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by

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OPTIMAL STOPPING PROBLEM WITH CONTROLLED RECALL

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This paper deals with the following discrete-time optimal stopping problem. For fixed search cost s, a random offer, $w \sim F(w)$, will be found for each time. This offer is either accepted, rejected, or "reserved" for recall later. The reserving cost for any offer depends on its value, regardless of how long the offer is reserved. The objective is to maximize the expected discounted net profit, provided that an offer must be accepted. The major finding is that no previously reserved offer should be accepted prior to the deadline of the search process.

1. Introduction

A problem of finding an optimal decision rule for accepting one of offers observed sequentially is usually called the optimal stopping problem. The subsequent offers are assumed to have certain stochastic values. The problem can be categorized into two groups in terms of its objective function: Maximization of the expected value of an accepted offer [1,4-15,18-21,23-25] and maximization of the probability of accepting the best offer [2,3,16,17,22]. The latter is usually referred to as the secretary problem. Optimal stopping problems are also classified into three models in view of the future availability of offers once inspected and passed up: Allowed recall, no recall, and uncertain recall. In allowed recall, the searcher is permitted to accept any offer once passed up [6,9,14,15,18,19]. In no recall, the passed up offer is instantly lost, unavailable forever; hence an offer to be accepted is limited to the latest one [2,3,5,6,8,12,13-15,18,20-25]. In uncertain recall, future availability is assumed to be probabilistic [4,10,11,16,17].

Not only in the models with allowed recall and no recall but in the model with uncertain recall, the recallability of each offer is determined independently of the searcher's will. In Karni et al.[10], the probability of a successful recall is assumed to decrease strictly with the time elapsed since offer appearance. Petrucelli[17] assumes that the probability is nonincreasing in the quantile of the offer as well as the passing of time. In Ikuta[4], the uncertainty is defined by the probability of a presently available offer becoming unavailable at the next point in time.

This paper gives the searcher the ability to control the recallabilities of offers. More precisely, we define a model of an optimal stopping problem where any offer is assumed to be forever recallable if some cost, called reserving cost in this paper, is paid for the offer and to be irrevocable if it is neither accepted nor reserved. The major finding in this paper is that accepting a reserved offer, however good it may be, will never become an optimal action, except at the deadline of the process.

In Section 2, we present the general structure of this model and its assumptions. Section 3 defines two functions used in subsequent sections. The optimal equation of this model is formulated in Section 4. In Section 5, the optimal decision rule is prescribed and its properties are revealed. In Section 6, we summarize how reserving cost or distribution of offers influences reserving action. Up to this point we deal with a finite planning horizon model. Its extension, an infinite planning horizon model, is discussed in Section 7. We find that the concept of reserving is similar to that of option buying in the option market. We state similarities and differences between these in Section 8.

2. Model

A searcher sequentially samples offers $w, w', \cdots$, which are i.i.d.random variables having a known distribution function F with a finite mean μ where the worst value of an offer is a and the best is b where $0 \leq a < b < \infty$. After having drawn an offer w, he must decide whether to (1) accept one of current available offers, or (2) reserve the current offer w, or (3) pass up the current offer w. Action (1) terminates the search process, while actions (2) and (3) force him to continue the search. Although he can continue the search as long as he wishes within a given planning horizon, he must accept an offer up to the deadline where offers once spurned cannot be taken up again (no recall). If he continues, a fixed expense $s > 0$ (search cost) must be paid to elicit a next offer. Reserving an offer w allows the searcher to recall and accept it at any time in the future but a reserving cost $r(w)$, depending on w, is required. Hence available offers at each time consist of the current offer w and all offers reserved so far. Obviously, the economically meaningful reserved offer is only that with the highest value. Let us call it the leading offer. Finally, let us assume:

1. $r(w)$ is nondecreasing and continuous in w and satisfies $r_a \leq r(w) \leq r_b$ with given r_a and r_b such that $0 < r_a \leq r_b < \infty$,
2. $a < -s+\beta\mu$

where β is a per-period discount factor with $0 < \beta \leq 1$. $-s+\beta\mu$ implies the expected marginal net profit from one more inspection and a is the value of the worst offer. Therefore, if $a \geq -s+\beta\mu$, no one is willing to engage in the search process. This will be theoretically ascertained in Section 5.

The objective here is to find an optimal decision rule to maximize the total expected discounted net profit, that is, the expectation of the present discounted value of an accepted offer minus that of the amount of the search costs and reserving costs paid up to the termination of the search with acceptance.

3. Preliminaries

For convenience of later discussions, we define, for all x,

$$S(x) = \int_a^b \max\{w, x\} dF(w), \tag{3.1}$$

$$K(x) = \beta \int_a^b \max\{w, x\} dF(w) - x - s \;=\; \beta S(x) - x - s. \tag{3.2}$$

Lemma 3.1

(a) *$S(x)$ is continuous, convex, and strictly increasing for $a \leq x$.*

(b) *$S(x) = \mu$ for $x \leq a$, $x < S(x)$ for $a < x < b$, and $S(x) = x$ for $b \leq x$.*

(c) *$K(x) = 0$ has a unique root h^o with $h^o < b$. And $a < h^o$ if and only if $a < -s+\beta\mu$.*

(d) *h^o is continuous and strictly increasing in β.*

Proof: See Ikuta[7] for the proofs of (a-c) ($S(x) = T(x)+x$ in Ikuta[7]). Assertion (d) is evident from the fact that $K(x)$ is continuous and strictly increasing in β. ∎

4. Optimal Equation

For convenience, we make a distinction between accepting the current offer w and accepting the leading offer x. Throughout this paper, w is used as the current offer and x as the leading offer. Let t indicate the point in time with $t = 0$ at the deadline and the time measured, equally spaced, backward from this point. Hence t also represents the number of periods remaining.

We define $v_t(x)$ as the maximum expected net profit attainable by starting from time t with a leading offer x. The backward induction argument then gives

$$v_0(x) = \int_a^b \max \left\{ \begin{array}{lcl} \text{AS} & : & w, \\ \text{PS} & : & x \end{array} \right\} dF(w), \tag{4.1}$$

$$v_t(x) = \int_a^b \max \left\{ \begin{array}{lcl} \text{AS} & : & w, \\ \text{RC} & : & -r(w)-s+\beta v_{t-1}(w), \\ \text{PS} & : & x, \\ \text{PC} & : & -s+\beta v_{t-1}(x) \end{array} \right\} dF(w), \qquad t \geq 1 \tag{4.2}$$

where symbols AS, RC, PS, and PC refer to the possible actions: "accept the current offer w and stop the search", "reserve w and continue the search", "pass up w and stop searching by accepting the leading offer x", and "pass up w and continue searching", respectively.

Originally, the expression corresponding to RC should be written $-r(w)-s+\beta v_{t-1}(\max\{w, x\})$. However, reserving a current offer w with $w < x$ produces $-r(w)-s+\beta v_{t-1}(x)$, which is less than $-s+\beta v_{t-1}(x)$, the expression of PC. Accordingly, it is not optimal to reserve an offer w such as $w < x$, and the expression of RC can be written as in Eq.(4.2).

Lemma 4.1

(a) *$v_t(x)$ is continuous, convex, and nondecreasing in x, and nondecreasing in t.*

(b) *$v_t(x) > x$ for $x < b$, $v_t(b) = b$, and $v_t(x) \geq \mu$ for all x.*

(c) *$\beta v_t(x) - x$ is strictly decreasing in x.*

Proof: (a) The former part is established by induction starting with the fact that the assertion is true for $t = 0$ since $v_0(x) = S(x)$, and the latter part, with the fact of $v_0(x) \leq v_1(x)$ from Eqs.(4.1) and (4.2).

(b) By Lemma 3.1(b), it follows that $v_0(x) > x$ for $x < b$ and $v_0(b) = b$. Hence we obtain the first assertion $v_t(x) > x$ for $x < b$ from (a). Letting $v_{t-1}(b) = b$, we get $-s+\beta v_{t-1}(b) \leq b$ and $-r(w)-s+\beta v_{t-1}(w) \leq b$ for $w \leq b$ from (a). Coupling all of these facts, we can confirm that, for $x = b$, each term in braces of Eq.(4.2) is less than or equal to b, and thus we get $v_t(b) = b$. The last assertion is obvious from (a) and the fact of $v_0(a) = S(a) = \mu$.

(c) Choosing any x_1 and x_2 with $x_1 < x_2 \leq b$, we obtain

$$\beta \frac{v_t(x_2) - v_t(x_1)}{x_2 - x_1} \leq \beta \frac{v_t(b) - v_t(x_1)}{b - x_1} \tag{4.3}$$

because $\beta v_t(x)$ becomes convex in x. According to (b), it follows that

$$\beta \frac{v_t(b) - v_t(x_1)}{b - x_1} = \beta \frac{b - v_t(x_1)}{b - x_1} < \beta \frac{b - x_1}{b - x_1} = \beta \leq 1. \tag{4.4}$$

Eqs.(4.4) and (4.3) yields $\beta v_t(x_2)-\beta v_t(x_1) < x_2-x_1$, or $\beta v_t(x_2)-x_2 < \beta v_t(x_1)-x_1$. ∎

We further introduce the following two functions $z_t^o(x)$ and $z_t^r(w)$:

$$z_t^o(x) = \max\{x,\ -s+\beta v_{t-1}(x)\}, \quad t \geq 1, \tag{4.5}$$

$$z_t^r(w) = \max\{w,\ -r(w)-s+\beta v_{t-1}(w)\}, \quad t \geq 1 \tag{4.6}$$

where $z_0^o(x) = x$ and $z_0^r(w) = w$. The former consists of the third and fourth terms in braces of Eq.(4.2), and the latter, the first and second. Their fundamental properties stated in the following lemma are simply derived from their definitions and Lemma 4.1.

Lemma 4.2

(a) $z_t^o(x)$ *is continuous, convex, and nondecreasing in* x*, and nondecreasing in* t*.*

(b) $z_t^r(w)$ *is continuous in* w *and nondecreasing in* t*.*

(c) $z_t^r(x) \leq z_t^o(x)$ *for all* x*.*

The function $z_t^o(x)$ is interpreted as the maximum expected net profit by first passing up the current offer w and then doing what is optimal, and $z_t^r(w)$ the maximum expected net profit by first deciding either to accept or reserve w and then proceeding optimally. Suppose that we have just drawn an offer w at time t with the leading offer x. Then, if w satisfies $z_t^o(x) \leq z_t^r(w)$, we can expect to gain a higher reward by not rejecting the offer w than rejecting. Let a set of such offers, or a set of all offers which we should not reject, be denoted by $W_t(x)$, that is, $W_t(x) = \{w \mid z_t^o(x) \leq z_t^r(w)\},\ t \geq 0$. These enable us to rewrite $v_t(x)$ as

$$v_t(x) = \int_a^b \max\{z_t^r(w),\ z_t^o(x)\}dF(w) \tag{4.7}$$

$$= \int_{W_t(x)} z_t^r(w)dF(w) + \int_{W_t(x)^c} z_t^o(x)dF(w). \tag{4.8}$$

We also define h_t^o and h_t^r as the values of x and w which equate two terms in braces of Eq.(4.5) and Eq.(4.6), respectively. So that h_t^o is a point of indifference between accepting x (PS) and continuing the search (PC). The quantity h_t^r is a point of indifference between accepting w (AS) and reserving w (RC).

Therefore the optimal decision rule is described by using h_t^o, h_t^r, and $W_t(x)$. We shall reveal some of their properties in the next section.

Remark. If $r(w) = 0$ for all w, or nothing is lost in reserving offers, we will reserve all offers. So that our model is virtually reduced to the conventional model with allowed recall (see Ikuta[6]). If $r(w) \geq w$ for all w, or offers cost more than their values, we will never reserve offers. This implies that our model is eventually reduced to the conventional model with no recall (see Sakaguchi[21]).

5. Optimal Decision Rule

This section is devoted to describing the optimal decision rule, characterized by h_t^o, h_t^r, and $W_t(x)$, and examining its properties.

Lemma 5.1

(a) *Both* h_t^o *and* h_t^r *exist uniquely with* $-s+\beta\mu \le h_t^o < b$ *and* $-r_b-s+\beta\mu \le h_t^r < h_t^o$.

(b) $x \le w$ *for all* $w \in W_t(x)$.

Proof: (a) Let us introduce the following two functions: $g_t^o(x) = -s+\beta v_{t-1}(x)-x$ and $g_t^r(w) = -r(w)-s+\beta v_{t-1}(w)-w$. Evidently h_t^o and h_t^r are respective roots of $g_t^o(x) = 0$ and $g_t^r(w) = 0$. Lemma 4.1(a,c) affords that both g_t^o and g_t^r are continuous and strictly decreasing. As to h_t^o, it follows from Lemma 4.1(b) that $g_t^o(-s+\beta\mu) = \beta v_{t-1}(-s+\beta\mu) - \beta\mu \ge 0$ and $g_t^o(b) = -s+\beta b-b < 0$, and thus $g_t^o(x) = 0$ has a unique root h_t^o on $[-s+\beta\mu, b)$. In the same fashion we have the assertion with respect to h_t^r.

(b) It is easy to show, by use of Lemma 4.2(a-c) and the intermediate-value theorem, that $W_t(x)$ exists for all x. The proof is by contraposition. Letting $w < x$ implies $-r(w)-s+\beta v_{t-1}(w) < -s+\beta v_{t-1}(x)$ by Lemma 4.1(a) and $r(w) > 0$, which yields $z_t^r(w) < z_t^o(x)$. Hence no w where $w < x$ belongs to $W_t(x)$. ∎

Note that h_t^o satisfies $a < h_t^o < b$ by Assumption 2, whereas it is possible that $h_t^r \le a$. This is, however, not important because the case $h_t^r \le a$ indicates only that AC is always preferable to RC. This can be confirmed from the optimal decision rule(I.b) stated later.

From the proof of Lemma 5.1(a) we have the following corollary:

Corollary 5.2 *For any* $t \ge 1$,

$$\text{(a) } z_t^o(x) = \begin{cases} -s+\beta v_{t-1}(x), & x < h_t^o, \\ -s+\beta v_{t-1}(x) = x, & x = h_t^o, \\ x, & h_t^o < x. \end{cases} \qquad \text{(b) } z_t^r(w) = \begin{cases} -r(w)-s+\beta v_{t-1}(w), & w < h_t^r, \\ -r(w)-s+\beta v_{t-1}(w) = w, & w = h_t^r, \\ w, & h_t^r < w. \end{cases}$$

From Corollary 5.2 and definitions of h_t^o, h_t^r, and $W_t(x)$, the optimal decision rule can be prescribed as follows.

Optimal Decision Rule: Let us be at time t with the leading offer x and have just drawn a current offer w:

(I) In the case of $x < h_t^o$:

- (a) If $w \in W_t(x)$ and $h_t^r \le w$, it is optimal to accept the current offer w.
- (b) If $w \in W_t(x)$ and $w < h_t^r$, it is optimal to reserve the current offer w.
- (c) Otherwise ($w \notin W_t(x)$), it is optimal to pass up the current offer w and continue the search.

(II) In the case of $h_t^o \le x$:

- (a) If $w \in W_t(x)$, it is optimal to accept the current offer w.
- (b) Otherwise ($w \notin W_t(x)$), it is optimal to pass up the current offer w and terminate the search by accepting the leading offer x.

A graphic representation of this rule is provided at the end of this section.

We here note that reserving an offer never becomes an optimal action in the case (II) because all $w \in W_t(x)$ satisfy $h_t^r < w$ due to $h_t^r < h_t^o$ from Lemma 5.1(a,b).

Although the rule is classified into two cases, the following theorem shows that the case (II) never occurs, implying that however high a value the leading offer has, it should not be accepted prior to the deadline.

Theorem 5.3

(a) *h_t^o is a constant, which is given by a unique root h^o of $K(x) = 0$.*

(b) *h_t^r is nondecreasing in t.*

Proof: (a) To begin with, we demonstrate that $h_t^o = h_{t+1}^o$ if and only if $v_{t-1}(h_t^o) = v_t(h_t^o)$. Assuming $v_{t-1}(h_t^o) = v_t(h_t^o)$, it follows that $g_{t+1}^o(h_t^o) = g_t^o(h_t^o) = 0$ where g_t^o is as in the proof of Lemma 5.1(a). Hence h_{t+1}^o, a unique root of $g_{t+1}^o(x) = 0$, must be equal to h_t^o. The converse can be also verified by a similar argument.

We can rearrange $v_0(h_1^o)$ and $v_1(h_1^o)$ as follows:

$$v_0(h_1^o) = \int_a^b \max\{w,\ h_1^o\} dF(w)$$
$$= \int_a^{h_1^o} h_1^o \, dF(w) + \int_{h_1^o}^b w \, dF(w), \tag{5.1}$$
$$v_1(h_1^o) = \int_a^b \max\{z_1^r(w),\ z_1^o(h_1^o)\} dF(w)$$
$$= \int_a^b \max\{z_1^r(w),\ h_1^o\} dF(w) \tag{5.2}$$
$$= \int_a^{h_1^o} h_1^o \, dF(w) + \int_{h_1^o}^b w \, dF(w) \tag{5.3}$$

where Corollary 5.2(a) is applied to get Eq.(5.2). Eq.(5.3) follows from Lemma 8.1 and Corollary 5.2(b). Therefore $v_0(h_1^o) = v_1(h_1^o)$, that is, $h_1^o = h_2^o$ holds.

In order to show $h_{t+1}^o = h_{t+2}^o$, let an induction hypothesis be $h_t^o = h_{t+1}^o$, or equivalently, $v_{t-1}(h_t^o) = v_t(h_t^o)$. In exactly the same way as the above, it follows that

$$v_t(h_{t+1}^o) = v_t(h_t^o) = \int_a^{h_t^o} h_t^o \, dF(w) + \int_{h_t^o}^b w \, dF(w)$$
$$= \int_a^{h_{t+1}^o} h_{t+1}^o \, dF(w) + \int_{h_{t+1}^o}^b w \, dF(w) = v_{t+1}(h_{t+1}^o),$$

therefore, $h_{t+1}^o = h_{t+2}^o$ holds.

Consequently h_t^o is independent of t. Furthermore, the value of h_t^o is equal to h^o, a unique root of $K(x) = 0$, because $g_1^o(x)$ and $K(x)$ coincide.

(b) It follows that $0 = g_t^r(h_t^r) \le g_{t+1}^r(h_t^r)$ for all t by Lemma 4.1(a), and this leads us to $h_t^r \le h_{t+1}^r$ since $g_{t+1}^r(w)$ is strictly decreasing in w. ∎

The acceptable value of the leading offer is required to be at least h_t^o at any time. It is, however, impossible for any leading offer itself to become an acceptable offer as time goes on because h_t^o is independent of time t (Theorem 5.3(a)). So the only way to retain an acceptable leading offer is to reserve an offer w with $h_t^o \le w$ sometime in the past. This is, however, also impossible because even if such an offer appears, the optimal action at that time is to accept it, not reserve it ($h_t^o \le w$ implies $h_t^r < w$). Therefore we never retain offers exceeding h_t^o, that is, the case (II) never occurs. Hence we should never recall and accept the leading offer before the deadline. This fact differs from the result of the conventional model with no recall, that is, the optimal decision rule reduces the lowest value of acceptable offers with passing time. (Theorem 5.3(b) is in accord with this point. see Sakaguchi[21].) Therefore the aim to reserve an offer is only to avoid the risk that an offer appearing on the deadline has a very low value. Having no permission to recall past offers, we must accept the offer even if it is the worst one.

Remark. What effect does the lack of Assumption 2 produce? If $-s+\beta\mu \leq a$, then $h_t^o \leq a$ by Lemma 3.1(c), and thus $h_t^r < a$ by Lemma 5.1(a). From these and Lemma 8.2(a), the optimal decision rule is always to accept the offer you first draw.

So far, we have examined the critical levels of leading offers to be accepted (h_t^o) and current offers to be accepted (h_t^r). Next, let us discuss the critical level of offers which should be reserved.

From now on, we use h^o instead of h_t^o.

Theorem 5.4

(a) $W_t(x_1) \supseteq W_t(x_2)$ *for any* x_1 *and* x_2 *such as* $x_1 < x_2$.

(b) $W_t(x) \supseteq W_{t+1}(x)$ *for all* x.

Proof: (a) Using $z_t^o(x_1) \leq z_t^o(x_2)$ by Lemma 4.2(a) with recalling $W_t(x_2) = \{w \mid z_t^o(x_2) \leq z_t^r(w)\}$, we obtain $z_t^o(x_1) \leq z_t^r(w)$ for every $w \in W_t(x_2)$, so $W_t(x_1) \supseteq W_t(x_2)$.

(b) From Lemma 5.1(b), or $x \leq w$ for all $w \in W_{t+1}(x)$, and Lemma 8.3, it follows for $w \in W_{t+1}(x)$ that $z_{t+1}^r(w) - z_t^r(w) \leq z_{t+1}^o(x) - z_t^o(x)$, or equivalently,

$$z_{t+1}^r(w) - z_{t+1}^o(x) \leq z_t^r(w) - z_t^o(x). \tag{5.4}$$

Since $W_{t+1}(x)$ is a set of w such as $0 \leq z_{t+1}^r(w) - z_{t+1}^o(x)$, all $w \in W_{t+1}(x)$ satisfy $0 \leq z_t^r(w) - z_t^o(x)$ from Eq.(5.4). That is, every $w \in W_{t+1}(x)$ belongs to $W_t(x)$. ∎

Letting $R_t(x) = W_t(x) \cap \{w \mid w < h_t^r\}$ and $A_t(x) = W_t(x) \cap \{w \mid h_t^r \leq w\}$, we immediately have $R_t(x_1) \supseteq R_t(x_2)$ and $A_t(x_1) \supseteq A_t(x_2)$ for $x_1 < x_2$ from assertion (a). The former, $R_t(x_1) \supseteq R_t(x_2)$, states that a better leading offer narrows down the range of offers to be reserved and a similar reasoning holds for the latter.

Assertion (b) yields $A_t(x) \supseteq A_{t+1}(x)$, but not $R_t(x) \supseteq R_{t+1}(x)$. Because although the assertion gives $\min R_t(x) \leq \min R_{t+1}(x)$, Theorem 5.3(b) shows $h_t^r \leq h_{t+1}^r$, or equivalently, $\max R_t(x) \leq \max R_{t+1}(x)$. Accordingly it is possible that $R_t(x)$, the range of offers to be reserved, disappears in spite of the approach of the deadline. This problem is taken up in the next section.

The remainder of this section is devoted to a graphic representation of the optimal decision rule using an example where $F(w) = w$ on $[0, 1]$ ($a = 0$ and $b = 1$), $\beta = 0.97$, $s = 0.005$, and $r(w) = 0.01$ for $w < 0.4$, $0.4w - 0.35$ for $0.4 \leq w < 0.6$, and 0.19 for $0.6 \leq w$. Assume we are at time 1 and have the leading offer of value 0.300. In this case we get $h^o = 0.760$ and $h_1^r = 0.501$.

Figure 1 illustrates $z_1^o(x)$ (the upper curve) and $z_1^r(w)$ (the lower curve with a dip) where the horizontal axis represents both the leading offer x and the current offer w. The thick line on the horizontal axis indicates $W_1(0.300) = \{w \mid 0.332 \leq w \leq 0.448,\ 0.523 \leq w\}$, corresponding to the

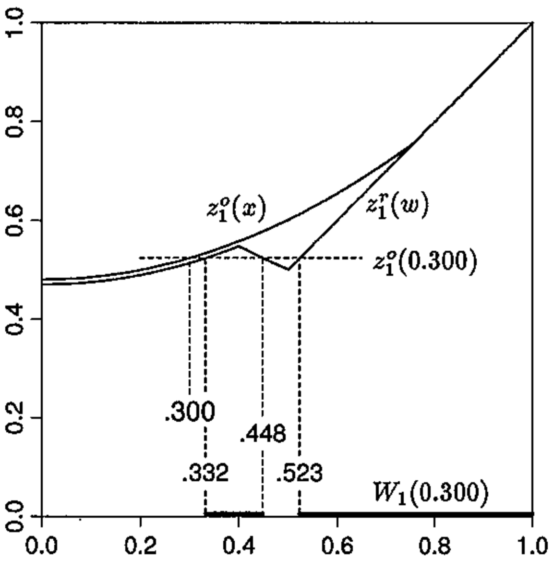

Figure 1. $z_1^o(x)$, $z_1^r(w)$, and $W_1(0.300)$.

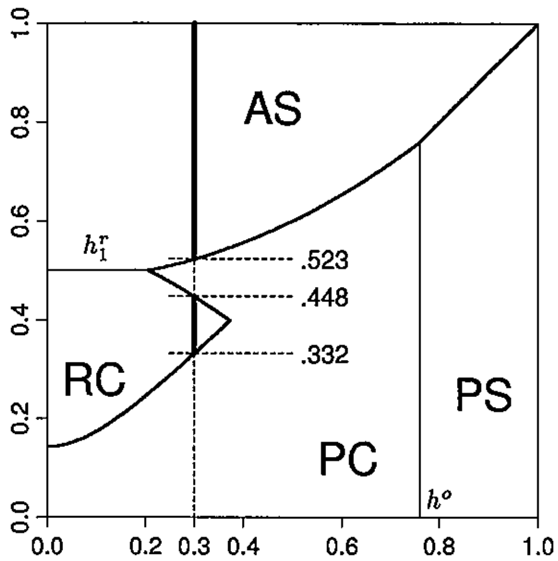

Figure 2. Optimal decision rule for $t = 1$.

vertical thick line in Figure 2. By repeating this argument for other leading offers x, the optimal decision rule can be schematized as Figure 2 where the horizontal axis represents the leading offer x and the vertical axis the current offer w.

Figure 2 tells us that if the current offer w is such that $w < 0.332$, PC is optimal. If $0.332 \leq w \leq 0.448$, RC is optimal. If $0.448 < w < 0.523$, PC again becomes optimal. Otherwise, AS is optimal.

As seen above, a situation may occur where $W_t(x)$ is not simply connected. In the light of the many numerical calculations the author made, the phenomenon tends to occur when $r(w)$ is steady or increases slightly up to a certain w and then rises steeply. Moreover, it can be theoretically verified that the phenomenon never occurs if $r(w)$ is concave due to the fact that $z_t^r(w)$ becomes convex.

6. Reserving Region

Let us define $\mathcal{R}_t = \{(x, w) \mid w \in R_t(x)\}$. This indicates that if $(x, w) \in \mathcal{R}_t$, reserving the current offer w becomes optimal. So let us call the set the reserving region.

In this section, we examine the properties of the reserving region. To begin with, we show the necessary and sufficient condition for which the reserving region becomes empty.

Lemma 6.1 *$\mathcal{R}_t = \phi$ if and only if $\beta\{v_{t-1}(w) - v_{t-1}(a)\} \leq r(w)$ for all $w \leq -s + \beta v_{t-1}(a)$.*

Proof: From Lemma 5.4(a), the necessary and sufficient condition for $\mathcal{R}_t = \phi$ is $R_t(a) = \phi$. This indicates that no offers are worth reserving when the leading offer is a.

Letting $\alpha_t = -s + \beta v_t(a)$, we obtain $a < \alpha_t$ for all t by Assumption 2 and Lemma 4.1(a). Hence, from Eq.(4.2), it is necessary and sufficient for $R_t(a) = \phi$ that, for all w, $-r(w) - s + \beta v_{t-1}(w) \leq \max\{w, \ \alpha_{t-1}\}$, or equivalently, $-s + \beta v_{t-1}(w) - \max\{w, \ \alpha_{t-1}\} \leq r(w)$. The left-hand side of the inequality becomes $-s + \beta v_{t-1}(w) - \alpha_{t-1}$ for $w \leq \alpha_{t-1}$ and $-s + \beta v_{t-1}(w) - w$ for $\alpha_{t-1} \leq w$. Since the former is nondecreasing in w by Lemma 4.1(a) and the latter is strictly decreasing by Lemma 4.1(c),

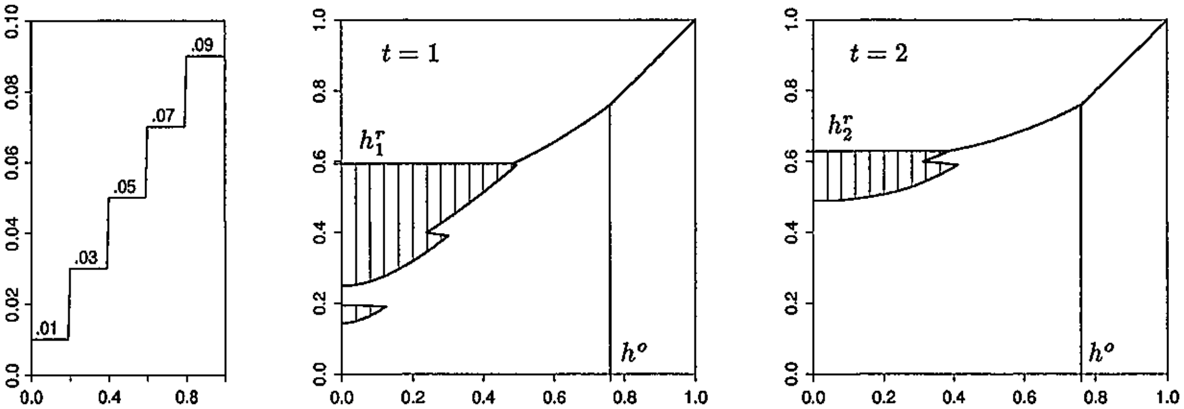

Figure 3. Example A.

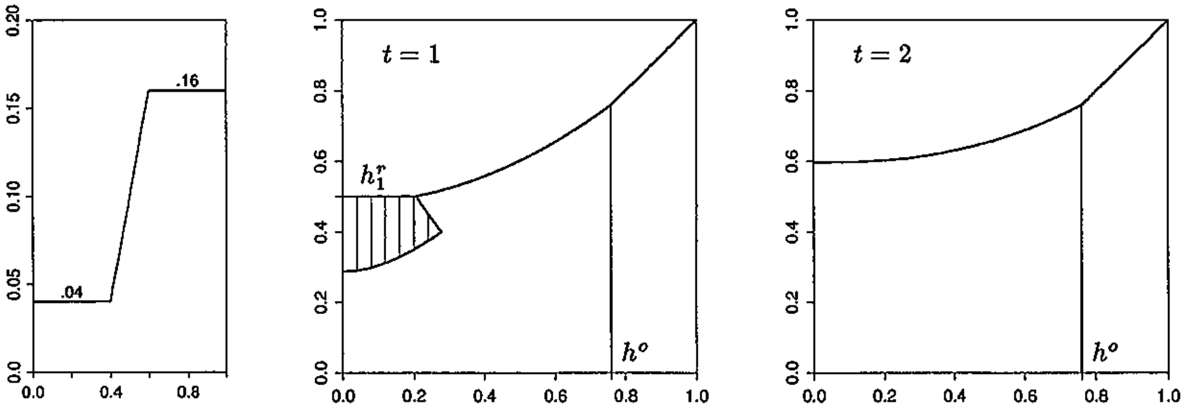

Figure 4. Example B.

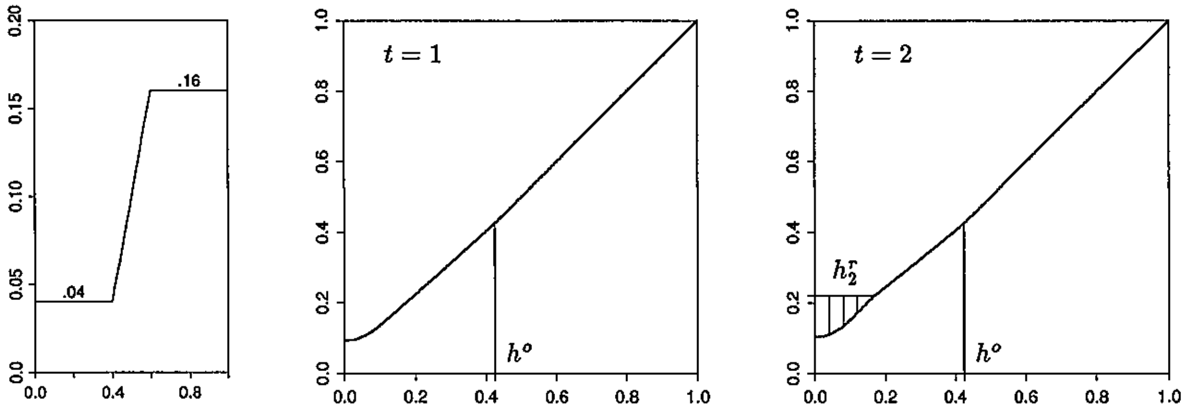

Figure 5. Example C.

$-s+\beta v_{t-1}(w)-\max\{w,\ \alpha_{t-1}\}$ is maximized at $w=\alpha_{t-1}$. Therefore, since $r(w)$ is nondecreasing, $R_t(a)=\phi$ holds if and only if $-s+\beta v_{t-1}(w)-\alpha_{t-1}\le r(w)$ for $w\le\alpha_{t-1}$. ∎

In some cases, the reserving region disappears even when the next time is the deadline, although it does exist at a time when two or more searches remain. Roughly speaking, such a thing tends to occur when $-s+\beta\mu$ is small for fixed a and b. This is because the smaller $-s+\beta\mu$ is, the smaller h_t^r is. That is, the less the search attracts us, the sooner we want to quit the search.

Let us give three examples (Figures 3 to 5) with $a=0$, $b=1$, $\beta=0.97$, and $s=0.005$. In Examples A and B, $F(w)=w$, hence $-s+\beta\mu=0.48$ and in Example C, $F(w)=9w$ for $w<0.1$

and $(w+8)/9$ for $0.1 \le w$, hence $-s+\beta\mu = 0.092$. $r(w)$ in each example is plotted in the diagram on the left (Examples B and C use the same $r(w)$). The diagrams in the middle and on the right indicate the optimal stopping rules for $t = 1$ and $t = 2$ where the shaded portions are the reserving regions.

7. Infinite Planning Horizon

Let us now attempt to extend the discussion into an infinite planning horizon.

From the following theorem and the fact that leading offers are always less than h^o, the optimal decision rule with an infinite planning horizon becomes that if $h^o \le w$, accept the current offer w, or else continue the search without reserving it. This is the same result as in the conventional optimal stopping problem with an infinite planning horizon (see Ikuta[6] and Sakaguchi[21]).

Theorem 7.1 *$v_t(x)$ converges to $\max\{(h^o+s)/\beta,\ S(x)\}$, $W_t(x)$ to $\{w \mid \max\{h^o, x\} \le w\}$, and h_t^r to the solution of $w = -r(w)+h^o$, as $t \to \infty$.*

Proof: Let $v(x)$ denote a function to which $v_t(x)$ converges as $t \to \infty$. Then We can affirm $v(x) = S(x)$ for $h^o \le x$ from Lemma 8.2(b). From this and the fact that $v(x)$ is nondecreasing in x by Lemma 4.1(a), we also obtain $v(x) \le S(h^o)$ $(= (h^o+s)/\beta)$ for $x \le h^o$. Now, letting $z^o(x)$ and $z^r(w)$ be functions to which $z_t^o(x)$ and $z_t^r(w)$ converge, we have

$$v(x) = \int_a^b \max\{z^r(w),\ z^o(x)\} dF(w), \tag{7.1}$$

$$z^o(x) = \max\{x,\ -s+\beta v(x)\}, \tag{7.2}$$

$$z^r(w) = \max\{w,\ -r(w)-s+\beta v(w)\}. \tag{7.3}$$

Accordingly, $W(x)$, a set to which $W_t(x)$ converges, can be written $W(x) = \{w \mid z^o(x) \le z^r(w)\}$. To prove the theorem, we consider two cases: (i) $\beta < 1$ and (ii) $\beta = 1$.

(i) In the case of $\beta < 1$, it suffices to verify that only the function $v(x) = (h^o+s)/\beta$ for $x \le h^o$ satisfies Eq.(7.1). For this, we first check that it satisfies Eq.(7.1) and then show that the other finite function $\bar{v}_t(x)$ for $x \le h^o$ cannot satisfy the equation.

Inserting $v(x) = (h^o+s)/\beta$ into Eq.(7.2), we have $z^o(x) = \max\{x, h^o\} = h^o$ for $x \le h^o$, and thus $W(x) = \{w \mid h^o \le z^r(w)\}$ for $x \le h^o$. Under the assumption of $v(w) = (h^o+s)/\beta$ for $w \le h^o$, it follows that $z^r(w) = \max\{w,\ -r(w)+h^o\} < h^o$ for all $w < h^o$. Furthermore, $h^o \le z^r(w)$ $(= w)$ for all $w \ge h^o$ from Eq.(7.3). Hence we have $W(x) = \{w \mid h^o \le w\}$ for $x \le h^o$. For these reasons, the right-hand side of Eq.(7.1) for $x \le h^o$ becomes

$$\begin{aligned} \int_{W(x)} z^r(w) dF(w) + \int_{W(x)^c} z^o(x) dF(w) &= \int_{h^o}^b w\ dF(w) + \int_a^{h^o} h^o\ dF(w) \\ &= \int_a^b \max\{h^o, w\} dF(w) \\ &= S(h^o)\ =\ (h^o+s)/\beta. \end{aligned} \tag{7.4}$$

Therefore, $v(x) = (h^o+s)/\beta$ for $x \le h^o$ satisfies Eq.(7.1).

Next, assume that another finite function $\bar{v}(x)$ for $x \le h^o$ also satisfies Eq.(7.1), or,

$$\bar{v}(x) = \int_a^b \max\{\bar{z}^r(w),\ \bar{z}^o(x)\} dF(w) \tag{7.5}$$

where $\bar{z}^r(w) = \max\{w,\ -r(w)-s+\beta\bar{v}(w)\}$ and $\bar{z}^o(x) = \max\{x,\ -s+\beta\bar{v}(x)\}$. Let $\Delta = \sup_x |v(x) - \bar{v}(x)|$ where clearly $0 < \Delta < \infty$. By using the general formula

$$|\max\{a_1, b_1\} - \max\{a_2, b_2\}| \le \max\{|a_1 - a_2|, |b_1 - b_2|\}, \tag{7.6}$$

we immediately get from Eqs.(7.1) and (7.5) that

$$|v(x) - \bar{v}(x)| \le \int_a^b \max\{|z^r(w) - \bar{z}^r(w)|,\ |z^o(x) - \bar{z}^o(x)|\}dF(w). \tag{7.7}$$

Since $|z^r(w) - \bar{z}^r(w)| \le \beta\Delta$ and $|z^o(x) - \bar{z}^o(x)| \le \beta\Delta$ from Eq.(7.6), it follows from Eq.(7.7) that $|v(x) - \bar{v}(x)| \le \beta\Delta$, yielding $\Delta \le \beta\Delta$. Since this is contrary to $\beta < 1$, no functions except $v(x) = (h^o+s)/\beta$ for $x \le h^o$ satisfy Eq.(7.1).

(ii) In the case of $\beta = 1$, we use the notation $v(x, \beta)$ and $h^o(\beta)$ instead of $v(x)$ and h^o.

As stated in the beginning of the proof, $v(x, 1) \le (h^o(1)+s)/1$ holds for $x \le h^o(1)$. Simply inductive argument gives $v(x, \beta) \le v(x, 1)$ for all x. It is already verified that $(h^o(\beta)+s)/\beta \le v(x, \beta)$ for all x whenever $\beta < 1$. Combining these three facts, we obtain

$$(h^o(\beta)+s)/\beta \le v(x, \beta) \le v(x, 1) \le (h^o(1)+s)/1 \tag{7.8}$$

for $x \le h^o(1)$. Lemma 3.1(d) affords that $(h^o(\beta)+s) \to (h^o(1)+s)/1$ as $\beta \to 1$. Hence it follows that $v(x, 1) = (h^o(1)+s)/1$ from Eq.(7.8).

From (i) and (ii), it follows that $v_t(x)$ converges to $(h^o+s)/\beta$ for $x \le h^o$, and consequently, $v(x) = \max\{(h^o+s)/\beta,\ S(x)\}$. For such $v(x)$, we have $W(x) = \{w \mid h^o \le w\}$ for $x \le h^o$ as seen in (i). From this and Lemma 8.2(a), we obtain $W(x) = \{w \mid \max\{h^o, x\} \le w\}$. Finally, since $z^r(w) = \max\{w,\ -r(w)+h^o\}$, it follows that h_t^r converges to h^r which satisfies $h^r = -r(h^r)+h^o$. ∎

8. Reserving and Option Buying

The purpose of reserving an offer is to keep it recallable in the future and that of buying an option is to make a profit on it while limiting the risk.

Both actions are similar in the point that they can be regarded as a trade of the right to choose whether or not to do a specified trade. In this model, the specified trade means accepting the reserved offer by the deadline, and in option buying, it means buying (call option) or selling (put option) the underlying asset at the exercise price by the expiration date. To acquire the right, we must pay the reserving cost or the premium and what we may lose by each action is limited to at worst the reserving cost or the premium.

On the other hand, both actions are different in the point that although offers with the same value always require the same reserving cost, options with the same underlying asset, exercise price, and expiration date have premiums varying with the market fluctuation. Furthermore, the profit from any reserved offer is fixed at the moment of reserving, whereas that from any option is not fixed until the moment of exercising or reselling. This is because the profit by accepting a reserved offer depends only on its value and this remains unchanged, while that by exercising an option depends not only on the exercise price but on the underlying asset price at the moment of exercising. In addition to this reason, reserved offers are assumed not to be resold to someone else, whereas held options can make a profit by being resold in the market. Hence, it also can be said that only the leading offer is to be kept in mind, while all previously bought options are to be so. Finally, although we deal with the reserving cost paid in the past as the sunk cost (see Eqs.(4.1) and (4.2)), the premium should not be dealt as such since it is a critical factor in reselling.

For the reasons mentioned above, reserving an offer is apparently similar to buying an option, but essentially different, so it is hard to directly apply the results of this study to the option trading as it is. However, it should prove interesting when this study is extended in the direction of the option trading problem.

Acknowledgments

I wish to express my gratitude to Associate Professor S.Ikuta of the University of Tsukuba for his valuable advice in this investigation.

Appendix: Supplementary Lemmas

Lemma 8.1 *$z_t^r(w) < h_t^o$ for $w < h_t^o$.*

Proof: For all $w < h_t^o$, it follows from Lemma 4.1(a) and Corollary 5.2(a) that

$$-r(w)-s+\beta v_{t-1}(w) \le -r(w)-s+\beta v_{t-1}(h_t^o) < -s+\beta v_{t-1}(h_t^o) = h_t^o.$$

Hence, if $w < h_t^o$, then two components of $z_t^r(w)$ are both less than h_t^o. ∎

Lemma 8.2 *For $h^o \le x$,* (a) *$W_t(x) = \{w \mid x \le w\}$ and* (b) *$v_t(x) = S(x)$.*

Proof: (a) From Corollary 5.2(a,b), if $h^o \le x \le w$, then $z_t^o(x) \le z_t^r(w)$, and hence $\{w \mid x \le w\} \subseteq W_t(x)$ for $h^o \le x$. Conversely, Lemma 5.1(b) shows $W_t(x) \subseteq \{w \mid x \le w\}$ for all x.

(b) We have already shown the case of $t = 0$ in Lemma 4.1(a). For $t \ge 1$, by using the assertion (a), it follows that, for $h^o \le x$,

$$v_t(x) = \int_x^b z_t^r(w)dF(w) + \int_a^x z_t^o(x)dF(w) = \int_x^b w\ dF(w) + \int_a^x x\ dF(w) \tag{8.1}$$

where the right-hand equation follows from Lemma 5.2(b,a). Since the last expression in Eq.(8.1) becomes $S(x)$, we have completed the proof. ∎

Lemma 8.3 *$z_{t+1}^r(w)-z_t^r(w) \le z_{t+1}^o(x)-z_t^o(x)$ for all w and x such that $w \ge x$.*

Proof: To prove this lemma, it is enough to show (a) $z_{t+1}^r(w)-z_t^r(w) \le z_{t+1}^o(w)-z_t^o(w)$ for all w and (b) $z_{t+1}^o(x)-z_t^o(x)$ is nonincreasing in x.

(a) For any w, the inequality $z_{t+1}^r(w)-z_t^r(w) \le z_{t+1}^o(w)-z_t^o(w)$ is clearly of the form

$$\max\{w,\ -r+\theta\} - \max\{w,\ -r+\eta\} \le \max\{w,\ \theta\} - \max\{w,\ \eta\} \tag{8.2}$$

where $r > 0$ and $\eta \le \theta$ by Lemma 4.1(a).

(b) Corollary 5.2(a) affords that $z^o_{t+1}(x) - z^o_t(x)$ becomes $\beta\{v_t(x)-v_{t-1}(x)\}$ for $x \le h^o$ and 0 for $h^o \le x$. So it suffices to show that $v_t(x)-v_{t-1}(x)$ is nonincreasing in $x \le h^o$.

For any x_1 and x_2 with $x_1 < x_2 \le h^o$, we obtain

$$v_t(x_1) = \int_{W_1} z^r_t(w)dF(w) + \int_{W^c_1} z^o_t(x_1)dF(w), \tag{8.3}$$

$$v_t(x_2) = \int_{W_2} z^r_t(w)dF(w) + \int_{W^c_2} z^o_t(x_2)dF(w) \tag{8.4}$$

where $W_1 = W_t(x_1)$ and $W_2 = W_t(x_2)$. From Theorem 5.4(a), or $W_1 \supseteq W_2$, it follows that

$$v_t(x_2) - v_t(x_1) = \int_{W^c_1} \{z^o_t(x_2) - z^o_t(x_1)\}dF(w) + \int_{W_1 \cap W^c_2} \{z^o_t(x_2) - z^r_t(w)\}dF(w). \tag{8.5}$$

Applying the fact of $z^o_t(x_1) \le z^r_t(w)$ for $w \in W_1$ to the second integral above, we obtain

$$\begin{aligned} v_t(x_2) - v_t(x_1) &\le \int_{W^c_1} \{z^o_t(x_2) - z^o_t(x_1)\}dF(w) + \int_{W_1 \cap W^c_2} \{z^o_t(x_2) - z^o_t(x_1)\}dF(w) \\ &= \{z^o_t(x_2) - z^o_t(x_1)\} \int_{W^c_2} dF(w) \;\le\; v_{t-1}(x_2) - v_{t-1}(x_1) \end{aligned}$$

where the last inequality follows from Corollary 5.2(a) and $\beta \int_{W^c_2} dF(w) \le 1$. Therefore it follows that $v_t(x_2) - v_{t-1}(x_2) \le v_t(x_1) - v_{t-1}(x_1)$ for all $x_1 < x_2 \le h^o$. Consequently, we also complete the proof of the latter assertion (b). ∎

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