x), \ldots, f_{\ell_f}(x)$ by

$$
\begin{aligned}
f_1(x) &= f'(x) = mx^{m-1} + (m-1)b_{m-1}x^{m-2} + \cdots + b_1; \\
f_{k-1}(x) &= q_k(x)f_k(x) - f_{k+1}(x) \quad \text{for } k = 1, \ldots, \ell_f - 1; \text{ and} \\
f_{\ell_f - 1}(x) &= q_{\ell_f}(x)f_{\ell_f}(x),
\end{aligned}
\tag{7.1}
$$

where each $-f_{k+1}(x)$ is obtained by the Euclid algorithm from $f_{k-1}(x)$ and $f_k(x)$, i.e., $q_k(x)$ is the quotient and $-f_{k+1}(x)$ is the remainder when polynomial $f_{k-1}(x)$ is divided by polynomial $f_k(x)$ (cf., van der Waerden [30], Section 68). We denote the number of changes in signs in the Sturm sequence of (7.2) at a real number α by $v_f(\alpha)$:

$$
f_0(\alpha), f_1(\alpha), \ldots, f_{\ell_f}(\alpha), \tag{7.2}
$$

where if some of these numbers are zero, we count the number of changes in signs ignoring 0's. It is known as *Sturm's theorem* (cf., van der Waerden [30], Section 68) that for any numbers α_1 and α_2 with $\alpha_1 < \alpha_2$, if $f(\alpha_1) \neq 0$ and $f(\alpha_2) \neq 0$, the numbers of real roots of $f(x) = 0$ between α_1 and α_2 is given as $v_f(\alpha_1) - v_f(\alpha_2)$. The total number of real roots of $f(x) = 0$ is given by choosing α_1 and α_2 large enough, which are called *bounds for roots.* One bound is given as $M_f = \max(|b_{m-1}|, \ldots, |b_0|) + 1$, i.e., $-M_f < x < M_f$ for all real x with $f(x) = 0$ (cf., van der Waerden [30], Section 68). It follows from Sturm's theorem that the number of real roots of $f(x) = 0$ that are less than α with $f(\alpha) \neq 0$ is given as

$$n_f(\alpha) = v_f(-M_f) - v_f(\alpha). \tag{7.3}$$

Therefore the *total number* n_f of real roots of $f(x) = 0$ is given as $n_f = n_f(M_f) = v_f(-M_f) - v_f(M_f)$. Sturm's theorem holds for any real closed field (cf., van der Waerden [30], Section 68).

It is important to notice that n_f is calculated from each polynomial $f \in P(m), m = 1, \ldots$ by the four arithmetic operations $+, -, \cdot, /$ on real numbers. Especially, when α is rational, all of these notions can be calculated in the ordered field of rational numbers.

7.2. The Sturm Axiom

We adopt the same list of symbols $\mathcal{L}_{\mathrm{ofr}}$ as in Section 6, but adopt axioms different from Axiom RT to determine the real algebraic numbers. Specifically, we formulate Sturm's method as an axiom (schema) instead of proving Sturm's theorem from some other axioms.

Let $f \in P(m), m = 1, \ldots$ The Sturm sequence $f_0(x), \ldots, f_{\ell_f}(x)$ is described as $\mathrm{f}_0(a), \ldots, \mathrm{f}_{\ell_f}(a)$ in our formalized language, where a is a free variable. Then "the change in signs in the Sturm sequence $\mathrm{f}_0(a), \ldots, \mathrm{f}_{\ell_f}(a)$ occurs at $\mathrm{f}_{s_1}(a), \ldots, \mathrm{f}_{s_k}(a)$" is described as

$$\begin{gathered} \textstyle\bigwedge_{\tau=0}^{k-1} \left((\mathrm{f}_{s_\tau}(a) \cdot \mathrm{f}_{s_{\tau+1}}(a) < 0) \wedge \left(\bigwedge_{t=1}^{s_{\tau+1}-s_\tau-1} \mathrm{f}_{s_\tau}(a) \cdot \mathrm{f}_{s_\tau+t}(a) \geq 0 \right) \right) \\ \wedge \left(\textstyle\bigwedge_{t=1}^{\ell_f - s_k} \mathrm{f}_{s_k}(a) \cdot \mathrm{f}_{s_k+t}(a) \geq 0 \right), \end{gathered} \tag{7.4}$$

where f_{s_0} is always fixed to be f_0. Since we add $\neg \mathrm{f}(a) = 0$ later, we can start with f_{s_0} in the first parentheses of (7.4). We denote this formula by $[\mathrm{f}_{s_1}(a), \ldots, \mathrm{f}_{s_k}(a)]$. Then we translate "$v_f(\alpha) = k$" ($k \leq \ell_f$), i.e., "the number of changes in signs in the Sturm sequence at α is k", into our formalized language by

$$\bigvee \{[\mathrm{f}_{s_1}(a), \ldots, \mathrm{f}_{s_k}(a)] : 0 < s_1 < \cdots < s_k \leq \ell_f\}. \tag{7.5}$$

The expression $[v_f(a) = k]$ denotes this formula if $k \leq \ell_f$ and $\mathbf{0} = \mathbf{1}$ if $k > \ell_f$.

The following formulae correspond to "$n_f(\alpha) = k$" and "$v_f(\alpha) - v_f(\beta) = 1$" :

$$\bigvee \{[v_f(-\mathrm{M}_f) = \tau + k] \bigwedge [v_f(a) = \tau] : \tau = 0, 1, \ldots, m\}; \tag{7.6}$$

$$\bigvee \{[v_f(a) = k + 1] \bigwedge [v_f(b) = k] : k = 0, 1, \ldots, m\}, \tag{7.7}$$

where $f \in P(m)$ and M_f is the rational numeral corresponding to the bound M_f given in Subsection 7.1. We denote these formulae by $[n_f(a) = k]$ and $[v_f(a) - v_f(b) = 1]$.

We also denote $\bigvee_{t<k}[n_f(a) = t]$ and $\bigvee_{t=k}^{m}[n_f(a) = t]$ by $[n_f(a) < k]$ and $[n_f(a) \geq k]$, respectively.

If we plug a rational numeral $\mathbf{r}$ with $\Phi_{\text{of}} \vdash_0 \neg \mathbf{f}(\mathbf{r}) = \mathbf{0}$ to the free variable a in $[v_f(a) = k]$, it follows from Lemma 2.4 and (2.2) that $\Phi_{\text{of}} \vdash_0 [v_f(\mathbf{r}) = k]$ if and only if $v_f(r) = k$, where r is the rational number corresponding to $\mathbf{r}$. The latter statement is evaluated in the ordered field, Q, of rational numbers. Similarly, we have,

Lemma 7.1. For any rational numbers r, r' with $f(r) \neq 0, f(r') \neq 0$ and for the corresponding rational numerals $\mathbf{r}, \mathbf{r}'$,

(1): $\Phi_{\text{of}} \vdash_0 [n_f(\mathbf{r}) = k]$ if and only if $n_f(r) = k$;

(2): $\Phi_{\text{of}} \vdash_0 [n_f(\mathbf{r}) < k]$ if and only if $n_f(r) < k$;

(3): $\Phi_{\text{of}} \vdash_0 [v_f(\mathbf{r}) - v_f(\mathbf{r}') = 1]$ if and only if $v_f(r) - v_f(r') = 1$.

Now we can formulate the Sturm axiom: for any $f \in \bigcup_{0<m<\omega} P(m)$,

(Sturm):(1): for $k = 1, ..., n_f$,

$$\mathbf{f}(\mathbf{e}_k^f) = \mathbf{0} \wedge \forall x \left(\neg \mathbf{f}(x) = \mathbf{0} \supset \left(([n_f(x) < k] \supset x < \mathbf{e}_k^f) \wedge [n_f(x) \geq k] \supset x > \mathbf{e}_k^f)\right)\right);$$

and for $k = n_f + 1, ..., m$, $\mathbf{e}_k^f = \mathbf{0}$;

(2): for any rational numerals $\mathbf{r}_1, \mathbf{r}_2$,

$$\forall x \forall y (\neg \mathbf{f}(\mathbf{r}_1) = \mathbf{0} \wedge \neg \mathbf{f}(\mathbf{r}_2) = \mathbf{0} \wedge [v_f(\mathbf{r}_1) - v_f(\mathbf{r}_2) = 1] \wedge$$
$$\mathbf{f}(x) = \mathbf{0} \wedge \mathbf{f}(y) = \mathbf{0} \wedge (\mathbf{r}_1 < x < \mathbf{r}_2) \wedge (\mathbf{r}_1 < x < \mathbf{r}_2) \supset x = y).$$

Sturm (1) means that for any $k = 1, ..., n_f$, symbol $\mathbf{e}_k^f$ is a roof of $\mathbf{f}(x) = \mathbf{0}$ and is located in the k-th position of the roots of $\mathbf{f}(x) = \mathbf{0}$, that $\mathbf{e}_k^f$ is fixed to be $\mathbf{0}$ if $k > n_f$. Sturm (2) means that the root located in the k-th position is unique whenever it exists. Lemma 7.1 guarantees that the instances of Sturm (2) are meaningful under the ordered field axioms Φ_{of}. Hence Sturm (1) and (2) determine the k-th root to be $\mathbf{e}_k^f$ under Φ_{of}. We denote the union of Φ_{of} and the set of all instances of Sturm (1) and (2) by Φ_{ofst}, and call $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{ofst}})$ the *ordered field theory with the Sturm axiom.*

The theory $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{ofst}})$ is properly weaker than $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{rcfr}})$, since every instance of Sturm is provable in $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{rcfr}})$ but conversely since no instance of Real-Closedness is provable in $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{ofst}})$. The structures $(\mathsf{A}; +, -, \times, / ; \geq; \psi)$ and $(\mathsf{R}; +, -, \times, / ; \geq; \psi)$ are models of $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{ofst}})$ as well as of $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{rcfr}})$. However, we can construct some models of $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{ofst}})$ in which no instance of Real-Closedness is true.

For the new theory $(\mathcal{P}^{\text{f}}(\mathcal{L}_{\text{ofr}}), \Phi_{\text{ofst}})$, the following form of decidability holds. Since the proof of this theorem needs proof-theoretic as well as model-theoretic arguments, we leave its proof to a discussion paper (Kaneko [8]).

Theorem 7.2 (Decidability for $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{ofst}})$). Let A be a universal or existential (i.e., it is expressed as $\forall x_1 ... \forall x_m B(x_1, ..., x_m)$ or $\exists x_1 ... \exists x_m B(x_1, ..., x_m)$ with no more quantifiers) closed formula in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}})$. Then $\Phi_{\mathrm{ofst}} \vdash_0 A$ or $\Phi_{\mathrm{ofst}} \vdash_0 \neg A$.

Theorem 7.2 states decidability only for universal or existential closed formulae. In this sense, Theorem 7.2 is weaker than the corresponding theorem (Theorem 6.2) for $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{rcfr}})$. Nevertheless, this theorem is sufficient for the consideration of the mere and specific existence of a Nash equilibrium. For example, it follows from Theorem 7.2 and the soundness theorem for classical logic that

$$\Phi_{\mathrm{ofst}} \vdash_0 \exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}), \tag{7.8}$$

since $\exists \overrightarrow{x} \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x})$ is true in the model $(\mathsf{A}; +, -, \times, / \ ; \geq; \psi)$. Then the question is whether this implies the existence of a profile $\overrightarrow{\mathrm{e}}$ of algebraic numerals such that $\Phi_{\mathrm{ofst}} \vdash_0 \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{\mathrm{e}})$. We have the affirmative answer to this question using Theorem 7.2, whose proof is essentially the same as the proof of Theorem 6.3.

Theorem 7.3 (Term-Existence for $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{ofst}})$). Let $\exists x_1 ... \exists x_m A(x_1, ..., x_m)$ be an existential closed formula in $\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}})$. Then if $\Phi_{\mathrm{ofst}} \vdash_0 \exists x_1 ... \exists x_m A(x_1, ..., x_m)$, then there are algebraic numerals $\mathrm{e}_1, ..., \mathrm{e}_m$ such that $\Phi_{\mathrm{ofst}} \vdash_0 A(\mathrm{e}_1, ..., \mathrm{e}_m)$.

Once these are proved, we obtain the result parallel to Theorem 6.A.

Theorem 7.A. Let $\mathcal{L}$ be $\mathcal{L}_{\mathrm{ofr}}$. Let g be any finite game. Then the following hold:

(1): $C(\Phi_{\mathrm{ofst}}) \vdash_\omega \exists \overrightarrow{x} C(\mathrm{Nash}_{\mathrm{g}}(\overrightarrow{x}))$;

(2): there is a profile $\overrightarrow{\mathrm{e}}$ of algebraic numerals such that $\Phi_{\mathrm{ofst}} \vdash_0 \mathrm{Nash}_{\mathrm{g}}(\overrightarrow{\mathrm{e}})$.

Thus, as far as the mere and specific existence of a Nash equilibrium is concerned, the ordered field theory, $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{ofst}})$, with the Sturm axiom fulfills all the desiderata discussed in Section 5. Hence the four arithmetic operations together with the axioms determining the real algebraic numbers suffices for our purpose of the consideration of the playability of a game.

Nevertheless, the theory $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{ofst}})$ is, probably, insufficient for the consideration of the specific existence for perfect equilibrium, since $\mathrm{Selten}_{\mathrm{g}}(\overrightarrow{a})$ itself involves some quantifiers – Lemma 3.1 does not hold for $\mathrm{Selten}_{\mathrm{g}}(\overrightarrow{a})$ and Theorem 7.2 is applied to existential and universal formulae. On the other hand, $(\mathcal{P}^{\mathrm{f}}(\mathcal{L}_{\mathrm{ofr}}), \Phi_{\mathrm{ofst}})$ suffices for Nash's bargaining theory.

8. Conclusion

We considered the problem of the playability of a finite game with mixed strategies. The epistemic axiomatization of final decision predictions leads to the requirement that existence is obtained in closed terms. Closed terms form permissible computation units in a mathematical theory. Each player can think of the game theory in an abstract manner, but to play his game, he is required to obtain his prediction expressed in terms of permissible computation units. In this sense, this specific existence is a language-dependent, while abstract logical statements are language-independent as far as they are expressed as formulae and appropriate axioms are given. Then Theorem 4.A stated that if some axioms are properly specified to determine computational units, then any other additional mathematical axioms are superfluous. From this result, the general version of the undecidability of Kaneko-Nagashima [9] was obtained. In Section 5, then we discussed the computations involved in playing a game, and provided the desiderata for our mathematical theory.

In Sections 6 and 7, we gave two theories satisfying the desiderata. The real closed field theory, $(\mathcal{P}^f(\mathcal{L}_{\rm ofr}), \Phi_{\rm rcfr})$, with real algebraic numbers given in Section 6 is complete – Theorem 6.2 –, and works for our purpose as well as for the other purposes such as the consideration of refinements. The ordered field theory $(\mathcal{P}^f(\mathcal{L}_{\rm ofr}), \Phi_{\rm ofst})$ with the Sturm axiom given in Section 7 is weaker than $(\mathcal{P}^f(\mathcal{L}_{\rm ofr}), \Phi_{\rm rcfr})$, but works for the consideration of Nash equilibrium. In both theories, the undecidability of Kaneko-Nagashima [12] is completely removed. Although the construction of the latter is more complicated, the second is purely constructive in the sense that it allows only the four arithmetic operations and the axioms determining real algebraic numbers.

Overall, our answer to the question of the playability of a game fulfills the desiderata discussed in Section 5. In this sense, our answer may be regarded as affirmative. Nevertheless, if we look at the details of our constructions of the theories, we would find that our theories may require tremendous numbers of steps for calculations involved. Returning to the epistemic consideration of a game, these calculations are required for each player. From this point of view, our answer should be said to be affirmative under no constraints on complexities of calculations. To have a better understanding of our problem, we would need more theoretical developments on complexities of logical and mathematical calculations. This remains open.

Appendix

Proof of Lemma 2.4. First, we assign a rational number $\eta(t)$ to every closed term t by induction on the structure of a term: i) $\eta(\mathbf{0}) = 0$ and $\eta(\mathbf{1}) = 1$; and if rational numbers $\eta(t_1)$ and $\eta(t_2)$ are already assigned to t_1 and t_2, then $\eta(t_1+t_2) = \eta(t_1)+\eta(t_2)$, $\eta(t_1-t_2) = \eta(t_1)-\eta(t_2)$, $\eta(t_1\cdot t_2) = \eta(t_1)\times\eta(t_2)$, and $\eta(t_1/t_2) = \eta(t_1)/\eta(t_2)$ if $\eta(t_2) \neq 0$, and $\eta(t_1/t_2) = 0$ if $\eta(t_2) = 0$.

Second, we will show, by induction on the structure of a term, that for any integer $m > 0$ and k, if $\eta(t) = k/m$, then $\Phi_{\rm of} \vdash_0 t = [k/m]$, which is Lemma 2.4.(1). For $\mathbf{0}$ and $\mathbf{1}$, we have $\Phi_{\rm of} \vdash_0 \mathbf{0} = [0/m]$ and $\Phi_{\rm of} \vdash_0 \mathbf{1} = [1/1]$. Consider a term $t_1 + t_2$

with $\eta(t_1+t_2) = k/m$ (k/m are irreducible). We make the induction hypothesis that if $\eta(t_1) = k_1/m_1$ and $\eta(t_2) = k_2/m_2$ ($m_1, m_2 > 0$ and $k_1/m_1, k_2/m_2$ are irreducible), then $\Phi_{of} \vdash_0 t_1 = [k_1/m_1]$ and $\Phi_{of} \vdash_0 t_2 = [k_2/m_2]$. It follows from this hypothesis that $\Phi_{of} \vdash_0 t_1 + t_2 = [k_1/m_1] + [k_2/m_2] = [k_1m_2 + k_2m_1]/[m_1m_2]$. From this, we have $\Phi_{of} \vdash_0 t_1 + t_2 = [k]/[m]$ if $k/m = \eta(t_1+t_2) = \eta(t_1) + \eta(t_2) = (k_1m_2 + k_2m_1)/m_1m_2$ (k/m is mutually irreducible).

In the cases of the other function symbols $-, \cdot$ and $/$, we can prove the assertion in similar manners.

From the above result, we have $\Phi_{of} \vdash_0 t_1 - t_2 = [k]/[m]$ if $\eta(t_1 - t_2) = k/m$ for integers k and $m > 0$. Thus $k \geq 0$ implies $\Phi_{of} \vdash_0 [k]/[m] \geq 0$, and $k < 0$ implies $\Phi_{of} \vdash_0 \neg[k]/[m] \geq 0$. This is equivalent to that $\eta(t_1) \geq \eta(t_2)$ implies $\Phi_{of} \vdash_0 t_1 \geq t_2$, and $\eta(t_1) < \eta(t_2)$ implies $\Phi_{of} \vdash_0 \neg(t_1 \geq t_2)$. Thus $\Phi_{of} \vdash_0 t_1 \geq t_2$ or $\Phi_{of} \vdash_0 \neg(t_1 \geq t_2)$. This is the second assertion. □

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