No. 735

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by

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June 1997

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Abstract In a well-known asset selling problem [4] as the optimal stopping problem, conventionally, a buyer has been assumed to offere a buying price. In this paper, contrary, a seller is assumed to offer a selling price.

1 Introduction

Suppose that there exist some homogeneous items that are to be sold out by a given deadline. If we progress with a contingency that those items remaining unsold at the deadline will be accepted by a dealer who is willing to buy all of them. The seller desires naturally enough to obtain as large a total revenue as possible, so he will try to find other buyers by paying a search cost. In this situation we ask ourselves, what is the best selling policy to maximize the total expected revenue ? This type of problem has been investigated over the years using different models of optimal stopping problems [1-12]. In all of them, the selling price is assumed to be that proposed by a buyer rather than a seller. In our paper we assume that the selling price is proposed by the seller.

A typical example of the problem is the following housing sale problem. Suppose that a real estate agent has a deadline to sell a certain number of houses by a certain date and the price is offered by the agent after a buyer has been found. The seller has by this time incurred some cost, say in advertising, for example. If some houses remain unsold at the deadline, the seller will sell all of them at a contracted price with the dealer. In this case, if the agent proposed too higher a price, the buyers will of course refuse while if too low a price is offered, the revenue will be so small as to render the deal useless. This problem of formulating a selling policy to maximize the total expected present discounted revenue by taking the number of unsold units and remaining periods of time into account, is the problem under consideration.

Other pricing problems such as in airline seat tickets with a fixed departure date, perishable items remaining unsold, and ecomomic value deteriorating products, say personal computers, fashionable clothes and so on carry with them, similar problems. The purpose of this paper is to provide a general model for such problems and examine the properties of the optimal decision policy.

2 Model

Consider the following discrete time sequential stochastic decision process with a finite planning horizon. First, for convenience, let points of time be numbered backward starting from the final point of time of the planning horizon time, 0, as 0, 1, and so on. Let an interval between two successive points of time, say time t and $t-1$, be called the period t.

Suppose there exist a certain number of items to be sold within a given planning horizon. Assume that if a fixed cost $c > 0$ (search cost) is paid at the beginning of a period, a buyer can be found at the next point in time. Assume that each buyer buys no more than one item. Let w denote the maximum permissible buying price of a buyer; that is, the buyer is willing to buy an item if and only if the selling price offered for the item is lower or equal to w. Here, let w be a random variable having a known continuous distribution function $F(w)$ with a finite expectation μ. Let $F(w)$ be strictly increasing for $a \leq w \leq b$ and assume that for a given a and b such that $0 < a < b < \infty$, $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \leq w < b$, and $F(w) = 1$ for $b \leq w$. Then, the probability that a buyer will buy the item, provided that a price x is obtained, can be expressed as

$$p(x) = 1 - F(x). \tag{2.1}$$

Here, $p(x) = 1$ for $x < a$, $0 < p(x) < 1$ for $a \leq x < b$, and $p(x) = 0$ for $b \leq x$. If i items remain unsold at time 0 (the deadline), they can be disposed of at a price $\alpha(i)$, which is nondecreasing and concave in i with $\alpha(0) = 0$. Let

$$\Delta\alpha(i) = \alpha(i) - \alpha(i-1) < b, \quad i \geq 1. \tag{2.2}$$

Furthermore, let $\beta \in (0, 1]$ be a per-period discount factor. Finally, we assume

$$\max_x p(x)x > c \tag{2.3}$$

which is a realistic assumption implying that the maximum expected revenue from dealing with a buyer is larger than the search cost paid.

The objective here is to find the optimal decision rule maximizing the total expected present discounted net profit over the planning horizon, the expectation of the total expected present discounted value of revenues obtained at each point in time minus the total expected present discounted value of search costs paid.

The decision policy of this model consists of the following two rules: The *search rule* prescribing whether or not to search for a buyer at the beginning of every period and the *pricing rule* prescribing how much should be proposed to a buyer when found.

3 Preliminaries

In this section we introduce the two functions, defined below, which will be used in the subsequent sections. For any real number ν and $\beta \in (0, 1)$ let us define

$$T(\nu) = \max_x p(x)(x - \nu), \tag{3.1}$$

$$K(\nu) = \beta \max\{T(\nu) - c, 0\} - (1 - \beta)\nu. \tag{3.2}$$

Let $x(\nu)$ denote the smallest x attaining the maximum $T(\nu)$, and let ν^* and h^* be the solutions of the equations, respectively, $T(\nu) - c = 0$ and $K(\nu) = 0$. The unique existences of ν^* and h^* will be proved in the lemmas below.

Lemma 3. 1 *we have*

(a) $a \le x(\nu)$ *for all* ν,

(b) 1. If $\nu < b$, then $T(\nu) > 0$ and $\nu < x(\nu) < b$,

2. if $\nu \ge b$, then $T(\nu) = 0$ and $x(\nu) = b$,

(c) 1. $T(\nu)$ is nonincreasing in ν,

2. $T(\nu)$ is strictly decreasing for $\nu \le b$,

3. $T(\nu) + \nu$ is nondecreasing in ν,

4. $T(\nu) + \nu$ is strictly increasing in $\nu > a$,

(d) $x(\nu)$ is nondecreasing in ν,

(e) If $\nu_1 \le \nu_2$, then $\max\{T(\nu_1) - c, 0\} - \max\{T(\nu_2) - c, 0\} \le \nu_2 - \nu_1$,

(f) If $\beta < 1$, $K(\nu)$ is strictly decreasing in ν.

PROOF (a) Clearly we have $p(x)(x-\nu) = x - \nu$ for $0 \le x \le a$, which is strictly increasing in x. Hence, it must be $a \le x(\nu)$.

(b1) If $\nu < b$, then $p(x)(x - \nu) \le 0$ for $x \le \nu$, $p(x)(x - \nu) > 0$ for $\nu < x < b$, and $p(x)(x - \nu) = 0$ for $b \le x$, hence $T(\nu) > 0$ and $\nu < x(\nu) < b$.

(b2) If $\nu \ge b$, then $p(x)(x - \nu) < 0$ for $x < b$ and $p(x)(x - \nu) = 0$ for $x \ge b$, hence $T(\nu) = 0$ and $x(\nu) = b$.

(c1) Immediate from the fact that $p(x)(x - \nu)$ is nonincreasing in ν for all x.

(c2) If $\nu' < \nu < b$, then $p(x(\nu)) > 0$ due to $x(\nu) < b$ from (b1). Hence we have

$$\begin{aligned} T(\nu) &= p(x(\nu))x(\nu) - p(x(\nu))\nu \\ &< p(x(\nu))x(\nu) - p(x(\nu))\nu' \\ &= p(x(\nu))(x(\nu) - v') \le T(\nu'). \end{aligned}$$

(c3) Immediate from the fact that $p(x)(x-\nu)+\nu$ $(= p(x)x + (1-p(x))\nu)$ is nondecreasing in ν for all x.

(c4) If $a < \nu < \nu'$, then $p(x(\nu)) < 1$ due to $x(\nu) > a$ from (b). Hence we have

$$\begin{aligned} T(\nu) + \nu &= p(x(\nu))(x(\nu) - \nu) + \nu \\ &= p(x(\nu))x(\nu) + (1 - p(x(\nu)))\nu \\ &< p(x(\nu))x(\nu) + (1 - p(x(\nu)))\nu' \\ &= p(x(\nu))(x(\nu) - \nu') + \nu' \le T(\nu') + \nu'. \end{aligned}$$

(d) For any $\xi > 0$ we have

$$\begin{aligned} T(\nu + \xi) &= \max_x p(x)(x - (\nu + \xi)) \\ &= p(x(\nu + \xi))(x(\nu + \xi) - (\nu + \xi)) \\ &= p(x(\nu + \xi))(x(\nu + \xi) - \nu) - p(x(\nu + \xi))\xi \end{aligned}$$

$$
\begin{aligned}
&\leq T(\nu) - p(x(\nu+\xi))\xi \\
&= p(x(\nu))(x(\nu)-\nu) - p(x(\nu+\xi))\xi \\
&= p(x(\nu))(x(\nu)-(\nu+\xi)) + \xi(p(x(\nu)) - p(x(\nu+\xi))) \\
&\leq T(\nu+\xi) + \xi(p(x(\nu)) - p(x(\nu+\xi))),
\end{aligned}
$$

from which we have $0 \leq p(x(\nu)) - p(x(\nu+\xi))$, that is, $p(x(\nu)) \geq p(x(\nu+\xi))$, therefore $x(\nu) \leq x(\nu+\xi)$ due to $a \leq x(\nu) \leq b$ and $a \leq x(\nu+\xi) \leq b$ from (a) and (b) and the fact that $p(x)$ is strictly decreasing on $a \leq x \leq b$.

(e) Suppose $\nu_1 \leq \nu_2$. Then, we have $T(\nu_1)-T(\nu_2) \leq \nu_2-\nu_1$ due to $T(\nu_1)+\nu_1 \leq T(\nu_2)+\nu_2$ from (c3). From this and by using the general formula $\max\{a_1,b_1\}-\max\{a_2,b_2\} \leq \max\{a_1-a_2, b_1-b_2\}$, it follows that $\max\{T(\nu_1)-c,0\}-\max\{T(\nu_2)-c,0\} \leq \max\{T(\nu_1)-T(\nu_2),0\} = T(\nu_1)-T(\nu_2) \leq \nu_2-\nu_1$ due to $T(\nu_1) \geq T(\nu_2)$ from (c1).

(f) Clearly from (c1). ■

Lemma 3. 2

(a) ν^* *uniquely exists where* $0 < \nu^* < b$.

(b) *If* $\beta < 1$, h^* *uniquely exists where* $0 < h^* < \nu^*$.

PROOF (a) $T(\nu)$ is strictly decreasing on $\nu \leq b$ from Lemma 3. 1 (c2) with $T(b)-c = -c < 0$ from Lemma 3. 1 (b2) and $T(0)-c = \max_x p(x)x > 0$ from the assumption Eq. (2.3) . Hence the statement holds true.

(b) $K(\nu)$ is strictly decreasing in ν for $\beta < 1$ from Lemma 3. 1 (f) with $K(0) = \beta\max\{T(0)-c,0\} = \beta\max_x p(x)x > 0$ and $K(\nu^*) = -(1-\beta)\nu^*$ from (a). Hence, the assertion holds true. ■

4 Optimal Equation

Let $v_t(i)$ be the maximum of the total expected present discounted values of the revenue obtained at each point in time minus the total expected present discounted value of search costs starting from time t with i unsold items. Then, clearly $v_0(i) = \alpha(i)$ for all i, $v_t(0) = 0$ for all t, and for $i \geq 1$ and $t \geq 1$

$$
v_t(i) = \max\left\{\begin{array}{ll} S: & \max_x\{p(x)(x+\beta v_{t-1}(i-1)-(i-1)h) \\ & +(1-p(x))(\beta v_{t-1}(i)-ih)\} - c, \\ N: & \beta v_{t-1}(i) - ih \end{array}\right\} \tag{4.1}
$$

where S and N means, respectively, "Search" and "Don't search" . Now, let

$$
z_t(i) = \beta(v_t(i) - v_t(i-1)), \quad t \geq 0,\ i \geq 1. \tag{4.2}
$$

Then, Eq. (4.1) can be expressed as follows:

$$
v_t(i) = \beta v_{t-1}(i) - ih + \max\{T(z_{t-1}(i)-h)-c, 0\}, \quad i \geq 1, \quad t \geq 1, \tag{4.3}
$$

For convenience of later discussions, we define $z_t(0) = h + b$ for $t \geq 0$, so $\max\{T(z_{t-1}(0) - h) - c, 0\} = 0$ for $t \geq 1$ from Lemma 3. 1 (b2). Therefore, by definition it follows that Eq. (4.3) holds true also for $i = 0$. From Eqs. (4.2) and (4.3) we have

$$\begin{aligned} z_t(i) = \beta(z_{t-1}(i) - h) + \beta(\max\{(T(z_{t-1}(i) - h) - c, 0\} \\ - \max\{T(z_{t-1}(i-1) - h) - c, 0\}), \quad i \geq 1, \quad t \geq 1, \end{aligned} \tag{4.4}$$

5 Optimal Price

The optimal pricing for any given $t \geq 0$ and $i \geq 1$ is given by the smallest x attaining the maximum $T(z_{t-1}(i) - h)$, denoted by $x_t(i) = x(z_{t-1}(i) - h)$. It is generally conjectured that the optimal prices are nondecreasing with the time remaining and nonincreasing with the number of the items remaining. The answers turn out to be affirmative only for the latter, in other words, the optimal price is not always nondecreasin in t. With the aid of the following lemma we will show the statements.

Lemma 5. 1 *We have*

(a) $z_t(i) < b$ *for all* t *and* i,

(b) $z_t(i)$ *is nonincreasing in* i *for all* t,

(c) *Suppose* $\beta < 1$. *Then*

1. if $\alpha(1) \geq h^*/\beta$ and $\Delta\alpha(i) \geq \beta\Delta\alpha(i-1)$ for $i \geq 2$ where $\Delta\alpha(i) = \alpha(i) - \alpha(i-1)$, then $z_t(i)$ is nonincreasing in t for all i,
2. if $h = 0$ and $\beta\alpha(1) \leq h^*$, then $z_t(1)$ is nondecreasing in t,

(d) If either "$h = 0$ and $\beta = 1$" or "$h = 0$ and $\alpha(i) = 0$" for all i, then $z_t(i)$ is nondecreasing in t for $i \geq 1$.

PROOF (a) It is clear from Eq. (4.2) that $z_0(i) = \beta\Delta\alpha(i) < b$ from the assumption. Suppose $z_{t-1}(i) < b$, since $\max\{T(z_{t-1}(i-1)) - c, 0\} \geq 0$ for $i \geq 1$, from Eq. (4.4) for $i \geq 1$ we have

$$\begin{aligned} z_t(i) &\leq \beta(z_{t-1}(i) - h + \max\{T(z_{t-1}(i) - h) - c, 0\}) \\ &= \beta \max\{T(z_{t-1}(i) - h) + z_{t-1}(i) - h - c, z_{t-1}(i) - h\} \\ &< \beta \max\{b - c, b\} = \beta b \leq b \end{aligned}$$

due to $T(z_{t-1}(i) - h) + z_{t-1}(i) - h < T(b) + b = b$ from Lemma 3. 1 (c4).

(b) The assertion for $t = 0$ is clear from the assumption that $\beta\Delta\alpha(i)$ $(= z_0(i))$ is nonincreasing in i. Suppose the assertion holds true for $t - 1$. Then, $z_{t-1}(i)$ is nonincreasing in i. From Eq. (4.4) we have

$$\begin{aligned} &z_t(i) - z_t(i-1) \\ &= \beta(z_{t-1}(i) - z_{t-1}(i-1)) \end{aligned}$$

$$+\ \beta(\max\{T(z_{t-1}(i)-h)-c,0\}-\max\{T(z_{t-1}(i-1)-h)-c,0\})$$
$$+\ \beta(\max\{T(z_{t-1}(i-2)-h)-c,0\}-\max\{T(z_{t-1}(i-1)-h)-c,0\}).$$

Here, note that from Lemma 3. 1 (e)

$$\max\{T(z_{t-1}(i)-h)-c,0\}-\max\{T(z_{t-1}(i-1)-h)-c,0\}\leq z_{t-1}(i-1)-z_{t-1}(i),\quad i\geq 2,$$

and from Lemma 3. 1 (c1)

$$\max\{T(z_{t-1}(i-2)-h)-c,0\}-\max\{T(z_{t-1}(i-1)-h)-c,0\}$$
$$\begin{cases}=-\max\{T(z_{t-1}(1)-h)-c,0\}\leq 0, & i=2,\\ \leq 0, & i\geq 3.\end{cases}$$

Consequently it follows that for $i\geq 2$

$$z_t(i)-z_t(i-1)\leq\beta(z_{t-1}(i))-z_{t-1}(i-1))+\beta(z_{t-1}(i-1)-z_{t-1}(i))=0.$$

(c1) By Eq. (4.4) and Lemma 3. 1 (c3), we have

$$\begin{aligned}z_1(1)&=\beta(z_0(1)-h)+\beta\max\{T(z_0(1)-h)-c,0\}\\&=\beta\max\{T(z_0(1)-h)+z_0(1)-h-c,z_0(1)-h\}\\&\leq\beta\max\{T(z_0(1))+z_0(1)-c,z_0(1)\}\\&=\beta\max\{T(z_0(1))-c,0\}-(1-\beta)z_0(1)+z_0(1)\\&=K(z_0(1))+z_0(1)\\&\leq z_0(1)\end{aligned}$$

since $K(z_0(1))=K(\beta\alpha(1))\leq K(h^*)=0$ from Lemma 3. 1 (f). For $i\geq 2$, since from Lemma 3. 1 (e), we have

$$\max\{T(z_0(i)-h)-c,0\}-\max\{T(z_0(i-1)-h)-c,0\}\leq z_0(i-1)-z_0(i),$$

it follows that

$$z_1(i)\leq\beta(z_0(i)-h)+\beta(z_0(i-1)-z_0(i))=\beta(z_0(i-1)-h)\leq z_0(i)$$

since $z_0(i-1)=\beta\Delta\alpha(i-1)\leq\beta\Delta\alpha(i)=\beta^{-1}z_0(i)$. Assume that $z_{t-1}(i)\leq z_{t-2}(i)$ for $i\geq 1$. From Eq. (4.4) we have

$$\begin{aligned}z_t(i)-z_{t-1}(i)&=\beta(z_{t-1}(i)-z_{t-2}(i))\\&\quad+\ \beta(\max\{T(z_{t-1}(i))-c,0\}-\max\{T(z_{t-2}(i))-c,0\})\\&\quad+\ \beta(\max\{T(z_{t-2}(i-1))-c,0\}-\max\{T(z_{t-1}(i-1))-c,0\}).\end{aligned}$$

From Lemma 3. 1 (e) we have

$$\max\{T(z_{t-1}(i))-c,0\}-\max\{T(z_{t-2}(i))-c,0\}\le z_{t-2}(i)-z_{t-1}(i),\quad i\ge 1,$$

and from Lemma 3. 1 (c1)

$$\max\{T(z_{t-2}(i-1))-c,0\}-\max\{T(z_{t-1}(i-1))-c,0\}\begin{cases}=0 & i=1,\\ \le 0, & i\ge 2.\end{cases}$$

Hence, it follows that $z_t(i)-z_{t-1}(i)\le\beta(z_{t-1}(i)-z_{t-2}(i))+\beta(z_{t-2}(i)-z_{t-1}(i))=0$. Thus $z_t(i)\le z_{t-1}(i)$ for $i\ge 1$.

(c2) From Eq. (4.4) we have

$$z_1(1)=\beta z_0(1)+\beta\max\{T(z_0(1))-c,0\}=K(z_0(1))+z_0(1)\ge z_0(1)$$

since $K(z_0(1))=K(\beta\alpha(1))\ge K(h^*)=0$ from Lemma 3. 1 (f). Assume that $z_{t-1}(1)\ge z_{t-2}(1)$ for $i\ge 1$, then from Eq. (4.4) we have

$$\begin{aligned}z_t(1)-z_{t-1}(1)&=\beta(z_{t-1}(1)-z_{t-2}(1))\\&\quad+\beta(\max\{T(z_{t-1}(1))-c,0\}-\max\{T(z_{t-2}(1))-c,0\})\\&\ge\beta(z_{t-1}(1)-z_{t-2}(1))+\beta(z_{t-2}(1)-z_{t-1}(1))=0\end{aligned}$$

from Lemma 3. 1 (e).

(d) Suppose $h=0$ and $\beta=1$. From Eq. (4.3) we have $v_t(i)\ge v_{t-1}(i)$ for all t and i due to $\max\{T(z_{t-1}(i))-c,0\}\ge 0$. Consequently, $z_t(1)-z_{t-1}(1)=v_t(1)-v_{t-1}(1)\ge 0$. For $i\ge 2$, from Eq. (4.4) we have

$$z_t(i)-z_{t-1}(i)=\max\{T(z_{t-1}(i))-c,0\}-\max\{T(z_{t-1}(i-1))-c,0\}\ge 0$$

from (b) and Lemma 3. 1 (c1). Hence $z_t(i)\ge z_{t-1}(i)$.

Suppose $h=0$ and $\alpha(i)=0$ for all i. Noting that $z_0(i)=\beta\Delta\alpha(i)=0$. From Eq. (4.4) , and by Lemma 3. 1 (c1) and (b) we have

$$z_1(i)-z_0(i)=\begin{cases}\max\{T(z_0(1))-c,0\}\ge 0 & i=1,\\ \max\{T(z_0(i))-c,0\}-\max\{T(z_0(i-1)-c,0\}\ge 0 & i\ge 2.\end{cases}$$

Hence, $z_1(i)\ge z_0(i)$ for all i. Assume that $z_{t-1}(i)\ge z_{t-2}(i)$ for $i\ge 1$. Then from Eq. (4.3) we have

$$\begin{aligned}&z_t(i)-z_{t-1}(i)\\&=\beta(z_{t-1}(i)-z_{t-2}(i))\\&\quad+\beta(\max\{T(z_{t-1}(i))-c,0\}-\max\{T(z_{t-2}(i))-c,0\})\\&\quad+\beta(\max\{T(z_{t-2}(i-1))-c,0\}-\max\{T(z_{t-1}(i-1))-c,0\}).\end{aligned}$$

From Lemma 3. 1 (e) we have

$$\max\{T(z_{t-1}(i)) - c, 0\} - \max\{T(z_{t-2}(i)) - c, 0\} \geq z_{t-2}(i) - z_{t-1}(i)$$

and from Lemma 3. 1 (c1) we have

$$\max\{T(z_{t-2}(i-1)) - c, 0\} - \max\{T(z_{t-1}(i-1)) - c, 0\} \geq 0.$$

Hence, it follows that

$$z_t(i) - z_{t-1}(i) \geq \beta(z_{t-1}(i) - z_{t-2}(i)) + \beta(z_{t-2}(i) - z_{t-1}(i)) = 0,$$

so $z_t(i) \geq z_{t-1}(i)$. ■

Theorem 5. 1 *We have*

(a) $a \leq x_t(i) < b$ *for all* t *and* i,

(b) $x_t(i)$ *is nonincreasing in* i *for all* t,

(c) *If* $\beta < 1$, *then*

 1. If $\beta\alpha(1) \geq h^*$ and $\Delta\alpha(i) \geq \beta\Delta\alpha(i-1)$ for $i \geq 2$, then $x_t(i)$ is nonincreasing in t.
 2. If $h = 0$ and $\beta\alpha(1) \leq h^*$, then $x_t(1)$ is nondecreasing in t.

(d) If either "$h = 0$ and $\beta = 1$" or if "$h = 0$ and $\alpha(i) = 0$" for all i, then $x_t(i)$ is nondecreasing in t for $i \geq 1$.

PROOF The proofs below are based on the fact that the monotoncity of $z_{t-1}(i)$ in t and i is inherited to $x_t(i)$ due to Lemma 3. 1 (d). (a) is immediate from Lemma 3. 1 (a), (b1) and (b2), and (b), (c) and (d) are immediate from, respectively, (b), (c) and (d) of Lemma 5. 1 . ■

From the above theorem we have the following corollaries.

Corollary 5. 1 *Suppose* $\beta < 1$ *and* $h = 0$. *Then*

(a) *if* $\beta\alpha(1) \geq h^*$, *then* $x_t(1)$ *is nonincreasing in* t, *or else nondecreasing in* t,

(b) *if* $\beta\alpha(1) = h^*$, *then* $x_t(1) = x(\alpha(1))$ *for all* t.

Corollary 5. 2 *Suppose* $\beta < 1$ *and* $\alpha(i) = \alpha i$ *with* $\alpha > 0$. *Then for any given* $i \geq 2$,

(a) *if* $\beta\alpha \geq h^*$, *then* $x_t(i)$ *is nonincreasing in* t,

(b) *if* $\beta\alpha \leq h^*$, *then* $x_t(i)$ *is not always nondecreasing in* t.

As an example of Corollary 5. 2 (b) we shall consider the following case. Let $\alpha(i) = i\alpha$ where $\alpha > 0$. Let $F(w)$ be a uniform distribution on $[0, 1]$. Since $T(\nu) = \max_x(1-x)(x-\nu)$, we have $x(\nu) = 0.5(1+\nu)$. Now $z_0(i) = \beta\alpha$ for $i \geq 1$ and $z_1(i) = \beta^2\alpha + \beta(\max\{T(\beta\alpha) - c, 0\} - \max\{T(\beta\alpha) - c, 0\}) = \beta^2\alpha$ for $i \geq 2$. Hence $z_1(i) < z_0(i)$. Consequently, $x_1(i) = x(z_0(i)) > x(z_1(i)) = x_2(i)$ for $i \geq 2$, implying that $x_t(i)$ is not always nondecreasing in t for $i \geq 2$. ■

Theorem 5. 2 *As* i *tends to* ∞, *the optimal price* $x_t(i)$ *converges to* $x_t = x(\beta(\beta^{t-1}\Delta\alpha - \sum_{n=0}^{t-2}\beta^n h))$ *for* $t \geq 2$ *and* $x_1 = x(\beta\Delta\alpha)$ *where* $\Delta\alpha = \lim_{i\to\infty}\Delta\alpha(i)$.

PROOF By definition $\Delta\alpha(i)$ is nonincreasing in i and nonnegative, so $\Delta\alpha(i)$ converges to a finite number. Now, we will show that $z_t(i)$ is lower-bounded in t and i. It is clear $z_0(i) = \beta\Delta\alpha(i) \geq -\beta\sum_{n=0}^{0}\beta^n h$. Assume that $z_{t-1}(i) \geq -\beta\sum_{n=0}^{t-1}\beta^n h$. Then, since $\max\{T(z_{t-1}(i)-h)-c,0\} - \max\{T(z_{t-1}(i-1)-h)-c,0\} \geq 0$ from Lemma 5. 1 (b) and Lemma 3. 1 (c1), it follows that

$$z_t(i) \geq \beta(z_{t-1}(i)-h) \geq \beta(-\beta\sum_{n=0}^{t-1}\beta^n h - h) = -\beta\sum_{n=0}^{t}\beta^n h \geq -\beta/(1-\beta). \tag{5.1}$$

Hence, $z_t(i)$ is lower bounded in t and i. Combining this and the nonincreasity of $z_t(i)$ in i from Lemma 5. 1 (b), $z_t(i)$ converges to a finite number. Let the limits of $\Delta\alpha(i)$ and $z_t(i)$ be defined by $\Delta\alpha$ and z_t, respectively. From Eq. (4.4) we have $z_t = \beta(z_{t-1}-h)$ for $t \geq 1$ and $z_0 = \beta\Delta\alpha$. Note, by induction, it follows that $z_t = \beta(\beta^t\Delta\alpha - \sum_{n=0}^{t-1}\beta^n h)$ for $t \geq 1$, hence $x_t = x(\beta(\beta^{t-1}\Delta\alpha - \sum_{n=0}^{t-2}\beta^n h))$ for $t \geq 2$ and $x_1 = x(\beta\Delta\alpha)$. ■

6 Optimal Search

In this section we will discuss the search rule to decide whether or not to search at the beginning of every period. From Eq. (4.3) the optimal searching rule can be stated as follows: If $T(z_{t-1}(i)-h)-c < 0$, then it is optimal not to search, or else to search. Thus, from Lemma 3. 1 (c1) this can be restated as follows: For any given i if $z_{t-1}(i) > \nu^*$, then it is optimal not to search, or else to search.

Lemma 6. 1

(a) *If $\Delta\alpha(i) \leq \nu^*/\beta$ for any given i, then $z_t(i) \leq \nu^*$ for all t.*
(b) *If $h=0$, $\beta=1$ and $\Delta\alpha(i) > \nu^*$ for any given i, then $z_t(i) > \nu^*$ for all t.*
(c) *If $\beta < 1$, $\Delta\alpha(i) > \nu^*/\beta$ for $i \geq 1$ and $\Delta\alpha(i) \geq \beta\Delta\alpha(i-1)$ for $i \geq 2$, then $z_t(i)$ converges, as t tends to ∞, to a finite number $z(i) \leq \nu^*$ with $z_0(i) > \nu^*$.*

PROOF (a) Since $z_0(i) = \beta\Delta\alpha(i) \leq \nu^*$, the assertion is clear for $t=0$. Assume that the assertion holds for $t-1$. Then, we have $z_{t-1}(i) - h \leq \nu^* - h \leq \nu^*$ for all i. Thus, by Lemma 3. 1 (c3) and Eq. (4.4) , it follows that

$$\begin{aligned} z_t(i) &\leq \beta(z_{t-1}(i) - h + \max\{T(z_{t-1}(i)-h)-c,0\}) \\ &= \beta\max\{z_{t-1}(i) - h + T(z_{t-1}(i)-h) - c, z_{t-1}(i) - h\} \\ &\leq \beta\max\{\nu^* + T(\nu^*) - c, \nu^*\} = \nu^*. \end{aligned}$$

(b) It is clear from the fact that $z_0(i) = \Delta\alpha(i) > \nu^*$ and $z_t(i)$ is nondecreasing in t from Lemma 5. 1 (d).

(c) It is clear that $z_0(i) = \beta\Delta\alpha(i) > \nu^* + h$. Since $z_t(i)$ is lower bounded in t and i from Eq. (5.1) and $z_t(i)$ is nonincreasing in t for all i from Lemma 5. 1 (b), it follows that $z_t(i)$ converges to a finite number $z(i)$ as t tends to ∞. From Eq. (4.4) we have

$$z(i) = \beta(z(i) - h) + \beta(\max\{T(z(i) - h) - c, 0\} - \max\{T(z(i-1) - h) - c, 0\}).$$

Now, assume $z(i) \geq \nu^* + h$ (> 0), so $z(i) - h \geq \nu^*$ for $i \geq 1$. Then, since $\max\{T(z(i) - h) - c, 0\} = \max\{T(z(i-1) - h) - c, 0\} = 0$ from Lemmas 5. 1 (c1), we have $z(i) = \beta(z(i) - h)$. Hence, $z(i) - h = -h/(1-\beta) - h < 0$, which contradicts the assumption. Thus, it must be $z(i) < \nu^* + h$. ■

Theorem 6. 1

(a) *If $\alpha(i) \leq \nu^*/\beta$ for a given i, then it is always optimal to search for the i.*

(b) *If $\beta = 1$, $h = 0$ and $\Delta\alpha(i) > \nu^*$ for a given i, then it is always optimal not to search for the i.*

(c) *If $\beta < 1$, $\Delta\alpha(i) > \nu^*$ for $i \geq 1$ and $\Delta\alpha(i) \geq \beta\Delta\alpha(i-1)$ for $i \geq 2$, then for a given i there exists $t(i)$ such that if $t \leq t(i)$, then it is optimal not to search, or else to search.*

Proof (a) Clear from the fact that $T(z_{t-1}(i)) - c \geq T(\nu^*) - c = 0$ due to $z_t(i) \leq \nu^*$ for all t from Lemma 6. 1 (a).

(b) Clear from the fact that $T(z_{t-1}(i)) - c \leq T(\nu^*) - c = 0$ due to $z_t(i) > \nu^*$ for all t from Lemma 6. 1 (b).

(c) Since $T(z_0(i)) - c \leq T(\nu^*) - c = 0$ from Lemma 6. 1 (c), it is optimal not to search at $t = 1$. Since $z_t(i) \leq \nu^*$ for any sufficiently large t from Lemma 6. 1 (c), we have $T(z_t(i)) - c \geq T(\nu^*) - c = 0$. From this and the fact that $z_t(i)$ is nonincreasing in t from Lemma 5. 1 (d1), it follows that there exists $t(i)$ such that if $t \leq t(i)$, then it is optimal not to search, or else to search. ■

7 Summary of Conclusion

For convenience, let us define the following two statements :

S_1: "$h = 0$ and $\beta = 1$" or "$h = 0$ and $\alpha(i) = 0$ for all i",

S_2: $\beta < 1$, $\beta\alpha(1) \geq h^*$, and $\Delta\alpha(i) \geq \beta\Delta\alpha(i-1)$ for $i \geq 2$.

Then, the conclusions that were obtained in the previous sections can be summerized as follows :

(a) Suppose $\Delta\alpha(i)$ is nonincreasing in i (concave). Then
   1. $x_t(i)$ is nonincreasing in i for all t,
   2. If S_1 is true, then $x_t(i)$ is nondecreasing in t for all i,
   3. If S_2 is true, then $x_t(i)$ is nonincreasing in t for all i,
   4. If both S_1 and S_2 are not ture, then nothing can be said as to the monotonicity of $x_t(i)$ in t. Indeed, we can demonostrate a case such that $x_t(i)$ is nonmonotonic in t (Figuer 7.1).

(b) Suppose $\Delta\alpha(i)$ is nondecreasing in i (convex). In this case, nothing can be said as to the monotonicity of $x_t(i)$ in not only t but also i. Indeed, we can demonostrate a case such that $x_t(i)$ is nonmonotonic in t and i (Figuer 7.2).

Now, in general, it can be conjectured that the optimal selling price $x_t(i)$ is nondecreasing with the remaining periods t and nonincreasing with the number of remaining items on hand. However, this does not always hold as above in (a3), (a4) and (b) (See. Figuer 7.1 and Figuer 7.2) .

Numerical examples: Let $\beta = 0.97$, $c = 0.02$, and $h = 0$, and let $F(w)$ be union distribution on $[0, 1]$.

• *Concave case:* Let $\Delta\alpha(i) = \Delta\alpha(i-1) - 0.0025$ with $\Delta\alpha(1) = 0.015$. Figuer 7.1 shows that $x_t(i)$ is is not monotone in t.

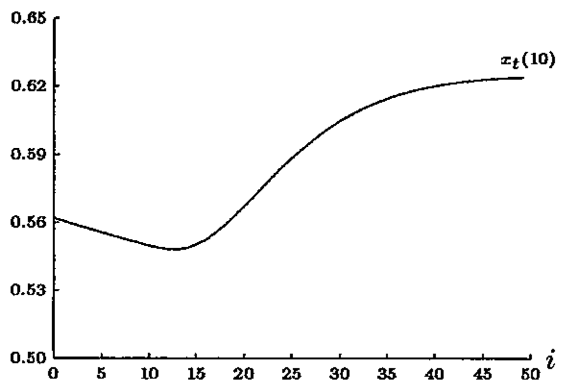

Figure 7.1: concave

• *Convex case:* Let $\Delta\alpha(i) = \Delta\alpha(i-1) + 0.0025$ with $\Delta\alpha(1) = 0.015$. Figuer 7.2 in an example shows that $x_t(i)$ is not monotone in t and i.

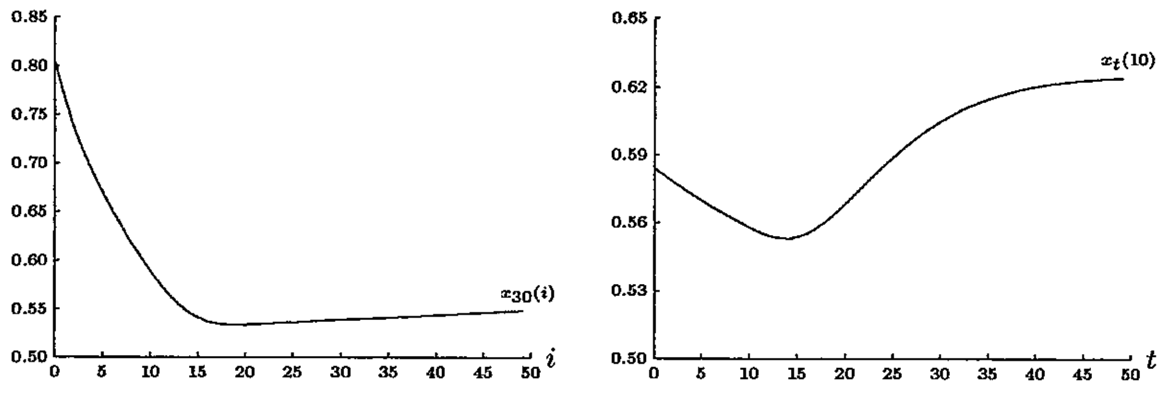

Figure 7.2: convex

8 Some limitations and future work

In this paper we presented a basic model with some assumptions. However, in order to make the model more realistic, it will be necessary to investigate the following variations:

1 . The seller of the items can dispose of a part or all of the remaining items at a known price $\alpha(i)$ at any point in time even before the deadline.

2. $F(w)$ depends on the search cost c paid.
3. A buying and selling problem.
4. Game theory problems with negotiations between sellers and buyers.

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