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On The Existence of a Unique Completely Labeled Simplex

by

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Abstract

For simplices P in the strong standard from, the Labeling Rule precludes the possibility of coexistence of completely labeled simplices of types I and II (see [4]). In this paper we give a new proof of the no-coexistenceness theorem of completely labeled simplices of types I and type II. Further we show an uniqueness theorem of a completely labeled simplex of type II in R^2 if P contains no integral points.

1 Introduction

Recently, Yang [4] proposed a simplicial algorithm for solving a class of NP-complete integer liner programming problems. Namely, his method is designed to test whether an arbitrary simplex in R^n contains an integral point or not. This method can be seen as an easy realization of Scarf's method [2] and [3] in the setting of triangulation. Scarf's method is built upon primitive sets and is very difficult to move from one primitive set to its adjacent one. We also point out that Yang's method can date back to the work of van der Laan and Talman [1]. In this paper we will investigate the properties of simplices in the two dimensional Euclidean space. We will show that there exists a unique completely labeled simplex of type II for any simplex P in the strong standard from. It seems to be very difficult to prove this interesting proporty in higer dimensional space although we conjecture the conclution holds true in higer dimension space.

This paper is organized as follows. In Section 2 we introduce some basic definitions. In Section 3 we present our main results.

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2 Preliminaries

We denote the 2-dimensional Euclidean space by R^2, the set of all integral (non-negative integral) points in R^2 by $Z^2(Z^2_+)$ and the set of integers $\{1, \ldots, n\}$ by I_n. In particular, I^i_n represents the set $I_n \backslash \{i\}$ for any $i \in I_n$. Given a set G in R^2, $\operatorname{Int} G$, $\operatorname{cl} G$ and ∂G denote the interior, the closure and the boundary of G in R^2, respectively. For any $x,\ y \in R^2, x > y$ means $x_i \geq y_i$ for all $i \in I_2$ with at least one inequality. The problem we consider is to test the integral feasibility of an 2-dimensional simplex given by

$$P = \{x \in R^2 \mid a_i^T x \leq b_i,\ \forall i \in I_3\},$$

where $a_i^T = (a_{i1}, a_{i2})$ is the i-th row of a 3×2 matrix A for each $i \in I_3$, and $b = (b_1, b_2, b_3)^T$ is a vector of R^3. Throughout the paper we suppose that a_1, a_2, a_3 and b are integral vectors, and that the following assumeption also holds.

Assumption 1 *The origin of R^2 is contained in the interior of the convex hull of the vectors a_1, a_2, a_3. In other words, there exists a unique vector $\lambda \in R^3_+$ such that $\Sigma_{i \in I_3} \lambda_i a_i = 0$ and $\Sigma_{i \in I_3} \lambda_i = 1$*

Let 3×2 matrix A satisfies the following conditions:

Assumption 2

$$(i) \qquad a_{ij} \leq 0 \quad \forall i \in I_3,\ \forall j \in I_2 \text{ with } j \neq i;$$

$$(ii) \qquad |a_{ij}| < a_{ii} \quad \forall i, j \in I_2 \text{ with } j \neq i.$$

A simplex P is said to be in the strong standard form if its associated matrix A satisfies Assumption 2. Now we introduce the following labeling rule.

Labeling rule: Let $L : Z^2 \to I_3 \cup \{0\}$ be a labeling function defined by

$$L(x) = \begin{cases} 0, & \text{if } a_i^T x - b_i \leq 0 \ \forall i \in I_3; \\ \min\{i \in I_3 \mid a_i^T x - b_i \geq a_j^T x - b_j \ \forall j \in I_3\}, & \text{if } a_i^T x - b_i > 0 \ \exists i \in I_3. \end{cases}$$

Remark 1. *The geometric context of the labeling function is quite intuitive, that is,*

$$(i)\ L(x) = 0 \ \text{ iff } \ x \in P \cap Z^2; \quad (ii)\ L(x) \in I_3 \ \text{ iff } \ x \in Z^2 \backslash P.$$

Given an 2-simplex σ with vertices $\{x^1, x^2, x^3\}$, let $L(\sigma) = \{L(x^i) \mid i \in I_3\}$. We introduce the following definition (see also Yang [4]).

Definition 1 *A simplex σ is called a completely labeled (c. l.) simplex if $|L(\sigma)| = 3$. In particular, σ is called a completely labeled of type II if $L(\sigma) = I_3$ and a completely labeled simplex of type I if $L(\sigma) = \{0\} \cup I_3^i$ for some $i \in I_3$.*

Observe that a c.l. simplex of type I has a vertex being an integral point in P. We recall the following theorem due to Yang [4].

Theorem 1 *Let P be a simplex in the strong standard form in R^2. Then there exists at last one completely labeled simplex of type I or II.*

3 Main results

Let $\Im$ be the K_1-triangulation of R^2 to be described later. This triangulation is such that the collection of the vercties of simplieces in $\Im$ is the set of all integer points of R^2. The $e(i)$ denotes the i-th unit vector in R^2 for $i = 1, 2$. Let

$$q(i) = -e(1),\ q(2) = -e(2) \ \text{ and } \ q(3) = e(1) + e(2).$$

If $x^1 \in Z^2$ and $\pi = (\pi(1), \pi(2))$ is a permutation of the elements of the set I_2, then denote the 2-simplex with vertices x^1, x^2, x^3 by $\sigma(x^1, \pi)$ where $x^{i+1} = x^i + q(\pi(i))$ for $i \in I_2$. The K_1-triangulation $\Im$ of R^2 is the collection of all such simplices. Given a 2-simplex $\sigma(x^1, \pi)$ in $\Im$, let

$$\vec{L}(\sigma) = (L(x^1), L(x^2), L(x^3))$$

and we call it the label of the simplex σ. Note that $\vec{L}(\sigma)$ is different from $L(\sigma)$.

For any $x \in R^2$ and $y \in R^2 \backslash \{x\}$, there exists $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in Z^3_+$ such that

$$y = x + \Sigma_{i \in I_3} \lambda_i q(i).$$

We define

$$\Lambda(y) = \{\lambda \in Z^3_+ \mid y = x + \Sigma_{i \in I_3} \lambda_i q(i)\}$$

and

$$\|\lambda\| = \lambda_1 + \lambda_2 + \lambda_3 \ \text{ for any } \lambda \in Z^3_+.$$

Note $\|\lambda\| \geq 1$ if $\lambda \in \Lambda(y)$. It is clear to see that there is an unique elment in $\Lambda(y)$ and we write

$$\lambda_y = \text{argmin}\,\{\|\lambda\| \mid \lambda \in \Lambda(y)\}.$$

It is clear that λ_y is uniquely determined by x and y. For simplicity we alway take $\lambda = \lambda_y$ in the following except for special declaration.

For any $\lambda \in \Lambda(y)$, we define

$$\text{supp}\,\lambda = \{i \in N \mid \lambda_i > 0\}.$$

Then we can easily show the following claim.

Claim. *For any* $\lambda \in \Lambda(y)$, $\lambda = \lambda_y$ *if only and if* $|supp\,\lambda| \leq 2$.

Now we give the following definition.

Definition 2 *For* $x \in Z^2$, *we define the regions* $G_i(x)$ *as follows.*

$$G_i(x) = \Big\{ y \in R^2 \backslash \{x\} \mid \lambda = \lambda_y \;\; s.t., \;\; \lambda_i = max\{\lambda_j \,|\, j \in I_3\} \Big\}$$

and

$$G_{i,j}(x) = \Big\{ y \in G_i(x) \mid \lambda = \lambda_y \;\; s.t., \;\; \lambda_j = max\{\lambda_k > 0 \mid k \in I_3^i\} \Big\}.$$

Then it is clear that $G_i(x) = \cup_{j \in I_3^i} G_{i,j}(x)$ and $R^2 \backslash \{x\} = \cup_{i \in I_3} G_i(x)$.

In this section we suppose that a simplex P is given in the strong standard form in R^2. Let A be a 3×2 matrix satisfying Asermption 2. The following remark will often be used later.

Remark 2.

$$a_i^T q(j) \geq 0 \quad if \quad i \neq j \;\; and \quad a_i^T q(j) \leq 0 \quad if \quad i = j \quad for \quad i, j \in I_3.$$

In particular, $a_i^T q(n+1) > 0$ *and* $a_i^T q(i) < 0$ *for* $i \in I_2$.

First we show the followig lemmas.

Lemma 1 *Let* $\sigma = \sigma(x^1, \pi)$ *be a c. l. simplex of type II. Then* $L(x^1) \neq 3$.

Proof. On the contrary, assume $L(x^1) = 3$. Since $x^3 = x^1 - q(3)$, we have

$$\begin{aligned} a_3^T x^3 - b_3 &= a_3^T x^1 - b_3 - a_3^T q(3) \\ &> a_i^T x^1 - b_i - a_i^T q(3) \\ &= a_i^T x^3 - b_i, \end{aligned}$$

where $i = 1, 2$. Then $L(x^3) \neq i$ for $i = 1, 2$. It is a contradiction. ∎

Lemma 2 *Let* $\sigma = \sigma(x^1, \pi)$ *be a c. l. simplex of type II in* R^2. *Then*

$$\vec{L}(\sigma) = (\pi(1), \pi(2), 3) \quad if \;\; L(x^1) = \pi(1)$$

and

$$\vec{L}(\sigma) = (\pi(2), 3, \pi(1)) \quad if \;\; L(x^1) = \pi(2)$$

where $(\pi(1), \pi(2)) \in \{(1, 2), (2, 1)\}$.

Proof. (i) Assume $L(x^1) = \pi(1)$. We only need to show that $L(x^2) = \pi(2)$. On the contrary, suppose $L(x^2) = 3$. Since $x^3 = x^2 + q(\pi(2))$, we have

$$\begin{aligned} a_3^T x^3 - b_3 &= a_3^T x^2 - b_3 + a_3^T q(\pi(2)) \\ &> a_{\pi(2)}^T x^2 - b_{\pi(2)} \\ &\geq a_{\pi(2)}^T x^2 - b_{\pi(2)} + a_{\pi(2)}^T q(\pi(2)) \\ &= a_{\pi(2)}^T x^3 - b_{\pi(2)}. \end{aligned}$$

Then $L(x^3) \neq \pi(2)$. This contradiction implies $\vec{L}(\sigma) = (\pi(1), \pi(2), 3)$.

(ii) Assume $L(x^1) = \pi(2)$. Since $x^2 = x^1 + q(\pi(1))$, we have

$$\begin{aligned} a_{\pi(2)}^T x^2 - b_{\pi(2)} &= a_{\pi(2)}^T x^1 - b_{\pi(2)} + a_{\pi(2)}^T q(\pi(1)) \\ &\geq a_{\pi(1)}^T x^1 - b_{\pi(1)} \\ &> a_{\pi(1)}^T x^1 - b_{\pi(1)} + a_{\pi(1)}^T q(\pi(1)) \\ &= a_{\pi(1)}^T x^2 - b_{\pi(1)}. \end{aligned}$$

Then $L(x^2) \neq \pi(1)$ and $\vec{L}(\sigma) = (\pi(2), 3, \pi(1))$.

Remark 3. *For $i \in I_2$ and $j = 3 - i$, the above Lemma 2 can be stated as follows.*

(a) *If $L(x^1) = i$ and $\pi = (i, j)$, then $\vec{L}(\sigma) = (i, j, 3)$.*

(b) *If $L(x^1) = i$ and $\pi = (j, i)$, then $\vec{L}(\sigma) = (i, 3, j)$.*

In the following proofs so long as no confusion arises, we replace $G_i(x)$ and $G_{i,j}(x)$ by G_i and $G_{i,j}$ for $i, j \in I_3$, respectively.

Let $\sigma(x^1, \pi)$ be a c.l. simplex of type II in R^2. It is clear that $L(x^1) = 1$ or $L(x^1) = 2$ by Lemma 1. For any $x \in R^2 \backslash \{x^1\}$, we have $x = x^1 + \Sigma_{j \in I_3} \lambda_j q(j)$, where $\lambda_j \geq 0$. In the proof of the following propositions and theorems, we assume that $i, j \in I_2$ with $i + j = 3$.

Proposition 1 *Let $\sigma(x^1, \pi)$ be a c. l. simplex of type II in R^2. Then*

$$L(x) \neq i \quad \textit{for any} \quad x \in Z^2 \cap \mathit{Int} G_i$$

with $i \in I_2$. In particular, if $\vec{L}(\sigma) \neq (j, 3, i)$, then

$$L(x) \neq i \quad \textit{for any} \quad x \in Z^2 \cap (G_i \backslash G_3)$$

with $i \in I_2$.

Proof. Choose $i \in I_2$ and now we show the first part of the proposition. We need to consider the

Case 1. $\lambda_i > \lambda_3 \geq \lambda_j = 0$, i.e., $x \in Z^2 \cap (G_{i,3} \backslash G_3)$. Then

$$x = x^1 + \lambda_i q(i) + \lambda_3 q(3). \tag{1}$$

There are three subcases.

(i) Let $L(x^1) = i$ and $\pi = (i, j)$. Then $\vec{L}(\sigma) = (i, j, 3)$ by Remark 3. Using (1), we have $x = x^2 + (\lambda_i - 1)q(i) + \lambda_3 q(3)$

$$\begin{aligned} a_j^T x - b_j &= a_j^T x^2 - b_j + a_j^T((\lambda_i - 1)q(i) + \lambda_3 q(3)) \\ &\geq a_j^T x^2 - b_j \\ &\geq a_i^T x^2 - b_i + (\lambda_i - 1)a_i^T(q(i) + q(3)) \\ &\geq a_i^T x^2 - b_i + a_i^T((\lambda_i - 1)q(i) + \lambda_3 q(3)) \\ &= a_i^T x - b_i. \end{aligned}$$

(ii) Let $L(x^1) = i$ and $\pi = (j, i)$. Then $\vec{L}(\sigma) = (i, 3, j)$ by Remark 3. Using (1), we have $x = x^3 + \lambda_i q(i) + (\lambda_3 + 1)q(3)$ and

$$\begin{aligned} a_j^T x - b_j &= a_j^T x^3 - b_j + a_j^T(\lambda_i q(i) + (\lambda_3 + 1)q(3)) \\ &\geq a_j^T x^3 - b_j \\ &\geq a_i^T x^3 - b_i + \lambda_i a_i^T(q(i) + q(3)) \\ &\geq a_i^T x^3 - b_i + a_i^T(\lambda_i q(i) + (\lambda_3 + 1)q(3)) \\ &= a_i^T x - b_i. \end{aligned}$$

(iii) $L(x^1) = j$ and $\pi = (\pi(1), \pi(2))$. Then

$$(a) \quad \vec{L}(\sigma) = (j, i, 3) \text{ if } \pi = (j, i)$$

and

$$(b) \quad \vec{L}(\sigma) = (j, 3, i) \text{ if } \pi = (i, j).$$

In both cases, we have

$$\begin{aligned} a_j^T x - b_j &= a_j^T x^1 - b_j + a_j^T(\lambda_i q(i) + \lambda_3 q(3)) \\ &\geq a_j^T x^1 - b_j \\ &\geq a_i^T x^1 - b_i + \lambda_3 a_i^T(q(i) + q(3)) \\ &\geq a_i^T x^1 - b_i + a_i^T(\lambda_i q(i) + \lambda_3 q(3)) \\ &= a_i^T x - b_i. \end{aligned}$$

Note that in the above inequality holds strictly if $j > i$. Therefore we have

$$L(x) \neq i \quad \text{for} \quad x \in Z^2 \cap (G_{i,3} \backslash G_3).$$

Case 2. $\lambda_i > \lambda_j > \lambda_3 = 0$, i.e., $x \in Z^2 \cap \operatorname{Int} G_{i,j}$. Then

$$(2) \qquad x = x^1 + \lambda_i q(i) + \lambda_j q(j).$$

There are three subcases:

(i) $L(x^1) = i$ and $\pi = (j, i)$. Then $\vec{L}(\sigma) = (i, 3, j)$ by Remark 3. Using (2), we have $x = x^2 + \lambda_i q(i) + (\lambda_j - 1)q(j)$ and

$$\begin{aligned} a_3^T x - b_3 &= a_3^T x^2 - b_3 + a_3^T(\lambda_i q(i) + (\lambda_j - 1)q(j)) \\ &> a_i^T x^2 - b_i \\ &\geq a_i^T x^2 - b_i + (\lambda_j - 1)a_i^T(q(i) + q(j)) \\ &\geq a_i^T x^2 - b_i + a_i^T(\lambda_i q(i) + (\lambda_j - 1)q(j)) \\ &= a_i^T x - b_i. \end{aligned}$$

(ii) $L(x^1) = j$ and $\pi = (i, j)$. Then $\vec{L}(\sigma) = (j, 3, i)$ by Remark 3. Using (2), we have $x = x^2 + (\lambda_i - 1)q(i) + \lambda_j q(j)$ and

$$\begin{aligned} a_3^T x - b_3 &= a_3^T x^2 - b_3 + a_3^T((\lambda_i - 1)q(i) + \lambda_j q(j)) \\ &> a_i^T x^2 - b_i \\ &\geq a_i^T x^2 - b_i + \lambda_j a_i^T(q(i) + q(j)) \\ &\geq a_i^T x^2 - b_i + a_i^T((\lambda_i - 1)q(i) + \lambda_j q(j)) \\ &= a_i^T x - b_i. \end{aligned}$$

(iii) $L(x^3) = 3$ and $\pi = (\pi(1), \pi(2))$. Then

$$(a) \qquad \vec{L}(\sigma) = (i, j, 3) \quad \text{if} \quad \pi = (i, j)$$

and

$$(b) \qquad \vec{L}(\sigma) = (j, i, 3) \quad \text{if} \quad \pi = (j, i).$$

In both cases, we have

$$x = x^3 + (\lambda_i - 1)q(i) + (\lambda_j - 1)q(j)$$

and

$$\begin{aligned} a_3^T x - b_3 &= a_3^T x^3 - b_3 + a_3^T((\lambda_i - 1)q(i)(\lambda_j - 1)q(j)) \\ &> a_i^T x^3 - b_i \\ &\geq a_i^T x^3 - b_i + (\lambda_j - 1)a_i^T(q(i) \\ &\geq a_i^T x^3 - b_i + a_i^T((\lambda_i - 1)q(i) + ((\lambda_j - 1)q(j)) \\ &= a_i^T x - b_i \end{aligned}$$

It follows from (i), (ii) and (iii) that $L(x) \neq i$ for any $x \in Z^2 \cap \mathrm{Int}\, G_{i,j}$. Note

$$\mathrm{Int}\, G_i = (G_{i,3} \backslash G_3) \cup \mathrm{Int}\, G_{i,j}.$$

Therefore $L((x) \neq i$ for any $x \in Z^2 \cap \mathrm{Int}\, G_i$.

Finally we show the second part of the proposition. Assume $\vec{L}(\sigma) \neq (j, 3, i)$. We need to consider the case that

$$\lambda_i = \lambda_j > 0 \quad \text{and} \quad \lambda_3 = 0,$$

that is, $x \in Z^2 \cap G_{i,j} \cap G_{j,i}$. We discuss the following two cases.

(i) Let $\vec{L}(\sigma) = (i, 3, j)$, namely, $L(x^1) = i$ and $\pi = (j, i)$. Then

$$x = x^2 + \lambda\, q(i) + (\lambda - 1)q(j)$$

and

$$\begin{aligned} a_3^T x - b_3 &= a_3^T x^2 - b_3 + a_3^T(\lambda\, q(i) + (\lambda - 1)q(j)) \\ &\geq a_3^T x^2 - b_3 \\ &> a_i^T x^2 - b_i + (\lambda - 1)a_i^T(q(i) + q(j)) \\ &\geq a_i^T x^2 - b_i + a_i^T(\lambda\, q(i) + (\lambda - 1)q(j)) \\ &= a_i^T x - b_i. \end{aligned}$$

(ii) Let $\vec{L}(\sigma) = (\pi(i), \pi(j), 3)$, i. e., $L(x^3) = 3$ and $\pi = (\pi(i), \pi(j))$ by Lemma 2. Then $x = x^3 + (\lambda - 1)(q(i) + q(j)$ and

$$\begin{aligned} a_3^T x - b_3 &= a_3^T x^3 - b_3 + (\lambda - 1)a_3^T(\lambda\, q(i) + q(j)) \\ &\geq a_3^T x^3 - b_3 \\ &> a_i^T x^3 - b_i + (\lambda - 1)a_i^T(q(i) + q(j)) \\ &= a_i^T x - b_i. \end{aligned}$$

That is,

$$L(x) \neq i \quad \text{for each} \quad x \in Z^2 \cap G_{i,j} \cap G_{j,i}.$$

This proof is completed. ∎

Proposition 2 *Let $\sigma(x^1, \pi)$ be a c. l. simplex of type II in R^2. Then $L(x) \neq 3$ for any $x \in Z^2 \cap G_3$.*

Proof. we consider the following two cases.

Case 1. $\lambda_3 \geq \lambda_i > \lambda_j = 0$, i.e., Then

$$x = x^1 + \lambda_i q(i) + \lambda_3 q(3). \tag{3}$$

There are three subcases;

(i) $L(x^1) = i$ and $\pi = (i, j)$. Then $\vec{L}(\sigma) = (i, j, 3)$ by Remark 3. Since $x = x^2 + (\lambda_i - 1)q(i) + \lambda_3 q(3)$, we have

$$\begin{aligned} a_j^T x - b_j &= a_j^T x^2 - b_j + a_j^T((\lambda_i - 1)q(i) + \lambda_3 q(3)) \\ &\geq a_3^T x^2 - b_3 \\ &\geq a_3^T x^2 - b_3 + \lambda_3 a_3^T(q(i) + q(3)) \\ &\geq a_3^T x^2 - b_3 + a_3^T((\lambda_i - 1)q(i) + \lambda_3 q(3)) \\ &= a_3^T x - b_3. \end{aligned}$$

(ii) $L(x^1) = i$ and $\pi = (j, i)$. Then $\vec{L}(\sigma) = (i, 3, j)$ by Remark 3. Since $x = x^3 + \lambda_i q(i) + (\lambda_3 + 1)q(3)$, we have

$$\begin{aligned} a_j^T x - b_j &= a_j^T x^3 - b_j + a_j^T(\lambda_i q(i) + (\lambda_3 + 1)q(3)) \\ &\geq a_3^T x^3 - b_3 \\ &\geq a_3^T x^3 - b_3 + \lambda_i a_3^T(q(i) + q(3)) \\ &\geq a_3^T x^3 - b_3 + a_3^T(\lambda_i q(i) + (\lambda_3 + 1)q(3)) \\ &= a_3^T x - b_3. \end{aligned}$$

(iii) $L(x^1) = j$ and $\pi = (\pi(1), \pi(2))$. Then (3) holds and

$$\begin{aligned} a_j^T x - b_j &= a_j^T x^1 - b_j + a_j^T(\lambda_i q(i) + \lambda_3 q(3)) \\ &\geq a_3^T x^1 - b_3 \\ &\geq a_3^T x^1 - b_3 + \lambda_i a_3^T(q(i) + q(3)) \\ &\geq a_3^T x^1 - b_3 + a_3^T(\lambda_i q(i) + \lambda_3 q(3)) \\ &= a_3^T x - b_3. \end{aligned}$$

Then we obtain the following inequality by (i), (ii) and (iii).

$$a_j^T x - b_j \geq a_3^T x - b_3 \quad \text{for } x \in Z^2 \cap (G_{3,i} \backslash G_{3,j}).$$

Case 2. $\lambda_3 > \lambda_i = \lambda_j = 0$, i.e., $x \in Z^2 \cap G_{3,i} \cap G_{3,j}$. According to Lemma 1 and 2, we need to consider the case that $L(x^1) = i$ and $\pi = (\pi(1), \pi(2))$. Then $x = x^1 + \lambda_3 q(3)$ and

$$\begin{aligned} a_i^T x - b_i &= a_i^T x^1 - b_i + \lambda_3 a_i^T q(3) \\ &> a_i^T x^1 - b_i \\ &\geq a_3^T x^1 - b_3 + \lambda_3 a_3^T q(3) \\ &= a_3^T x - b_3. \end{aligned}$$

Then we obtain the following strict inequality

$$a_i^T x - b_i > a_3^T x - b_3.$$

Note

$$G_3 = (G_{3,i} \backslash G_{3,j}) \cup (G_{3,j} \backslash G_{3,i}) \cup (G_{3,i} \cap G_{3,j}).$$

Therefore $L(x) \neq 3$ for $x \in Z^2 \cap G_3$. This proof is completed. ∎

Finally, we give the following main theorems.

Theorem 2 *Let P be a simplex in the strong standard form in R^2. then there does not exist any c. l. simplex of type I, namely, P contains no integral points.*

Proof. Let $\sigma(x^1, \pi)$ be a c. l. simplex of type II in R^2, and $\pi = (\pi(1), \pi(2))$. For any $x^0 \in Z^2$, if $x = x^1$, then $L(x^0) = L(x^1) \neq 0$. Let $x^0 \in Z^2 \backslash \{x^1\}$, $\pi(3) = 3$ and write

$$x^1 = x^0 + \Sigma_{i \in I_3} \lambda_{\pi(i)} q(\pi(i)),$$

where $\lambda = (\lambda_{\pi(1)}, \lambda_{\pi(2)}, \lambda_3) = \lambda_{x^1} > 0$. Put

$$l = \min\{i \mid \lambda_{\pi(i)} \geq \lambda_{\pi(j)} \text{ for all } \forall j \in I_3\}.$$

Then $\lambda_{\pi(l)} \geq 1$ and there are $k, s \in I_3$ such that

$$L(x^k) = \pi(l) = s \in I_3.$$

For $x^k \in Z^2$, there exist $\lambda' = (\lambda'_{\pi(1)}, \lambda'_{\pi(2)}, \lambda'_3) = \lambda_{x^k} > 0$ such that

$$x^k = x^0 + \Sigma_{i \in I_3} \lambda'_{\pi(i)} q(\pi(i)).$$

Note that

$$\lambda'_s = \lambda'_{\pi(l)} = \max\{\lambda'_{\pi(i)} \mid i \in I_3\}.$$

(i) Let $s \neq 3$. Without loss of generality, we can assume $l = 1$, that is, $\pi(1) = s$. Then

$$\begin{aligned}
a_s{}^T x^0 - b_s &= a_s{}^T x^k - b_s - a_s^T \cdot \Sigma_{i \in I_3} \lambda'_{\pi(i)} q(\pi(i)) \\
&> \lambda'_s a_{ss} + \lambda'_{\pi(2)} a_{s\pi(2)} - \lambda'_3 \cdot \Sigma_{i \in I_2} a_{s\pi(i)} \\
&\geq \lambda'_s a_{ss} + \lambda'_s a_{s\pi(2)} - \lambda'_3 \cdot \Sigma_{i \in I_2} a_{s\pi(i)} \\
&\geq (\lambda'_s - \lambda'_3) \Sigma_{i \in I_2} a_{s\pi(i)} \\
&\geq 0
\end{aligned}$$

(ii) Let $s = 3$. Then

$$\begin{aligned}
a_3^T x^0 - b_3 &= a_3^T x^k - b_3 - a_3^T \cdot \Sigma_{i \in I_3} \lambda'_{\pi(i)} q(\pi(i)) \\
&> \Sigma_{i \in I_2} \lambda'_{\pi(i)} a_{3\pi(i)} - \lambda'_3 \cdot \Sigma_{i \in I_2} a_{3\pi(i)} \\
&\geq (\lambda'_3 - \lambda'_3) \Sigma_{i \in I_2} a_{3\pi(i)} \\
&= 0.
\end{aligned}$$

This implies that $L(x^0) \neq 0$ and conclude $x^0 \notin P \cap Z^2$ from (i) and (ii). Since $x^0 \in Z^2$ is arbitrary, we have $P \cap Z^2 = \phi$. Therefore there does not exist any c. l. simplex of type I. ∎

By the above theorem and Theorem 1, we immediately derive the following corollary.

Corollary 1 *Let P be a simplex in the strong standard form in R^2. If P contains an integral point, then there is no c. l. simplex of type II, that is, the coexistence of a c. l. simplex of type I and II is impossible. Moreover, P contains an integral point if and only if there are no c. l. simplices of type II.*

Now we show that there exists a unique completely labeled simplex of type II if P contains no integral point.

Theorem 3 *Let P be a simplex in the strong standard form in R^2. If P contains no integral point, then there exists a unique c. l. simplex of type II.*

Proof. Since $P \cap Z^2 = \phi$, there exist only c. l. simplices of type II by Corollary 1. Now we show the uniqueness as follows.

On the contrary, assume that there exist two c. l. simplices of type II, say,

$$\sigma = \sigma(x^1, \pi) \text{ and } \sigma' = \sigma(y^1, \rho).$$

If $\sigma \cap \sigma' = \phi$, it is obvious that there exists a unique index $i \in I_3$ such that

$$\sigma' \subset \operatorname{Int} G_i(x^1) \text{ or } \sigma \subset \operatorname{Int} G_i(y^1).$$

However it is impossible by Propositions 1 and 2.

Let $\sigma \cap \sigma' \neq \phi$. Since $L(x^1) \neq 3$ and $L(y^1) \neq 3$, we have

$$(4) \quad \sigma' \not\subset G_3(x^1) \text{ and } \sigma \not\subset G_3(y^1)$$

and

$$(5) \quad \sigma' \not\subset \operatorname{Int} G_i(x^1) \text{ and } \sigma \not\subset \operatorname{Int} G_i(y^1) \text{ for } i \in I_2.$$

Then we only need to consider the case that there is at least one $k \in I_3$ such that

$$y^k \in \partial(\operatorname{cl} G_i(x^1)) \quad \text{for some} \quad i \in I_2.$$

We discuss the following two cases.

Case 1. Let $y^1 = x^1 + q(i) + q(3)$ with $i \in I_2$. There are two subcases.

(i) Let $\rho = (i, j)$, then $\pi = (i, j)$ because $\sigma \cap \sigma' \neq \phi$.

Since x^1, $x^3 \in$ Int $G_j(y^1)$, $L(x^1) \neq j$ and $L(x^3) \neq j$ by Proposition 1. But $\rho = (i, j)$ implies $L(y^3) \neq j$ by Remark 3. Then $L(x^2) = L(y^3) \neq j$. It is a contradiction.

(ii) Let $\rho = (j, i)$. Then $\pi = (i, j)$ because $\sigma \not\subset \text{Int}\, G_j(y^1)$.

Since $L(y^2) = L(x^1) \neq 3$, then $\vec{L}(\sigma') \neq (i, 3, j)$ by the (b) of Remark 3. Therefore $L(x^k) \neq j$ for each $k \in I_3$ by Proposition 1. It is a contradiction.

Case 2. We only need to consider the following two subcases by (4) and (5).

(i) Let $y^1 = x^1$ and $\rho = (\rho(1), \rho(2))$. Then $\pi = (\rho(2), \rho(1))$.

Since $L(y^1) = L(x^1)$ and $\rho \neq \pi$, then $L(y^3) \neq L(x^3)$ by Remark 3. This contradicts the fact that $y^3 = x^3$.

(ii) Let $y^2 = x^1 + q(1) + q(2)$ and $\rho = (\rho(1), \rho(2))$. Then $L(y^1) = \rho(1)$, because $L(y^1) \neq 3$ and $y^1 \notin \text{Int}\, G_{\rho(2)}(x^1)$. Using (a) of Remark 3, we have

$$L(x^3) = L(y^2) = \rho(2).$$

This implies $\pi = (\rho(2), \rho(1))$, whence $x^2 = y^1$. Using (a) of Remark 3 for σ', we have

$$L(x^2) = L(y^1) = \rho(1).$$

Hence $L(x^1) = 3$. This contradicts Lemma 1 and the proof is completed. ∎

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