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A New Extension of Caristi-Kirk's Fixed Point Theorem

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Abstract

The important fixed point theorem for nonlinear mappings in metric spaces is given by Caristi and Kirk [9]. Kasahara [7] extend this theorem to a common fixed point theorem on collections of mappings. In this paper, we give a new extension of Caristi-Kirk's theorem. Our results contain some results of above and other authors.

1 Introduction

The important fixed point theorem for nonlinear mappings which need not be continuous in metric spaces is given by Caristi and Kirk(see [9] and [2]). We recall the theorem as follows:

Theorem 1 *Let (X, d) be a complete metric space and $f : X \to X$ an arbitrary map. Suppose there exist a lower semicontinuous function ϕ mapping X into the set of the nonnegative real numbers such that*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)) \quad \textit{for each } x \in X,$$

then f has a fixed point in X.

The advantage of Theorem 1 lies in the fact that it typically applies to non-continuous mappings. Its proof is an immediate adaptation of the proof of Theorem 2.1 of [2] (also see [4]). Their result sharpened a normal.solvability result of Browder [1] and was applied to prove some general mapping theorems in metric and Banach spaces. After that Kasahara [7] extend this theorem to a common fixed point theorem on collections of mappings. On the other hand, Downing and Kirk [4] give a generalization of Theorem 1 with applications to nonlinear mapping theory. In this paper, to improve and unify the results of above authors, we give a new extension of Caristi-Kirk's theorem. Our results contain some results of above and other authors(see [3], [4], [7], [8] and [11]).

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2 Preliminaries

We denote the set of all positive integers, the of all nonnegative integers and the set of all nonnegative real numbers by $\mathbf{N}$, ω and $\mathbf{R}_+$, respectively. We write $\hat{\mathbf{R}}_+ = \mathbf{R}_+ \cup \{\infty\}$. Let $\mathcal{L}$ be a collection of subsets of the set $X^\omega \times X$. Now we give certain definitions(see [8] and [10]).

Definition 1 *A pair $(X, \mathcal{L})$ is called a L-space if the following two conditions are satisfied:*

(a) $(\{x_n\}_{n\in\omega}, x) \in \mathcal{L}$ *if* $x_n = x \in X$ *for all* $n \in \omega$;

(b) $(\{x_{n_i}\}_{i\in\omega}, x) \in \mathcal{L}$ *if* $(\{x_n\}_{n\in\omega}, x) \in \mathcal{L}$ *with* $\{x_{n_i}\}_{i\in\omega}$ *is subsequence of* $\{x_n\}_{n\in\omega}$.

The L-space $(X, \mathcal{L})$ is said to be separated if each sequence in X converges to at most one point of X. Let $d : X \times X \to \hat{\mathbf{R}}_+$ be a function. The L-space $(X, \mathcal{L})$ is said to be d-complete if each sequence $\{x_n\}_{n\in\omega} \in X$ with $\Sigma_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$ converges to at least one point of X. □

Definition 2 *A nonnegative extended real valued function $d : X \times X \to \hat{\mathbf{R}}_+$ is called to a premetric metric on X if the following conditions are satisfied:*

$$(P1) \quad d(x, x) = 0 \text{ for every } x \in X;$$
$$(P2) \quad d(x, y) \leq d(x, z) + d(z, y) \text{ for every } x, y, z \in X.$$

The pair (X, d) is called a premetric space. □

Definition 3 *A sequence $\{x_n\}_{n\in\omega}$ in a premetric space (X, d) is called to be convergent to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ and we write And (X, d) is called complete if each Cauchy sequence converges to at least one point of X.* □

We denote the subset of the form $(\{x_n\}_{n\in\omega}, x)$ with $x_n \xrightarrow{d} x$ as $n \to \infty$ by $\mathcal{L}_d$. If (X, d) is a premetric space, then it is obvious that $(X, \mathcal{L}_d)$ is an L-space and that the completeness of (X, d) implies the completeness of $(X, \mathcal{L}_d)$. Note that a complete metric space is a complete premetric space.

Definition 4 *A space X is called Kasahara space if it is a nonempty L-space $(X, \mathcal{L})$ which is d-complete for a premetric d in X such that the function $x \mapsto d(x, y)$ is lower semicontinuous for each $y \in X$ and $d(x, y) = 0$ implies $x = y$.* □

It is clear that a nonempty complete metric space is Kasahara space. In [7] and [8], Kasahara gives some fixed points on such type of spaces.

Definition 5 *Let (X, d) and (Y, d') be Kasahara spaces. The map $g : X \to Y$ is called to closed if for any a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in X, $x_\alpha \to x^*$ and $g(x_\alpha) \to y^*$ imply $g(x^*) = y^*$.* □

3 Main results

In this section, we give our main results as follows.

Theorem 2 *Let $\Im$ be a family of mappings of an ordered set $(X, \preceq)$ into itself such that there exists a $p(x) \in \mathrm{N}$ for each $x \in X$ such that*

$$x \preceq f^{p(x)}(x) \quad \textit{for each } f \in \Im.$$

Suppose there exists an element $e \in X$ such that each chain C which contains e has a least upper bounded in C. Then there exists $x_0 \in X$ and $p(x_0) \in \mathrm{N}$ such that

$$f^{p(x_0)}(x_0) = x_0 \quad \textit{for each } f \in \Im.$$

Proof. It is clear that there exists a maximal chain C in X containing e by Zorn's Lemma. Let x_0 be a upper bound of C in X. By the maximality of C, we have $x_0 \in C$. Then there is $p(x_0) \in \mathrm{N}$ such that

$$x_0 \preceq f^{p(x_0)}(x_0) \quad \text{for each } f \in \Im.$$

Since C is a maximal chain in X and x_0 maximal element in C, then

$$f^{p(x_0)}(x_0) \in C \quad \text{and} \quad f^{p(x_0)}(x_0) \preceq x_0 \quad \text{for each } f \in \Im.$$

Therefore

$$f^{p(x_0)}(x_0) = x_0 \quad \text{for each } f \in \Im. \qquad \square$$

The next theorem is the main result of this section. Now we use Theorem 2 to prove the theorem:

Theorem 3 *Let (X, d) and (Y, d') be Kasahara spaces. The map $g : X \to Y$ is closed and $\Im$ denotes a collection of selfmaps of X. Suppose there exist a lower semicontinuous function $\phi : g(X) \to \mathbf{R}_+$ and a constant $c > 0$ such that for each $f \in \Im$, the following two conditions are satisfied:*

$$(i) \qquad \phi(g(x)) \neq \phi(g(f(x))) \quad \textit{if } f(x) \neq x;$$

(ii) For each $x \in X$ there exists a $p(x) \in \mathrm{N}$ such that

$$(1) \qquad \max\{d(x, f^{p(x)}(x)), cd'(g(x), g(f^{p(x)}(x)))\} \leq \phi(g(x)) - \phi(g(f(x))).$$

Then $\Im$ has a common fixed point.

Proof. First we declare that there exists $p(x) \in \mathrm{N}$ for each $x \in X$ such that

$$(2) \qquad \phi(g(f^{p(x)}(x))) \leq \phi(g(f(x))) \ \text{ for each } f \in \Im.$$

In fact, the following inequality is always true for each $x \in X$ and $f \in \Im$ by above condition (ii),

$$\phi(g(f(x))) \leq \phi(g(x)).$$

For any $n \in \mathrm{N}$ and $f \in \Im$, we replace to x by $f^{n-1}(x)$ in the above inequality, then

$$\phi(g(f^n(x))) \leq \phi(g(f^{n-1}(x))).$$

We repeat the $p(x) - 1$ time to using the inequality, then obtain (2). And using (1) and (2), we have

$$(3) \qquad \begin{aligned} &\max\{d(x, f^{p(x)}(x)), cd'(gx, g(f^{p(x)}(x)))\} \\ &\leq \phi(g(x)) - \phi(g(f(x))) \\ &\leq \phi(g(x)) - \phi(g(f^{p(x)}(x))). \end{aligned}$$

Next we define the relation $\preceq_\phi$ on X as follows: For any $x, y \in X$,

$$x \preceq_\phi y \iff \max\{d(x,y), cd'(g(x), g(y))\} \leq \phi(g(x)) - \phi(g(y)).$$

It is easy to see that $\preceq_\phi$ is a order on X. According to (3), for each $x \in X$, there exists $p(x) \in \mathrm{N}$ such that

$$x \preceq_\phi f^{p(x)}(x) \ \text{ for any } f \in \Im.$$

Now we fixed a $e \in X$ and assume that $C = \{x_\alpha \mid \alpha \in \Lambda\}$ is a chain which containing e in $(X, \preceq_\phi)$, where Λ is a totally ordered set such that

$$\alpha' \succeq \alpha \iff x_\alpha \preceq_\phi x_{\alpha'}.$$

Then the set $\{\phi(g(x_\alpha)) \mid \alpha \in \Lambda\}$ is a decreasing net, that is,

$$\phi(g(x_\alpha)) \leq \phi(g(x_{\alpha'})) \ \text{ if } \alpha' \succeq \alpha.$$

Thus there is nonnegative real number

$$\beta = \inf\{\phi(g(x_\alpha)) \mid \alpha \in \Lambda\}.$$

For arbitrary $\varepsilon > 0$, there exists $\alpha_0 \in \Lambda$ such that

$$0 \leq \beta \leq \phi(g(x_\alpha)) \leq \beta + \varepsilon \ \text{ for each } \alpha \succeq \alpha_0.$$

For any $\alpha' \succeq \alpha \succeq \alpha_0$, we have

$$x_\alpha \preceq_\phi x'_\alpha \ \Rightarrow \phi(g(x'_\alpha)) \leq \phi(g(x_\alpha)),$$

and

$$(4) \qquad \max\{d(x_{\alpha'}, x_\alpha), cd'(g(x_{\alpha'}), g(x_\alpha))\} \\ \leq \phi(g(x_\alpha)) - \phi(g(x_{\alpha'})) \\ \leq \beta + \varepsilon - \beta = \varepsilon.$$

Thus $\{x_\alpha \mid \alpha \in \Lambda\}$ and $\{g(x_\alpha) \mid \alpha \in \Lambda\}$ are Cauchy nets in X and Y, respectively. Hence there exist $x^* \in X$ and $y^* \in Y$ such that

$$\lim_\alpha x_\alpha = x^* \text{ and } \lim_\alpha g(x_\alpha) = y^*,$$

whence $g(x^*) = y^*$ because g is closed.

Since ϕ is lower semicontinuous, we have

$$(5) \quad \phi(g(x^*)) = \phi(\lim_\alpha g(x_\alpha) \leq \lim_\alpha \phi(g(x_\alpha)) \leq \beta + \varepsilon.$$

Put $\lim_{\alpha' \to \infty}$ at both ends of first inequality of (4) and use (5), we obtain

$$\max\{d(x^*, x_\alpha), cd'(g(x^*), g(x_\alpha))\} \leq \phi(g(x_\alpha)) - \phi(g(x^*)).$$

Therefore $x_\alpha \preceq_\phi x^*$ for any $\alpha \in \Lambda$. That is, C has an upper bound in X.

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Therefore $x_\alpha \preceq_\phi x^*$ for any $\alpha \in \Lambda$. That is, C has an upper bound in X.
By Theorem 2, there exist $\hat{x} \in X$ and $p(\hat{x}) \in \mathbb{N}$ such that

$$f^{p(\hat{x})}(\hat{x}) = \hat{x} \text{ for each } f \in \Im.$$

Therefore

$$(6) \quad \phi(g(\hat{x})) = \phi(g(f^{p(\hat{x})}(\hat{x}))) \text{ for each } f \in \Im.$$

Finally we show that $\hat{x}$ is a common fixed point of the collection $\Im$. On the contrary, assume that there exists $\tilde{f} \in \Im$ such that $\tilde{f}(\hat{x}) \neq \hat{x}$. According to (3) and (6), we have

$$0 \leq \phi(g(\hat{x})) - \phi(g(\tilde{f}(\hat{x}))) \leq \phi(g(\hat{x})) - \phi(g(\tilde{f}^{p(\hat{x})}(\hat{x}))) \leq 0,$$

i. e., $\phi(g(\hat{x})) = \phi(g(\tilde{f}(\hat{x})))$.
This is a contradiction by the condition (i) of Theorem 3. The proof is completed. □

Remark 1. In above Theorem, if we let $c = 1$ and $p(x) = 1$ for each $x \in X$, then the main result of Park [10] is obtained as the direct corollary. Since Kasahara's [7] Theorem is a special case of the theorem of Park [10], our results also contain Kasahara's result.

Remark 2. If we consider the case that X and Y are complete metric spaces and $\Im$ only contains one element f in Theorem 2. Then the main theorem of Guo [5] is obtained immediately. And the theorem of Downing and Kirk [4] is obtained also, because their main Theorem is the special case of Guo's results.

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