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by

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Correlation of consecutive interdeparture times characterizes the output process in a queueing system. We present a recursive procedure for calculating the joint distribution of an arbitrary number of consecutive interdeparture times in M/G/1 and M/G/1/K queues. We then obtain explicitly the covariances of nonadjacent interdeparture times, and display the correlation coefficients that reveal the long-range dependence.

Key words: Queues, M/G/1, M/G/1/K, output process, interdeparture time.

1. Introduction

In a network of queues such as those appearing in communication networks and manufacturing systems, the customers departing from a queue constitute arrival streams for other queues. It is therefore important to characterize the output processes in queueing systems [6, chap. 6]. The output process of an M/G/1 queue with infinite or finite capacity can be formulated as a Markov renewal process for (a) the number of customers left behind in the queue by departing customers, and (b) the time intervals between two successive departures. The marginal process for (a) is a simple discrete-time, discrete-valued Markov chain. The marginal process for (b), which consists of consecutive interdeparture times, is more complicated because of their long-memory correlation structure except for a few special cases.

The past studies for the output process in an M/G/1 queue with an infinite capacity include the following. Burke [1] and Finch [7] showed that the output process of an M/M/1 queue is a Poisson process at the same rate as the arrival process. Jenkins [9] analyzed the correlation of consecutive interdeparture times for an M/E_m/1 queue, where E_m denotes the Erlang-m distribution. For an M/G/1 queue, Conolly [2, sec. 5.5.1] (see also Takagi [12, sec. 1.5]) gives the joint distribution for two consecutive interdeparture times τ_1 and τ_2, from which the covariance $\mathrm{Cov}[\tau_1, \tau_2]$ is derived. Daley [3] derives the generating function for the sequence $\{\mathrm{Cov}[\tau_1, \tau_n]; n = 2, 3, \ldots\}$, where τ_n is the $n-1$st interdeparture time after τ_1. Daley [4] and Reynolds [11] present surveys of the available results. In the present paper, we show a procedure for calculating the joint distribution for the arbitrary number n consecutive interdeparture times $\tau_1, \tau_2, \ldots, \tau_n$ from which we can obtain $\mathrm{Cov}[\tau_1, \tau_n]$.

For finite-capacity M/G/1 queues, which we denote by M/G/1/K where K is the capacity including a customer in service, Daley and Shanbhag [5] and King [10] showed that $\mathrm{Cov}[\tau_1, \tau_n] = 0$ for $n \geq 2$ in M/G/1/1 and M/D/1/2 queues, and that $\mathrm{Cov}[\tau_1, \tau_n] = 0$ for $n \geq 3$ in an M/G/1/2 queue. See also Takagi [13, sec. 5.2]. Rather recently, Ishikawa [8] proved an interesting result that $\mathrm{Cov}[\tau_1, \tau_n] = a_1^{n-3}\mathrm{Cov}[\tau_1, \tau_3]$ for $n \geq 4$ in an M/G/1/3 queue, where a_1 is the probability that exactly one customer arrives during a service time. He also derived an explicit expression for $\text{Cov}[\tau_1, \tau_n]$ for $n \le K$ in an M/M/1/K queue. His analysis is based on the formulation of a Markov renewal process as mentioned above. Our paper provides a different approach by taking advantage of a recursive structure in the set of interdeparture times.

The rest of the paper is organized as follows. In Section 2, we introduce notation and give as a preliminary a set of equations for calculating the queue size distribution at departure times. In Section 3, we show a procedure for calculating the joint distribution of n consecutive interdeparture times from that of $n-1$ consecutive interdeparture times. In Sections 4 and 5, we present the explicit expressions for the covariances of nonadjacent interdeparture times in M/G/1 and M/G/1/K queues, respectively. We also display some numerical results for the covariances, and give a few remarks.

2. Queue Size Distribution at Departure Times

We deal with an M/G/1 queue with a Poisson arrival process at rate λ, independent and identically distributed service times, and a single server in the steady state. We also consider an M/G/1/K queue with similar settings and a finite capacity such that at most K customers, including one in service, can be accommodated in the system at a time. The density function and its Laplace transform for the service time are denoted by $b(x)$ and $B^*(s)$, respectively, so that $B^*(s) := \int_0^\infty e^{-sx} b(x)dx$. The mean b and the second moment $b^{(2)}$ of the service time are then given by $b = -B^{*(1)}(0)$ and $b^{(2)} = B^{*(2)}(0)$, respectively, where $B^{*(i)}(s) := d^i B^*(s)/ds^i$ for $i = 1, 2, \ldots$. When the service times are exponentially distributed, the service rate is denoted by μ so that $b = 1/\mu$.

In the steady state, let π_k be the probability that k customers are left behind in the queue by a departing customer. If $K = 1$, we have $\pi_0 = 1$. Otherwise, the set $\{\pi_k; 0 \le k \le K-1\}$ satisfies

$$\pi_k = \pi_0 a_k + \sum_{j=0}^{k+1} \pi_j a_{k-j+1} \qquad 0 \le k \le K-2 \tag{1}$$

$$\sum_{k=0}^{K-1} \pi_k = 1 \tag{2}$$

where a_k denotes the probability that k customers arrive during a service time, and is given by

$$a_k = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x)dx = \frac{(-\lambda)^k}{k!} B^{*(k)}(\lambda) \qquad k \ge 0 \tag{3}$$

Thus we can calculate π_k/π_0 $(1 \le k \le K-1)$ by (1), and then evaluate π_0 by (2). If $K = \infty$, we have $\pi_0 = 1 - \lambda b$ which is assumed to be positive for the stability of the queue.

3. Joint Distribution of Consecutive Interdeparture Times

The time interval between two successive points in time at which customers depart from the queue after the service is completed is called an interdeparture time. Note that we exclude those points in time at which customers are blocked upon arrivals in the M/G/1/K queue. The Laplace transform $\Delta^*(s)$ of the density function for the length τ of a single interdeparture time is then given by

$$\Delta^*(s) = \pi_0 \frac{\lambda}{s+\lambda} B^*(s) + (1-\pi_0)B^*(s) = \left(1 - \frac{\pi_0 s}{s+\lambda}\right) B^*(s) \tag{4}$$

from which we get the mean and the second moment of the interdeparture time as

$$E[\tau] = b + \frac{\pi_0}{\lambda} \quad ; \quad E[\tau^2] = b^{(2)} + \frac{2\pi_0(1+\lambda b)}{\lambda^2} \tag{5}$$

When $K = \infty$, we have $E[\tau] = 1/\lambda$ as no customers are lost or created.

We denote by $\Delta^*(s, s')$ the Laplace transform of the joint density function for the lengths τ and τ' of two consecutive interdeparture times. By considering several conditions in the queue size at departure times, we can obtain [2, 13]

$$\Delta^*(s,s') = \left\{ B^*(s) - \frac{\pi_0}{s+\lambda}\left[sB^*(s) + \frac{\lambda s' B^*(s+\lambda)}{s'+\lambda}\right] - \frac{\pi_1 s' B^*(s+\lambda)}{s'+\lambda} \right\} B^*(s') \tag{6}$$

with $\pi_1 = [1 - B^*(\lambda)]\pi_0/B^*(\lambda)$. From (6) we get the covariance of the two consecutive interdeparture times as

$$\mathrm{Cov}[\tau, \tau'] = \frac{\pi_0}{\lambda}\left[\frac{B^*(\lambda)}{\lambda} - \frac{B^{*(1)}(\lambda)}{B^*(\lambda)} - b - \frac{\pi_0}{\lambda} \right] \tag{7}$$

for $K \geq 2$. For $K = 1$ (an M/G/1/1 loss system), we simply have

$$\Delta^*(s) = \frac{\lambda}{s+\lambda} B^*(s) \quad ; \quad \Delta^*(s,s') = \Delta^*(s)\Delta^*(s') \quad ; \quad \mathrm{Cov}[\tau,\tau'] = 0 \tag{8}$$

as the departure process is a renewal process. For an M/M/1 queue, we have

$$\Delta^*(s) = \frac{\lambda}{s+\lambda} \quad ; \quad \Delta^*(s,s') = \Delta^*(s)\Delta^*(s') \quad ; \quad \mathrm{Cov}[\tau,\tau'] = 0 \tag{9}$$

as the departure process is a Poisson process at rate λ.

In order to obtain the covariance of nonadjacent interdeparture times, we have to consider the joint distribution for more than two consecutive interdeparture times. Let $\Delta_n^*(s_n, s_{n-1}, \ldots, s_1)$ be the Laplace transform of the joint density function for n consecutive interdeparture times τ_1 through τ_n where the transform parameter s_i corresponds to τ_{n+1-i} $(1 \leq i \leq n)$. Also, let $\Delta_{n:k}^*(s_n, s_{n-1}, \ldots, s_1)$ and $\Delta_{n:\geq n}^*(s_n, s_{n-1}, \ldots, s_1)$ be the similar Laplace transforms on the condition that there are k and n or more customers, respectively, in the queue at a departure time. Thus unconditioning yields

$$\Delta_n^*(s_n, s_{n-1}, \ldots, s_1) = \sum_{k=0}^{n-1} \pi_k \Delta_{n:k}^*(s_n, s_{n-1}, \ldots, s_1) + \left(1 - \sum_{k=0}^{n-1} \pi_k\right) \Delta_{n:\geq n}^*(s_n, s_{n-1}, \ldots, s_1) \tag{10}$$

We will show that $\{\Delta_{n:k}^*(s_n, s_{n-1}, \ldots, s_1); 0 \leq k \leq n-1\}$ and $\Delta_{n:\geq n}^*(s_n, s_{n-1}, \ldots, s_1)$ can be expressed in terms of $\{\Delta_{n-1:k}^*(s_{n-1}, s_{n-2}, \ldots, s_1); 0 \leq k \leq n-2\}$ and $\Delta_{n-1:\geq n-1}^*(s_{n-1}, s_{n-2}, \ldots, s_1)$. Therefore, starting with

$$\Delta_{1:0}^*(s_1) = \frac{\lambda}{s_1+\lambda} B^*(s_1) \quad ; \quad \Delta_{1:\geq 1}^*(s_1) = B^*(s_1) \tag{11}$$

we can calculate $\{\Delta_{n:k}^*(s_n, s_{n-1}, \ldots, s_1); 0 \leq k \leq n-1\}$ and $\Delta_{n:\geq n}^*(s_n, s_{n-1}, \ldots, s_1)$ recursively with respect to n for an arbitrary value of n in principle. Substituting them into (10) we get $\Delta_n^*(s_n, s_{n-1}, \ldots, s_1)$, from which we can obtain the covariance of interdeparture times τ_1 and τ_n by

$$\mathrm{Cov}[\tau_1, \tau_n] = \left. \frac{\partial^2 \Delta_n^*(s_n, s_{n-1}, \ldots, s_1)}{\partial s_n \partial s_1} \right|_{s_n = s_{n-1} = \cdots = s_1 = 0} - (E[\tau])^2 \tag{12}$$

We now present the recursive procedure. Let us first consider an M/G/1 queue. Obviously we have

$$\Delta^*_{n:\geq n}(s_n, s_{n-1}, \ldots, s_1) = B^*(s_n)\Delta^*_{n-1:\geq n-1}(s_{n-1}, s_{n-2}, \ldots, s_1) = B^*(s_n)B^*(s_{n-1})\cdots B^*(s_1) \tag{13}$$

In order to express $\Delta^*_{n:k}(s_n, s_{n-1}, \ldots, s_1)$, $1 \leq k \leq n-1$, in terms of $\{\Delta^*_{n-1:j}(s_{n-1}, s_{n-2}, \ldots, s_1);\ 0 \leq j \leq n-2\}$ and $\Delta^*_{n-1:\geq n-1}(s_{n-1}, s_{n-2}, \ldots, s_1)$, we note that the joint Laplace transform of the density function for the length of a service time and the probability that j customers arrive in that service time is given by

$$\int_0^\infty e^{-sx}\frac{(\lambda x)^j}{j!}e^{-\lambda x}b(x)dx = \frac{(-\lambda)^j}{j!}B^{*(j)}(s+\lambda) \tag{14}$$

Thus we get the relation

$$\begin{aligned}\Delta^*_{n:k}(s_n, s_{n-1}, \ldots, s_1) &= \sum_{j=0}^{n-k-1}\frac{(-\lambda)^j}{j!}B^{*(j)}(s_n+\lambda)\Delta^*_{n-1:k+j-1}(s_{n-1}, s_{n-2}, \ldots, s_1)\\ &+\left[B^*(s_n) - \sum_{j=0}^{n-k-1}\frac{(-\lambda)^j}{j!}B^{*(j)}(s_n+\lambda)\right]\Delta^*_{n-1:\geq n-1}(s_{n-1}, s_{n-2}, \ldots, s_1)\\ &\qquad 1 \leq k \leq n-1\end{aligned} \tag{15}$$

If there is an idle period before the first interdeparture time, the joint Laplace transform of the density function for the length of the idle period and a service time and the probability that j customers arrive in that service time is given by

$$\int_0^\infty e^{-st}dt\int_0^t \lambda e^{-\lambda x}\frac{[\lambda(t-x)]^j}{j!}e^{-\lambda(t-x)}b(t-x)dx = \frac{\lambda}{s+\lambda}\cdot\frac{(-\lambda)^j}{j!}B^{*(j)}(s+\lambda) \tag{16}$$

Thus we get the relation

$$\begin{aligned}\Delta^*_{n:0}(s_n, s_{n-1}, \ldots, s_1) &= \frac{\lambda}{s_n+\lambda}\sum_{j=0}^{n-2}\frac{(-\lambda)^j}{j!}B^{*(j)}(s_n+\lambda)\Delta^*_{n-1:j}(s_{n-1}, s_{n-2}, \ldots, s_1)\\ &+\frac{\lambda}{s_n+\lambda}\left[B^*(s_n) - \sum_{j=0}^{n-2}\frac{(-\lambda)^j}{j!}B^{*(j)}(s_n+\lambda)\right]\Delta^*_{n-1:\geq n-1}(s_{n-1}, s_{n-2}, \ldots, s_1)\end{aligned} \tag{17}$$

Equations (13), (15), and (17) provide the recursive procedure for the M/G/1 queue.

In an M/G/1/K queue, equations (13), (15), and (17) hold for $n \leq K-1$. In addition, we have

$$\Delta^*_{K-1:K-1}(s_{K-1}, s_{K-2}, \ldots, s_1) = B^*(s_{K-1})B^*(s_{K-2})\cdots B^*(s_1) \tag{18}$$

For $n \geq K$, by similar arguments we get

$$\begin{aligned}\Delta^*_{n:k}(s_n, s_{n-1}, \ldots, s_1) &= \sum_{j=0}^{K-k-1}\frac{(-\lambda)^j}{j!}B^{*(j)}(s_n+\lambda)\Delta^*_{n-1:k+j-1}(s_{n-1}, s_{n-2}, \ldots, s_1)\\ &+\left[B^*(s_n) - \sum_{j=0}^{K-k-1}\frac{(-\lambda)^j}{j!}B^{*(j)}(s_n+\lambda)\right]\Delta^*_{n-1:K-1}(s_{n-1}, s_{n-2}, \ldots, s_1)\\ &\qquad 1 \leq k \leq K-1\end{aligned} \tag{19}$$

and

$$\Delta^*_{n:0}(s_n, s_{n-1}, \ldots, s_1) = \frac{\lambda}{s_n + \lambda} \sum_{j=0}^{K-2} \frac{(-\lambda)^j}{j!} B^{*(j)}(s_n + \lambda)\Delta^*_{n-1:j}(s_{n-1}, s_{n-2}, \ldots, s_1)$$
$$+ \frac{\lambda}{s_n + \lambda} \left[B^*(s_n) - \sum_{j=0}^{K-2} \frac{(-\lambda)^j}{j!} B^{*(j)}(s_n + \lambda) \right] \Delta^*_{n-1:K-1}(s_{n-1}, s_{n-2}, \ldots, s_1) \tag{20}$$

These equations complete the recursive procedure for the M/G/1/K queue.

4. Covariance of Nonadjacent Interdeparture Times in an M/G/1 Queue

By the recursive procedure given in Section 3, we can calculate the Laplace transform $\Delta^*_n(s_n, s_{n-1}, \ldots, s_1)$ for the joint distribution of n consecutive interdeparture times $\tau_1, \tau_2, \ldots, \tau_n$. Such calculation is made possible by symbolic formula manipulation software, for example, *Mathematica* [14]. We can then obtain the covariance of τ_1 and τ_n by (12).

In this section, we present the result for an M/G/1 queue. For the simplicity of notation, let us use $b_i := B^{*(i)}(\lambda)$ for $i = 0, 1, 2, \ldots$ in Sections 4 and 5. We have

$$\mathrm{Cov}[\tau_1, \tau_2] = \left(\frac{1}{\lambda} - b\right) \left[-\frac{1}{\lambda} + \frac{b_0}{\lambda} - \frac{b_1}{b_0} \right] \tag{21a}$$

$$\mathrm{Cov}[\tau_1, \tau_3] = \left(\frac{1}{\lambda} - b\right) \left[-\frac{1}{\lambda} + \frac{b_0^2}{\lambda} - \frac{b_1}{b_0} - b_0 b_1 - \frac{\lambda b_1^2}{b_0} + \lambda b_2 \right] \tag{21b}$$

$$\mathrm{Cov}[\tau_1, \tau_4] = \left(\frac{1}{\lambda} - b\right) \left[-\frac{1}{\lambda} + \frac{b_0^3}{\lambda} - \frac{b_1}{b_0} - 2b_0^2 b_1 - \lambda b_1^2 + \lambda b_0 b_1^2 \right.$$
$$\left. + \frac{\lambda^2 b_1^3}{b_0} + \lambda b_2 + \frac{\lambda}{2} b_0^2 b_2 + \frac{\lambda^2 b_1 b_2}{2b_0} - \frac{\lambda^2}{2} b_0 b_3 \right] \tag{21c}$$

$$\mathrm{Cov}[\tau_1, \tau_5] = \left(\frac{1}{\lambda} - b\right) \left[-\frac{1}{\lambda} + \frac{b_0^4}{\lambda} - \frac{b_1}{b_0} - 3b_0^3 b_1 + \frac{\lambda b_1^2}{b_0} - \lambda b_1^2 - \lambda b_0 b_1^2 + 3\lambda b_0^2 b_1^2 \right.$$
$$+ \frac{\lambda^2 b_1^3}{b_0} + 2\lambda^2 b_1^3 - \lambda^2 b_0 b_1^3 - \frac{\lambda^3 b_1^4}{b_0} + \lambda b_2 + \lambda b_0^3 b_2 + \frac{\lambda^2 b_1 b_2}{2b_0}$$
$$-\lambda^2 b_1 b_2 + \lambda^2 b_0 b_1 b_2 - \frac{3}{2}\lambda^2 b_0^2 b_1 b_2 - \frac{\lambda^3 b_1^2 b_2}{b_0} + \frac{3}{2}\lambda^3 b_1^2 b_2 - \frac{\lambda^3}{2} b_2^2$$
$$\left. + \frac{\lambda^3}{2} b_0 b_2^2 - \frac{\lambda^2}{2} b_0 b_3 - \frac{\lambda^2}{6} b_0^3 b_3 - \frac{\lambda^3 b_1^3}{b_0} + \frac{\lambda^3}{2} b_0 b_1 b_3 + \frac{\lambda^3}{6} b_0^2 b_4 \right] \tag{21d}$$

We note that (21a) is given in Conolly [2], while the results in (21b)–(21d) are new.

According to Daley [3], the sequence $\{\mathrm{Cov}[\tau_1, \tau_n]; n = 2, 3, \ldots\}$ for the M/G/1 queue satisfies the relations

$$\sum_{n=2}^{\infty} \mathrm{Cov}[\tau_1, \tau_n] = \frac{1}{2} \left(\frac{1}{\lambda^2} - \mathrm{Var}[\tau_1] \right) \tag{22}$$

$$\sum_{n=2}^{\infty} \mathrm{Cov}[\tau_1, \tau_n] z^{n-1} = \frac{1-\lambda b}{\lambda^2(1-z)} \left[\frac{w(z)-z}{1-w(z)} + \frac{zw'(z)-w(z)}{w(z)w'(z)} \right] \tag{23}$$

where $w(z)$ is the root of smallest modulus of the equation

$$w = zB^*[\lambda(1-w)] \tag{24}$$

and $w'(z) = dw(z)/dz$. Thus (21a)–(21d) could have been obtained from (23) and (24).

We display the correlation coefficients of the interdeparture times defined by

$$\rho(\tau_1, \tau_n) := \frac{\mathrm{Cov}[\tau_1, \tau_n]}{\mathrm{Var}[\tau]} \tag{25}$$

for the Erlang-m distribution of the service time

$$B^*(s) = \left(\frac{m}{s+m} \right)^m \tag{26}$$

which has the unit mean and the correlation coefficient $1/m$, where m is a positive integer. Thus the case $m = 1$ corresponds to an M/M/1 queue, and the case $m = \infty$ to an M/D/1 queue. With (26), from (22) we have

$$\sum_{n=2}^{\infty} \mathrm{Cov}[\tau_1, \tau_n] = \frac{1}{2}\left(1 - \frac{1}{m}\right) \tag{27}$$

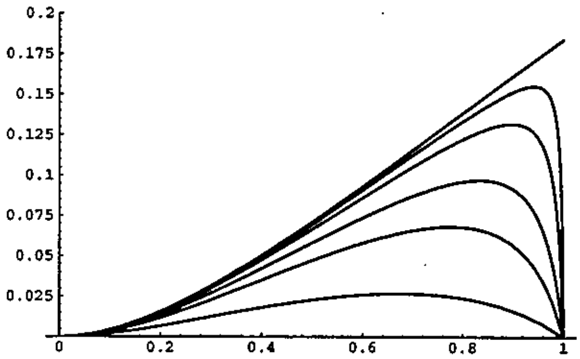
which is independent of the arrival rate λ. Jenkins [9] gives the expression for $\rho(\tau_1, \tau_3)$ for the M/E$_m$/1 queue, which agrees with our special case.

In Fig. 1(a)–(d), we plot $\rho(\tau_1, \tau_n)$ for $n = 2, 3, 4$ and 5, respectively, for an M/E$_m$/1 queue with $b = 1$ in the cases $m = 2, 5, 10, 30, 100$ and ∞ against λ. We make the following observations:

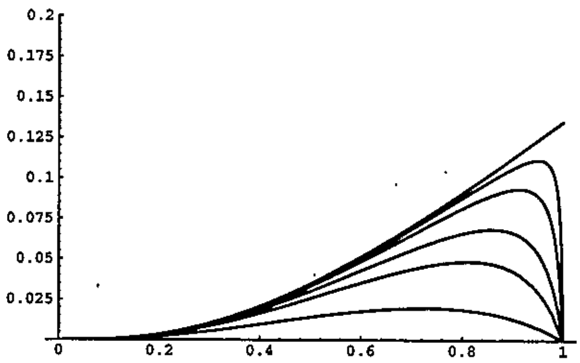
- $\rho(\tau_1, \tau_n)$ is always nonnegative. Thus, from (27) we always have

$$\lim_{n\to\infty} \rho(\tau_1, \tau_n) = 0 \tag{28}$$

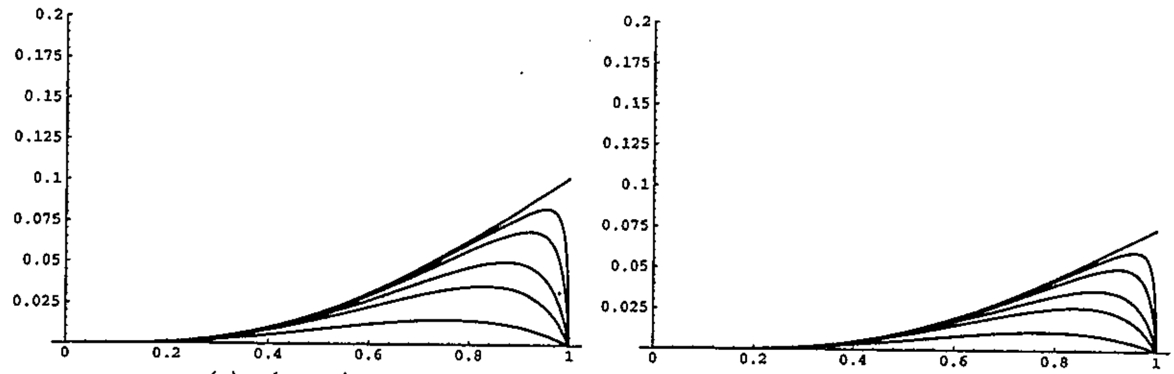
- Given n and m finite, $\rho(\tau_1, \tau_n)$ is a unimodal function of λ.
- Given λ, $\rho(\tau_1, \tau_n)$ increases with m, and decreases with n.



(a) $\rho(\tau_1, \tau_2)$



(b) $\rho(\tau_1, \tau_3)$



(c) $\rho(\tau_1, \tau_4)$ (d) $\rho(\tau_1, \tau_5)$

Fig.1. Correlation coefficients of interdeparture times in an M/E$_m$/1 queue
($m = 2, 5, 10, 30, 100$ and ∞ from below).

5. Covariance of Nonadjacent Interdeparture Times in an M/G/1/K Queue

Let us also present the covariances of the interdeparture times for M/G/1/K queues, and display the numerical values of the correlation coefficients for the M/E$_m$/1/K queues such that the service time distribution is given by (26).

For an M/G/1/2 queue, we have

$$\mathrm{Cov}[\tau_1, \tau_2] = -\frac{bb_0 + b_1}{\lambda} \quad ; \quad \mathrm{Cov}[\tau_1, \tau_n] = 0 \qquad n \geq 3 \tag{29}$$

In particular, for an M/D/1/2 queue we have $b_0 = e^{-\lambda b}$ and $b_1 = -be^{-\lambda b}$ so that $\mathrm{Cov}[\tau_1, \tau_2] = 0$ as noted previously [5, 10]. Fig. 2 shows that the correlation is negative for the M/E$_m$/1/2 queue.

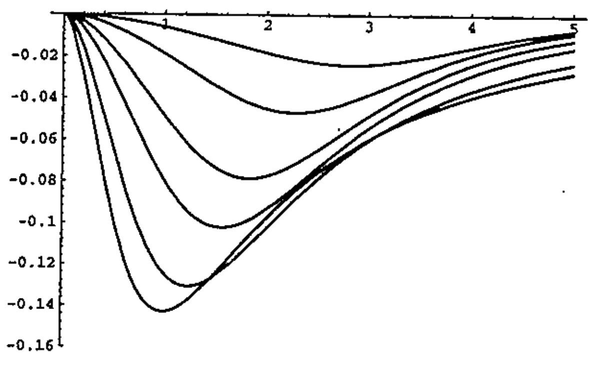


Fig. 2. Correlation coefficients $\rho(\tau_1, \tau_2)$ of interdeparture times in an M/E$_m$/1/2 queue
($m = 1, 2, 5, 10, 30$ and 100 from below at $\lambda = 1$).

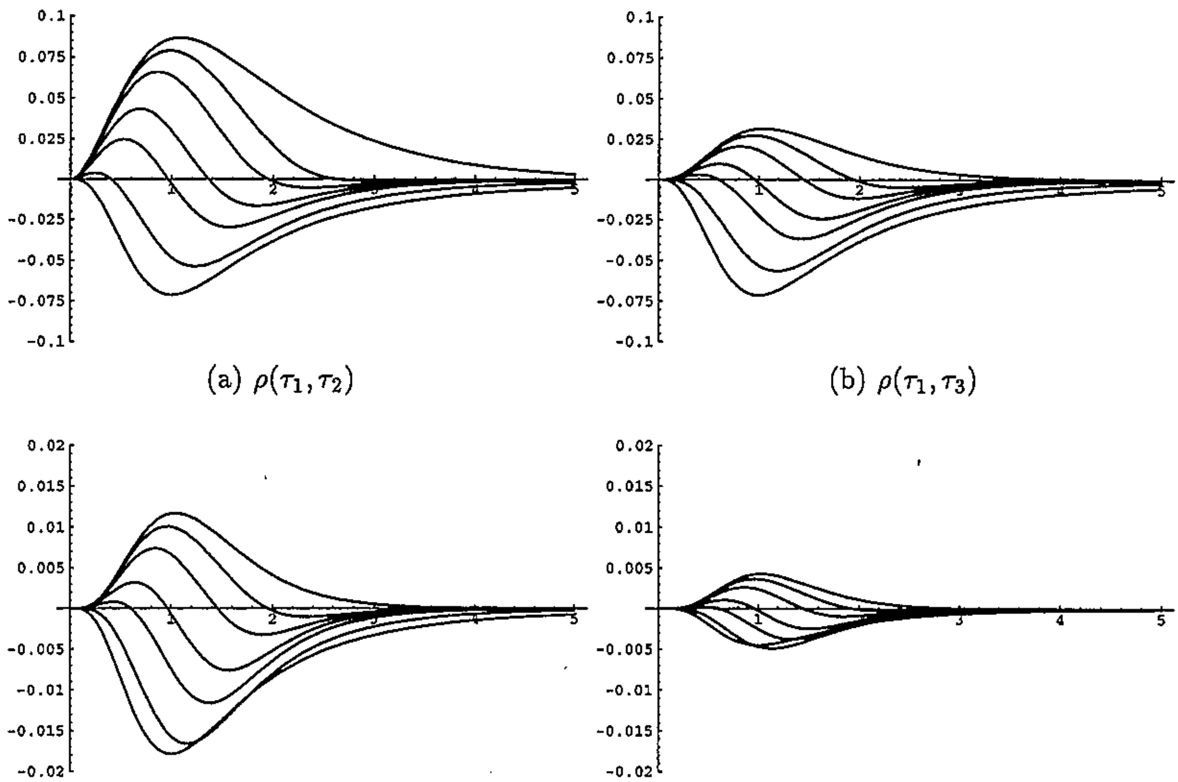
For an M/G/3 queue, our result agrees with Ishikawa [8] as

$$\mathrm{Cov}[\tau_1, \tau_2] = \frac{b_0(-\lambda bb_0 + b_0^2 - b_0^3 - \lambda b_1 - \lambda^2 bb_0 b_1 + \lambda b_0^2 b_1 - \lambda^2 b_1^2)}{\lambda^2(1 + \lambda b_1)^2} \tag{30a}$$

$$\mathrm{Cov}[\tau_1, \tau_3] = \frac{b_0(-bb_0 - b_1 - \lambda bb_0 b_1 - b_0^2 b_1 + b_0^3 b_1 - 2\lambda b_1^2 - \lambda b_0^2 b_1^2 - \lambda^2 b_1^3 + \lambda b_0 b_2 + \lambda^2 b_0 b_1 b_2)}{\lambda(1 + \lambda b_1)^2} \tag{30b}$$

$$\mathrm{Cov}[\tau_1, \tau_n] = a_1^{n-3}\mathrm{Cov}[\tau_1, \tau_3] \qquad n \geq 4 \tag{30c}$$

where $a_1 = -\lambda b_1$ is the probability that exactly one customer arrives during a service time. In Fig. 3(a)–(d), we plot $\rho(\tau_1, \tau_n)$ for $n = 2, 3, 4$ and 5, respectively, for an $\mathrm{M/E}_m/1/3$ queue with $b = 1$ in the cases $m = 1, 2, 5, 10, 30, 100$ and ∞. Here the correlation coefficient can be both positive and negative but vanishes at $\lambda = 0$ and as $\lambda \to \infty$.



(a) $\rho(\tau_1, \tau_2)$ (b) $\rho(\tau_1, \tau_3)$

(c) $\rho(\tau_1, \tau_4)$ (d) $\rho(\tau_1, \tau_5)$

Fig. 3. Correlation coefficients of interdeparture times in an $\mathrm{M/E}_m/1/3$ queue ($m = 1, 2, 5, 10, 30, 100$ and ∞ from below at $\lambda = 1$).

Finally, for an M/G/1/4 queue, we have

$$\mathrm{Cov}[\tau_1, \tau_n] = \frac{b_0^2 D_n}{(-2 - 4\lambda b_1 - 2\lambda^2 b_1^2 + \lambda^2 b_0 b_2)^2} \qquad n = 2, 3, 4, 5 \tag{31}$$

where

$$\begin{aligned}\frac{\lambda^2}{2} D_2 &= -2b\lambda b_0 + 2b_0^2 - 2b_0^4 - 2\lambda b_1 - 4b\lambda^2 b_0 b_1 + 4\lambda b_0^2 b_1 - 4\lambda^2 b_1^2 - 2b\lambda^3 b_0 b_1^2 + 2\lambda^2 b_0^2 b_1^2 \\ &\quad -2\lambda^3 b_1^3 + b\lambda^3 b_0^2 b_2 - \lambda^2 b_0^3 b_2 + \lambda^3 b_0 b_1 b_2\end{aligned} \tag{32a}$$

$$\begin{aligned}\frac{\lambda^2}{2} D_3 &= -2b\lambda b_0 + 2b_0^3 - 2b_0^4 - 2\lambda b_1 - 4b\lambda^2 b_0 b_1 - 2\lambda b_0^2 b_1 + 4\lambda b_0^3 b_1 - 6\lambda^2 b_1^2 - 2b\lambda^3 b_0 b_1^2 \\ &\quad -4\lambda^2 b_0^2 b_1^2 + 2\lambda^2 b_0^3 b_1^2 - 6\lambda^3 b_1^3 - 2\lambda^3 b_0^2 b_1^3 - 2\lambda^4 b_1^4 + 2\lambda^2 b_0 b_2 + b\lambda^3 b_0^2 b_2 - \lambda^2 b_0^4 b_2 \\ &\quad +5\lambda^3 b_0 b_1 b_2 + \lambda^3 b_0^3 b_1 b_2 + 3\lambda^4 b_0 b_1^2 b_2 - \lambda^4 b_0^2 b_2^2\end{aligned} \tag{32b}$$

$$\begin{aligned}\lambda D_4 &= -4bb_0 - 4b_1 - 8b\lambda b_0 b_1 - 8b_0^3 b_1 + 8b_0^4 b_1 - 12\lambda b_1^2 - 4b\lambda^2 b_0 b_1^2 + 4\lambda b_0^2 b_1^2 - 16\lambda b_0^3 b_1^2 \\ &+4\lambda b_0^4 b_1^2 - 12\lambda^2 b_1^3 + 8\lambda^2 b_0^2 b_1^3 - 8\lambda^2 b_0^3 b_1^3 - 4\lambda^3 b_1^4 + 4\lambda^3 b_0^2 b_1^4 + 4\lambda b_0 b_2 + 2b\lambda^2 b_0^2 b_2 \\ &+2\lambda b_0^3 b_2 - 2\lambda b_0^5 b_2 + 12\lambda^2 b_0 b_1 b_2 + 4\lambda^2 b_0^3 b_1 b_2 + 4\lambda^2 b_0^4 b_1 b_2 + 10\lambda^3 b_0 b_1^2 b_2 + 2\lambda^4 b_0 b_1^3 b_2 \\ &-2\lambda^3 b_0^2 b_2^2 - \lambda^3 b_0^4 b_2^2 - \lambda^4 b_0^2 b_1 b_2^2 - 2\lambda^2 b_0^2 b_3 - 4\lambda^3 b_0^2 b_1 b_3 - 2\lambda^4 b_0^2 b_1^2 b_3 + \lambda^4 b_0^3 b_2 b_3\end{aligned} \tag{32c}$$

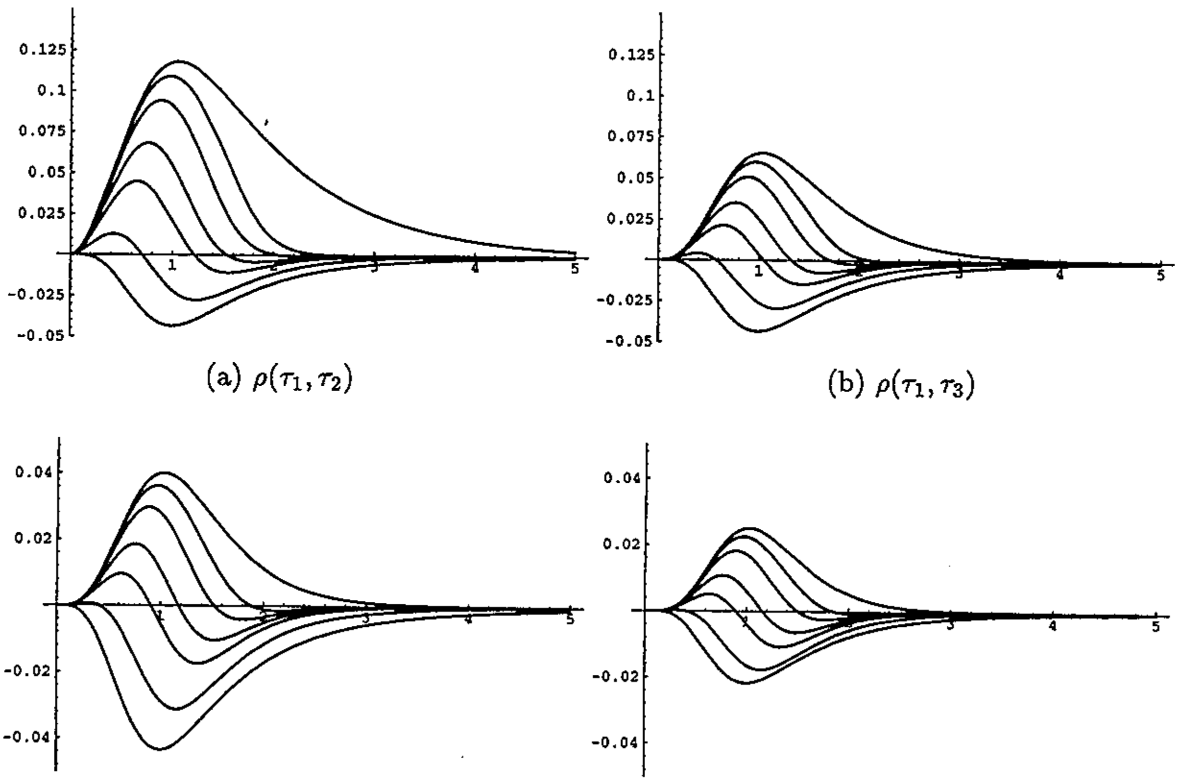
$$\begin{aligned}D_4 &= 8bb_0 b_1 + 8b_1^2 + 20b\lambda b_0 b_1^2 + 12b_0^3 b_1^2 - 12b_0^4 b_1^2 + 28\lambda b_1^3 + 16b\lambda^2 b_0 b_1^3 - 4\lambda b_0^2 b_1^3 + 24\lambda b_0^3 b_1^3 \\ &-8\lambda b_0^4 b_1^3 + 36\lambda^2 b_1^4 + 4b\lambda^3 b_0 b_1^4 - 8\lambda^2 b_0^2 b_1^4 + 12\lambda^2 b_0^3 b_1^4 + 20\lambda^3 b_1^5 - 4\lambda^3 b_0^2 b_1^5 + 4\lambda^4 b_1^6 \\ &-2b\lambda b_0^2 b_2 + 2b_0^4 b_2 - 2b_0^5 b_2 - 10\lambda b_0 b_1 b_2 - 8b\lambda^2 b_0^2 b_1 b_2 - 6\lambda b_0^3 b_1 b_2 + 4\lambda b_0^4 b_1 b_2 + 4\lambda b_0^5 b_1 b_2 \\ &-34\lambda^2 b_0 b_1^2 b_2 - 4b\lambda^3 b_0^2 b_1^2 b_2 - 12\lambda^2 b_0^3 b_1^2 b_2 - 4\lambda^2 b_0^4 b_1^2 b_2 - 36\lambda^3 b_0 b_1^3 b_2 - 4\lambda^3 b_0^3 b_1^3 b_2 \\ &-12\lambda^4 b_0 b_1^4 b_2 + 2\lambda^2 b_0^2 b_2^2 + b\lambda^3 b_0^3 b_2^2 - \lambda^2 b_0^5 b_2^2 + 9\lambda^3 b_0^2 b_1 b_2^2 + 3\lambda^3 b_0^4 b_1 b_2^2 + 7\lambda^4 b_0^2 b_1^2 b_2^2 \\ &-\lambda^4 b_0^3 b_2^3 + 4\lambda^2 b_0^2 b_1 b_3 + 8\lambda^3 b_0^2 b_1^2 b_3 + 4\lambda^4 b_0^2 b_1^3 b_3 - 2\lambda^4 b_0^3 b_1 b_2 b_3\end{aligned} \tag{32d}$$

We note that Ishikawa [8] derives $\mathrm{Cov}[\tau_1, \tau_n]$ for an M/M/1/K queue as

$$\mathrm{Cov}[\tau_1, \tau_n] = -\left(\frac{\lambda - \mu}{\lambda^K - \mu^K}\right)^2 (\lambda\mu)^{K-2} \qquad 2 \le n \le K \tag{33}$$

independent of n. We have confirmed that our results in (31) with (32a)–(32c) reduce to (33). However, Ishikawa's results for $\mathrm{Cov}[\tau_1, \tau_3]$ and $\mathrm{Cov}[\tau_1, \tau_4]$ do not. In fact, our results in (31) with (32b)–(32d) are different from the corresponding results by Ishikawa.

In Fig. 4(a)–(d), we plot $\rho(\tau_1, \tau_n)$ similarly. By comparing with Fig. 3(a)–(d), we see that the correlation coefficient generally increases with the capacity K.



(a) $\rho(\tau_1, \tau_2)$ (b) $\rho(\tau_1, \tau_3)$

(c) $\rho(\tau_1, \tau_4)$ (d) $\rho(\tau_1, \tau_5)$

Fig. 4. Correlation coefficients of interdeparture times in an M/E_m/1/4 queue ($m = 1, 2, 5, 10, 30, 100$ and ∞ from below at $\lambda = 1$).

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