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Dynamic Pricing Policy in a Buying Problem

by

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1 Introduction

Consider a problem of purchasing a fixed number of items where sellers of the items must be searched by paying some search cost. The objective of the buyer is to minimize the total expected present discounted pruchasing cost. This type of problem is related to the classical optimal stopping problem [1-9] where a buying price is offered by a seller instead a buyer. In our model, we assume that a buying price is offered by the buyer and each seller has a minimum permissible selling price, which is a random variable.

In the problem, two decisions are encompassed: 1. a searching rule, prescribing whether or not to search for a buyer, and 2. a pricing rule, prescribing how much should be offered to an appearing seller. The pricing rule must be determined with care since low pricing policy may cause loss of some deals with found sellers even though it will result in low total purchasing cost. Therefore, in this paper we present a model for such problem and examine its optimal decision rule so as to attain the above objective.

In addition, this problem can be applied to other related problem such as a problem of searching for a fixed number of works, engineers, specialists and so on.

2 Model

Consider the following discrete time sequential stochastic decision process with a finite planning horizon. First, for convenience, let points of time be numbered backward starting from the final point of time of the planning horizon, time 0, as 0, 1, and so on. Let an interval between two successive points of time, say time t and $t-1$, be called the period t.

Suppose there exist a certain number of items to be bought. Assume that if a fixed cost $c > 0$ (search cost) is paid at the beginning of a period, a seller can be found at the next point in time. Assume that each buyer buys no more than one item. Let w denote the minimum permissible

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Suppose there exist a certain number of items to be bought. Assume that if a fixed cost $c > 0$ (search cost) is paid at the beginning of a period, a seller can be found at the next point in time. Assume that each buyer buys no more than one item. Let w denote the minimum permissible selling price of a seller; that is, the seller is willing to sell an item if and only if the price offered for the item by buyer is buyer than w. Here, let w be a random variable having a known continuous distribution function $F(w)$ with a finite expectation μ, which is defined in the same ways as in the selling problem. Then, if the decision maker offers a price y to an appearing seller, the seller sells the item with probability

$$q(y) = F(y). \tag{2.1}$$

which $q(y)$ is strict increasing for $a \leq y \leq b$ with $q(y) = 0$ for $y \leq a$, $0 < q(y) < 1$ for $a < y < b$, and $q(y) = 1$ for $b \leq y$. Moreover, assume that if i items have not still been purchased at the time 0, they can be purchased from a dealer at a predetermined contract price $\gamma(i)$, which is nondecreasing and convex in i with $\gamma(0) = 0$ and $\gamma(i) > a$.

The objective here is to find the optimal decision rule to minimize the total expected present discounted net cost over the planning horizon, the expectation of the total expected present discounted value of cost paid at each point in time plus the total expected present discounted value of search costs paid.

The decision policy of this model consists of the following two rules: The *search rule* prescribing whether or not to search for a buyer at the beginning of every period and the *pricing rule* prescribing how much should be proposed to a buyer when found.

3 Preliminaries

In this section we introduce the following functions, which will be used in the subsequent sections. For any real number ν let us define

$$S(\nu) = \min_{y} q(y)(y - \nu). \tag{3.1}$$

Let $y(\nu)$ be the largest ν attaining the minimum $S(\nu)$ and τ^* be the solution of the equation $S(\nu) + c = 0$.

Lemma 3. 1

(a) 1. *$S(\nu) \leq 0$ for all ν,*

2. *$S(\nu) + \nu \geq 0$ for $\nu \geq 0$.*

(b) *$y(\nu) \leq b$ for all ν.*

(c) 1. *if $\nu > a$, then $S(\nu) < 0$ and $a < y(\nu) < \nu$,*

2. *If $\nu \leq a$, then $S(\nu) = 0$ and $y(\nu) = a$.*

(d) 1. *$S(\nu)$ is nonincreasing in ν,*

2. *$S(\nu)$ is strictly decreasing for $\nu > a$,*

3. *$S(\nu) + \nu$ is nondecreasing in ν.*

(e) *$y(\nu)$ is nondecreasing in ν.*

(f) *If $\nu_1 \geq \nu_2$, then* $\min\{S(\nu_1) + c, 0\} - \min\{S(\nu_2) + c, 0\} \geq \nu_2 - \nu_1$.

PROOF

(a1) $S(\nu) \leq q(a)(a - \nu) = 0$ for all ν due to $q(a) = 0$.

(a2) Immediate from the fact that $q(y)(y - \nu) + \nu = q(y)y + (1 - q(y))\nu \geq 0$ for all ν and y.

(b) Obviously if $y \geq b$, then $q(y)(y - \nu) = y - b \geq b - \nu = q(b)(b - \nu)$ due to $q(b) = 1$, hence it must be $y(\nu) \leq b$.

(c1) If $\nu > a$, then $q(y)(y - \nu) = 0$ for $y \leq a$, $q(y)(y - \nu) < 0$ for $a < y < \nu$, and $q(y)(y - \nu) \geq 0$ for $y \geq \nu$, hence $S(\nu) < 0$ and $a < y(\nu) < \nu$.

(c2) If $\nu \leq a$, then $q(y)(y - \nu) = 0$ for $y \leq a$ and $q(y)(y - \nu) > 0$ for $y > a$, hence $S(\nu) = 0$ and $y(\nu) = a$ from the definition of $y(\nu)$.

(d1) Obvious from the fact that $q(y)(y - \nu)$ is nonincreasing in ν for all y.

(d2) Suppose $a < \nu < \nu'$, then we have $q(y(\nu)) > 0$ due to $y(\nu) > a$ from (c1). Hence, we have

$$
\begin{aligned}
S(\nu) &= q(y(\nu))y(\nu) - q(y(\nu))\nu \\
&> q(y(\nu))y(\nu) - q(y(\nu))\nu' \\
&= q(y(\nu))(y(\nu) - \nu') \geq S(\nu').
\end{aligned}
$$

(d3) Immediate from the fact that $q(y)(y - \nu) + \nu$ ($= q(y)y + (1 - q(y))\nu$) is nondecreasing in ν for all y.

(e) For any $\xi > 0$ we have

$$
\begin{aligned}
S(\nu + \xi) &= \min_y q(y)(y - (\nu + \xi)) \\
&= q(y(\nu + \xi))(y(\nu + \xi) - (\nu + \xi)) \\
&= q(y(\nu + \xi))(y(\nu + \xi) - \nu) - q(y(\nu + \xi))\xi \\
&\geq S(\nu) - q(y(\nu + \xi))\xi \\
&= q(y(\nu))(y(\nu) - \nu) - q(y(\nu + \xi))\xi \\
&= q(y(\nu))(y(\nu) - (\nu + \xi)) + \xi(q(y(\nu)) - q(y(\nu + \xi))).
\end{aligned}
$$

Since $q(y(\nu))(y(\nu) - (\nu + \xi)) \geq S(\nu + \xi)$, eventually we have

$$
S(\nu + \xi) \geq S(\nu + \xi) + \xi(q(y(\nu)) - q(y(\nu + \xi))),
$$

from which we have $q(y(\nu))-q(y(\nu+\xi)) \leq 0$, that is, $q(y(\nu)) \leq q(y(\nu+\xi))$, therefore, $y(\nu) \leq y(\nu+\xi)$ from $a \leq y(\nu) \leq b$ and $a \leq y(\nu+\xi) < b$ due to (b) and (c) and the assumption that $q(y)$ is strictly increasing in $a \leq y \leq b$.

(f) Noting the formula $\min\{a_1, b_1\} - \min\{a_2, b_2\} \geq \min\{a_1 - a_2, b_1 - b_2\}$, for any real number $\nu_1 \geq \nu_2$ we have

$$\min\{S(\nu_1) + c, 0\} - \min\{S(\nu_2) + c, 0\} \geq \min\{S(\nu_1) - S(\nu_2), 0\}.$$

Here, $S(\nu_1) - S(\nu_2) \geq \nu_2 - \nu_1$ due to $S(\nu_1) + \nu_1 \geq S(\nu_2) + \nu_2$ for $\nu_1 \geq \nu_2$ from (d3), hence the result follows. ■

Lemma 3. 2 *τ^* uniquely exist with $a < \tau^* \leq b + c$.*

PROOF $S(\nu)$ is continuous and strictly decreasing for $\nu > a$ from Lemma 3. 1 (d2) with $S(a)+c = c > 0$ and $S(b+c)+c = \min_y q(y)(y-b-c) \leq q(b)(b-b-c)+c = 0$ due to $q(b) = 1$. Hence, the statement holds. ■

4 Optimal Equation

Let N be the total number of units should be purchased. Let $v_t(i)$ be the minimum total expected present discounted purchasing cost when i units should be purchased and t periods remain until the end of the planning horizon.

$$v_0(i) = \gamma(i), \quad i \geq 0, \tag{4.1}$$

$$v_t(0) = \beta v_{t-1}(0) - Nh, \quad t \geq 0, \tag{4.2}$$

$$v_t(i) = \min \left\{ \begin{array}{ll} S: & \min_y\{q(y)(y + \beta v_{t-1}(i-1) + (N-i+1)h) \\ & \quad + (1-q(y))(\beta v_{t-1}(i) + (N-i)h)\} + c, \\ P: & \beta v_{t-1}(i) + (N-i)h \end{array} \right\} t \geq 1,\ i \geq 1 \tag{4.3}$$

where S and P means, respectively, "Search" and "Do not search". Now, let us denote

$$z_t(i) = \beta(v_t(i) - v_t(i-1)), \quad i \geq 1, \quad t \geq 0 \tag{4.4}$$

Then, $v_t(i)$ can be rewritten as follow:

$$v_t(i) = \beta v_{t-1}(i) + (N-i)h + \min\{S(z_{t-1}(i) - h) + c, 0\}, \quad i \geq 1, \quad t \geq 1. \tag{4.5}$$

For convenience, let $z_t(0) = 0$ for all $t \geq 0$, so $S(z_t(0)) = 0$ from Lemma 3. 1 (c2). By definition it follows that Eq. (4.5) also holds for $i = 0$. Then, from Eqs. (4.4) and (4.5) we have

$$z_t(i) = \beta(z_{t-1}(i) - h) + \beta(\min\{S(z_{t-1}(i) - h) + c, 0\} - \min\{S(z_{t-1}(i-1) - h) + c, 0\}), \quad i \geq 1. \quad t \geq 1. \tag{4.6}$$

Theorem 4. 1 *We have*

(a) $z_t(i)$ *is nondecreasing in* i *for all* t.

(b) $z_t(i)$ *is nonincreasing in* t *for* $i \geq 1$.

PROOF

(a) The assertion for $t = 0$ is obvious from the fact that $z_0(i) - z_0(i-1) = \beta(\Delta\gamma(i) - \Delta\gamma(i-1)) \geq 0$ by definition. Assume that it holds for $t-1$. Then, from Eq. (4.6) we have

$$\begin{aligned} z_t(i) - z_t(i-1) &= \beta(z_{t-1}(i) - z_{t-1}(i-1)) \\ &+ \beta(\min\{S(z_{t-1}(i) - h) + c, 0\} - \min\{S(z_{t-1}(i-1) - h) + c, 0\}) \\ &+ \beta(\min\{S(z_{t-1}(i-2) - h) + c, 0\} - \min\{S(z_{t-1}(i-1) - h) + c, 0\}). \end{aligned}$$

Since from Lemma 3. 1 (f) we have

$$\min\{S(z_{t-1}(i) - h) + c, 0\} - \min\{S(z_{t-1}(i-1) - h) + c, 0\} \geq z_{t-1}(i-1) - z_{t-1}(i)$$

and from Lemma 3. 1 (d1) we have

$$\min\{S(z_{t-1}(i-2) - h) + c, 0\} - \min\{S(z_{t-1}(i-1) - h) + c, 0\} \begin{cases} = -\min\{S(z_{t-1}(1) - h) + c, 0\} \geq 0, & i = 2 \\ \geq 0, & i \geq 3, \end{cases}$$

it follows that

$$z_t(i) - z_t(i-1) \geq \beta(z_{t-1}(i) - z_{t-1}(i-1)) + \beta(z_{t-1}(i-1) - z_{t-1}(i)) = 0.$$

Therefore, the assertion is proved.

(b) From Eq. (4.6) we have

$$z_1(1) - z_0(1) = (\beta - 1)z_0(1) - \beta h + \beta \min\{S(z_0(1) - h) + c, 0\} \leq 0.$$

For $i \geq 2$, since

$$z_0(i) - h = \beta\Delta\gamma(i) - h \geq \beta\Delta\gamma(i-1) - h = z_0(i-1) - h$$

by definition, we have

$$\min\{S(z_0(i)-h)+c,0\}-\min\{S(z_0(i-1)-h)+c,0\}\le 0$$

from Lemma 3. 1 (d), hence

$$\begin{aligned} z_1(i)-z_0(i) &= (\beta-1)z_0(i)-\beta h \\ &\quad +\beta(\min\{S(z_0(i)-h)+c,0\}-\min\{S(z_0(i-1)-h)+c,0\}) \\ &\le (\beta-1)z_0(i)-\beta h\le 0. \end{aligned}$$

Assume that $z_{t-1}(i)\le z_{t-2}(i)$. From Eq. (4.6) we have

$$\begin{aligned} z_t(i)-z_{t-1}(i) &= \beta(z_{t-1}(i)-z_{t-2}(i)) \\ &\quad +\beta(\min\{S(z_{t-1}(i)-h)+c,0\}-\min\{S(z_{t-2}(i)-h)+c,0\}) \\ &\quad +\beta(\min\{S(z_{t-2}(i-1)-h)+c,0\}-\min\{S(z_{t-1}(i-1)-h)+c,0\}). \end{aligned}$$

Since from Lemma 3. 1 (d1) we have

$$\min\{S(z_{t-2}(i-1)-h)+c,0\}-\min\{S(z_{t-1}(i-1)-h)+c,0\}\le 0$$

and from Lemma 3. 1 (f) we have

$$\min\{S(z_{t-1}(i)-h)+c,0\}-\min\{S(z_{t-2}(i)-h)+c,0\}\le z_{t-2}(i)-z_{t-1}(i),$$

it follows that

$$z_t(i)-z_{t-1}(i)\le \beta(z_{t-1}(i)-z_{t-2}(i))-\beta(z_{t-2}(i)-z_{t-1}(i))=0. \quad \blacksquare$$

5 Optimal Prices

Let $y_t(i)$ be the optimal purchasing price when t periods of time remain and i units should be purchased, which is given by $y(z_{t-1}(i)-h)$.

Theorem 5. 1

(a) $a\le y_t(i)\le b$.

(b) $y_t(i)$ *is nondecreasing in* i *for all* t,

(c) $y_t(i)$ *is nonincreasing in* t *for* $i\ge 1$.

PROOF (a) Immediately from Lemma 3. 1 (b) and (c).

(b) Clearly from Lemma 3. 1 (e) and Theorem 4. 1 (a).

(c) Clearly from Lemma 3. 1 (e) and Theorem 4. 1 (b). ■

5.1 Optimal Search

From Eq. (4.5) we see that if $S(z_{t-1}(i)-h)+c>0$ for any given i, it is optimal not to search at the beginning of period t, or else to search. Then, from Eq. (4.5) , it is optimal not to search for $z_{t-1}(i)<\tau^*+h$, and to search for $z_{t-1}(i)\geq\tau^*+h$.

Theorem 5. 2

(a) *If $\beta\Delta\gamma(i)<\tau^*$ for a given i, then $z_t(i)<\tau^*$ for all t,*

(b) 1. *$z_t(i)$ converges to a finite number $z(i)$ as t tends to ∞,*

2. *if $\beta<1$, then $z(i)\leq 0$,*

(c) *If $\beta=1$, $h=0$, and $\Delta\gamma(i)\geq\tau^*$ for any given i, then $z_t(i)\geq\tau^*$ for all t.*

PROOF

(a) Clear from the fact that $z_0(i)=\beta\Delta\gamma(i)<\tau^*$ and $z_t(i)$ is nonincreasing in t from Lemma 4. 1 (b).

(b1) It is clear that $z_0(i)=\beta\Delta\gamma(i)\geq-\beta\sum_{n=0}^{0}\beta^n h$. Assume that $z_{t-1}(i)\geq-\beta\sum_{n=0}^{t-1}\beta^n h$. Since $z_{t-1}(i)-h\geq z_{t-1}(i-1)-h$, from Theorem 4. 1 (a) and Lemma 3. 1 (f) we have

$$\min\{S(z_{t-1}(i)-h)+c,0\}-\min\{S(z_{t-1}(i-1)-h)+c,0\}\geq\beta(z_{t-1}(i-1)-z_{t-1}(i)),$$

hence, from Eq. (4.6) we have

$$\begin{aligned}z_t(i)&\geq\beta(z_{t-1}(i)-h)+\beta(z_{t-1}(i-1)-z_{t-1}(i))\\&=\beta(z_{t-1}(i-1)-h)\\&\geq-\beta\sum_{n=0}^{t}\beta^n h>-h\beta/(1-\beta).\end{aligned}$$

Therefore, $z_t(i)$ is lower bounded in t and i. Then, since $z_t(i)$ is nonincreasing in t for all i from Theorem 4. 1 (b), it follows that $z_t(i)$ converges to a finite number $z(i)$ as t tends to ∞.

(b2) Since $\min\{S(z(i)-h)+c,0\}-\min\{S(z(i-1)-h)+c,0\}\leq 0$ from Theorem 4. 1 (a) and Lemma 3. 1 (d1), from Eq. (4.6) we have $z(i)\leq\beta(z(i)-h)$, so $z(i)\leq-h/(1-\beta)\leq 0$.

(c) Clearly $z_0(i)=\Delta\gamma(i)\geq\tau^*$, hence, the assertion holds for $t=0$. Assume that it holds true for $t-1$. Since $-\min\{S(z_{t-1}(i-1))+c,0\}\geq 0$, from Eq. (4.6) and Lemma 3. 1 (d3) we have

$$\begin{aligned}z_t(i)&\geq z_{t-1}(i)+\min\{S(z_{t-1}(i))+c,0\}\\&=\min\{S(z_{t-1}(i))+z_{t-1}(i)+c,z_{t-1}(i)\}\\&\geq\min\{S(\tau^*)+\tau^*+c,z_{t-1}(i)\}=\tau^*.\quad\blacksquare\end{aligned}$$

Theorem 5. 2 (a), (b) and (c) imply, respectively, the followings :

(a) If $c > 0$ and $\Delta\gamma(i) < r^*$ for a given i, it is always optimal not to search when there are i units need to be purchased.

(b) For a given i, there exists a set, $\{t(i)\}$, such that a search cost is invested for $t \leq t(i)$, and is not invested for $t > t(i)$. (See Figure 1(b))

(c) If $\beta = 1$, $h = 0$ and $\Delta\gamma(i) \geq r^*$ for any given i, it is always optimal to search when there are i units need to be purchased.

6 Summary of Conclusion

The conclusions that were obtained in the previous sections can be summarized as follows:

(a) Suppose $\Delta\gamma(i)$ is nondecreasing in i (convex). Then

1. $x_t(i)$ is nondecreasing in i for all t,
2. $x_t(i)$ is nonincreasing in t for all i,

(b) Suppose $\Delta\gamma(i)$ is nonincreasing in i (concave). In this case, nothing can be said as to the monotonicity of $x_t(i)$ in not only t but also i. Indeed, we can demonstrate a case such that $x_t(i)$ is nonmonotone in t and i (Figuer 6.1).

Now, in general, it can be conjectured that the optimal selling price $x_t(i)$ is nondecreasing with the remaining periods t and nonincreasing with the number of remaining items on hand. However, the numerical example below shows that this does not always hold true.

Numerical examples: Let $\beta = 1.00$, $c = 0.02$, and $h = 0$. Let $F(w)$ be union distribution on $[0, 1]$, $\Delta\gamma(i) = \Delta\gamma(i-1) - 0.0025$ with $\Delta\gamma(1) = 1.00$. Then, Figuer 6.1 shows that $x_t(i)$ is not monotone in t and i.

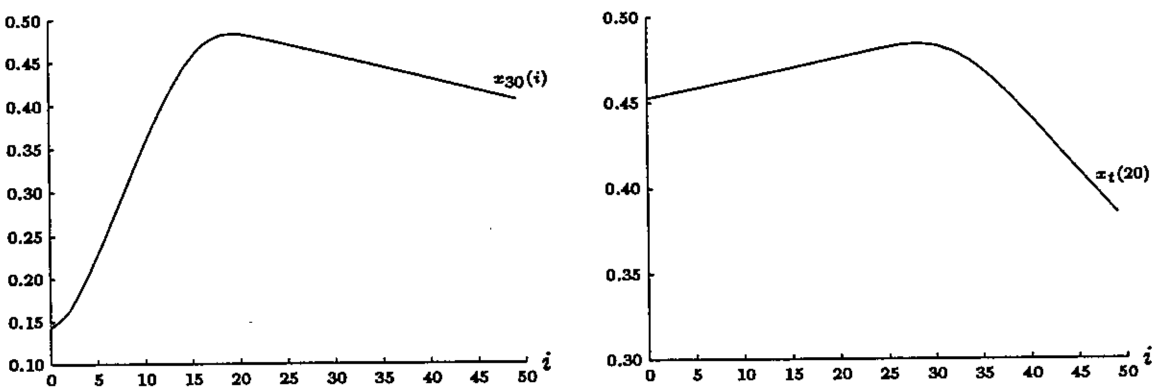

Figure 6.1: concave

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