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A Generalization of Subjective Expected Utility without Transitivity and Additivity

by

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Abstract

Recent developments in decision making under uncertainty focus on either of two relaxations of the subjective expected utility (SEU) models, i.e., nonadditivity of subjective state probabilities and nontransitivity of preferences. This paper examines an axiomatic characterization of a generalization of SEU models which does not presume both additivity and transitivity.

1 Introduction

Since the 1980's, there have been developed a number of alternative theories which axiomatically generalize subjective expected utility (SEU) models. Two line of research may be observed in those generalizations. The first is concerned with relaxation of the independence axiom (or the sure-thing principle) which leads to nonadditive state probabilities, occasionally called capacities, or probabilistically sophisticated preferences. Schmeidler (1984, 1989) was the first to recognize that Choquet (1953–54) integration provides an appropriate numerical representation in the SEU framework, referred to here as Choquet expected utility (CEU) model, to deal with nonadditivity of state probabilities. His discovery initiated several axiomatic refinements and generalizations of CEU model, including Chew and Wakker (1995), Gilboa (1987), Nakamura (1990, 1992, 1997a), Sarin and Wakker (1992), Wakker (1989a, b, 1991) and others.

Probabilistically sophisticated preferences were first axiomatized in the Savage's (1954) SEU framework by Machina and Schmeilder (1992) to demonstrate that those preferences could induce additivity of state probabilities and have an order-preserving numerical representation, which is monotonic with respect to stochastic

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dominance. Their axiomatization motivated research interest in subsequent axiomatic considerations of probabilistic sophistication in the SEU framework, including Grant (1995), Machina and Schmeidler (1995), Nakamura (1995b, 1997b), Sarin and Wakker (1993), and others.

The second line of research is concerned with relaxation of transitivity of preferences. Nontransitive preferences are sometimes observed in decision making under uncertainty (e.g., see Fishburn (1988, 1991)). They may also be caused by the psychological notion of regret (e.g., see regret theories independently developed by Bell (1982) and Loomes and Sugden (1982)). Fishburn (1984) axiomatically investigated preferences under uncertainty without assuming transitivity in the Anscombe and Aumann's (1963) SEU framework, and derived a skew-symmetric additive (SSA) representation of preferences, which is additively separable across the states (i.e., rataining the sure-thing principle), but nonseparable within each state (i.e., comparisons between outcomes may not be separable in the outcomes, thus allow for nontransitive preferences). The SSA repersentation is mathematically tantamount to the regret model.

Axiomatic refinements of SSA model in various set-ups have been examined by Fishburn (1988, 1989, 1990), Fishburn and LaValle (1987), Nakmaura (1995c), and Sugden (1993). Although SSA model induces additivity of state probabilities, it is shown that probabilistic sophistication reduces SSA model to SEU model (see Fishburn (1988) and Machina and Schmeidler (1992)). Because of that, Nakamura (1997b) recently developed a numerical representation for probabilistically sophisticated preferences without transitivity.

The aim of the present paper is to discuss a generalization of SEU model which does not presume both additivity and transitivity, and identify sufficient axioms for that generalization when the state space is finite. Our representational form requires that nontransitive preferences stem from people's nontransitive belief judgments about the degree to which the true state is contained in an event, i.e, a subset of the state space. For example, suppose that a decision maker feels that an event A is more probable than an event B, and that the event B is more probable than an event C. Then we do not require that the decision maker *should* judge that the event A is more probable than the event C. This nontransitivity of belief judgments causes nontransitivity of preferences.

Such nontransitive belief judgments were first investigated by Fishburn (1983a) in a comparative probability framework, and Fishburn (1983b) in a decision-theoretic framework. In both frameworks, he obtained necessary and sufficient axioms for the existence of a nonseparable numerical representation $\phi(A, B)$ of the degree of beliefs, whose positive value may be interpreted as the degree to which the event A is more probable than the event B. Notable features of that representation is that ϕ is skew-symmetric and conditionally additive, so that ϕ-representation reduces to additive probability representation when ϕ is separable, i.e., $\phi(A, B) = \pi(A) - \pi(B)$ for some set function π. It follows from conditional additivity and normalization that π must be a probability measure. We shall consider the more general representation in which ϕ is conditionally monotonic in place of conditionally additive. When $\phi(A, B) =$

$\pi(A) - \pi(B)$ for some set function π, it follows from conditionl monotonicity that π must be a capacity, so that our representation reduces to CEU model.

The paper is organized as follows. Section 2 introduces a model that does not presume both additivity and tansitivity. Then Section 3 presents sufficient axioms for our model when the state space is finite. Section 4 proves the existence of a utility function u and a skew-symmetric monotone bivariate function ϕ. Section 5 completes the proof of the representation theorem. Finally, Section 6 concludes the paper.

2 A Nontransitive CEU Model

Let S be the set of states. By Γ_S, we denote an algebra of subsets of S. Let $\emptyset$ be the empty set. Elements of Γ_S are called events. By A^c, we denote the complement of an event A. When an event consists of a single state s, we write the event as s instead of $\{s\}$. Acts are functions which map S into the set X of consequences. Let $f(S)$ denote the range of f, i.e., $\{f(s) \in X : s \in S\}$. A constant act is an act $f \in \mathcal{F}$ such that $f(S) = \{x\}$ for some $x \in X$. Every $x \in X$ will be identified with a constant act. Let $\mathcal{F}$ denote the set of all acts, so $X \subseteq \mathcal{F}$.

Let $\succeq$ be the binary preference relation on $\mathcal{F}$ with $\sim$ and $\succ$ defined in the usual way: for $f, g \in \mathcal{F}$, $f \sim g$ if $f \succeq g$ and $g \succeq f$; $f \succ g$ if $f \succeq g$ and not$(g \succeq f)$. When we write $x \succeq y$ for $x, y \in X$, it means that $f \succeq g$ when $f(S) = \{x\}$ and $g(S) = \{y\}$.

A real valued function π on Γ_S is said to be a *capacity* when $\pi(\emptyset) = 0$, $\pi(S) = 1$, and π is monotone, i.e., for all $A, B \in \Gamma_S$, $\pi(A) \geq \pi(B)$ if $A \supseteq B$. A real valued function u on X is said to be a utility function when for all $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$. Given a utility function u and an act $f \in \mathcal{F}$, let

$$S_f^u(\tau) = \{s \in S : u(f(s)) \geq \tau\}.$$

Then the Choquet expected utility of an act $f \in \mathcal{F}$ with respect to a capacity π on Γ_S is defined as follows.

$$E_C(f, u, \pi) = \int_0^{+\infty} \pi(S_f^u(\tau))d\tau - \int_{-\infty}^0 (1 - \pi(S_f^u(\tau)))d\tau.$$

This means that expected utilities are calculated with respect to decumulative distribution functions $\pi(S_f^u(\tau))$ induced by π through f.

A CEU model yields that there exist a utility function u on X and a capacity π on Γ_S such that for all $f, g \in \mathcal{F}$,

$$f \succeq g \Longleftrightarrow E_C(f, u, \pi) \geq E_C(g, u, \pi),$$

where u is unique up to a positive linear transformation and π is unique. Schmeidler (1984, 1989) first developed an axiomatic characterization of this model by using an auxiliary devise, i.e., lottery acts introduced by Anscombe and Aumann (1963). Gilboa's (1987) axiomatization applied Savage's act-oriented framework. His representation requires that X includes at least three consequences that are mutually

indifferent, S is continuously divisible, and a utility function is bounded. Nakamura (1997a) relaxes both continuous divisibility of the state space and the boundedness of a utility function. Wakker (1989a) developed topological approaches when S is finite. On the other hand, Nakamura (1990, 1992) and Wakker (1991) developed algebraic approaches in that case.

A notable feature of CEU model is nonadditivity of π. A qualitative probability relation $\succ^*$ on Γ_S, read as *is more probable than*, is usually defined through a definite preference for one *binary act* over another as follows. For events $A, B \in \Gamma_S$ and consequences $x, y \in X$ with $x \succ y$, if

$$f(s) = \begin{cases} x & \text{for } s \in A \\ y & \text{for } s \in A^c, \end{cases} \qquad g(s) = \begin{cases} x & \text{for } s \in B \\ y & \text{for } s \in B^c, \end{cases}$$

then $A \succ^* B$ if and only if $f \succ g$. Then CEU model leads to the following qualitative probability representation. For all $A, B \in \Gamma_S$,

$$A \succ^* B \Longleftrightarrow \pi(A) > \pi(B).$$

While $A \succ^* B$ and $A \cap C = B \cap C = \emptyset$, nonadditivity of π allows for $B \cup C \succ^* A \cup C$ which is inconsistent with SEU model.

To introduce our nontransitive version of CEU model, let ϕ be a real valued bivariate function on $\Gamma_S \times \Gamma_S$ such that $\phi(S, \emptyset) > 0$ and for all $A, B \in \Gamma_S$,

$$\begin{aligned} \text{skew-symmetry:} \quad & \phi(A, B) = -\phi(B, A), \\ \text{monotonicity:} \quad & A \supseteq B \Longrightarrow \phi(A, B) \geq 0. \end{aligned}$$

Then a nontransitive CEU model yields that there exist a utility function u on X and a skew-symmetric monotone bivariate function ϕ on $\Gamma_S \times \Gamma_S$ such that for all $f, g \in \mathcal{F}$,

$$f \succ g \Longleftrightarrow \int_{-\infty}^{+\infty} \phi(S_f^u(\tau), S_g^u(\tau)) d\tau > 0.$$

When ϕ is additively decomposed, i.e., $\phi(A, B) = \pi(A) - \pi(B)$ for all $A, B \in \Gamma_S$ and a capacity π on Γ_S, it is easy to see that this model reduces to a CEU model. As in CEU model, u is unique up to a positive linear transformation. However, ϕ on $\Gamma_S \times \Gamma_S$ cannot be uniquely determined, since we can assign any positive number to $\phi(S, \emptyset)$ regardless of the other $\phi(A, B)$ values. We shall discuss more about the uniqueness of ϕ later in the section.

A notable feature of nontransitive CEU model is nontransitivity of the qualitative probability relation in addition to including CEU model as a special case. That model leads to the following nonseparable qualitative probability representation. For all $A, B \in \Gamma_S$,

$$A \succ^* B \Longleftrightarrow \phi(A, B) > 0.$$

Although preferences over the consequence space X are transitive, i.e., for all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$, we may have $f \succeq g$, $g \succeq h$, and $h \succ f$ for some acts $f, g, h \in \mathcal{F}$, because of nonseparability of ϕ.

Different approaches have been developed by Fishburn (1983a, b) to derive axiomatically the nonseparable qualitative probability representation. When S is finite, Fishburn (1983a) provides necessary and sufficient axioms for the existence of ϕ under the comparative probability framework with the following additional properties: $\phi(S,\emptyset)=1$, and for all $A,B,C\in\Gamma_S$,

$$\text{conditional additivity:}\quad \phi(A\cup B,C)+\phi(\emptyset,C)=\phi(A,C)+\phi(B,C)\quad \text{whenever } A\cap B=\emptyset.$$

To obtain a uniqueness result for conditionally additive ϕ, Fishburn (1983b) devised a randomization mechanism over Γ_S, so that normalization leads to a unique ϕ.

This paper establishes an axiomatic characterization of nontransitive CEU model when S is finite. For $f\in\mathcal{F}$ and $x\in X$, let $f^{-1}(x)=\{s\in S: f(s)\succeq x\}$. Given $f,g\in\mathcal{F}$, suppose that $f(S)\cup g(S)=\{x_1,\ldots,x_m\}$ and $x_1\succeq\ldots\succeq x_m$. Then the finite-state version of nontransitive CEU model is stated as follows.

$$f\succ g\Longleftrightarrow\sum_{i=1}^{m-1}\phi(f^{-1}(x_i),g^{-1}(x_i))(u(x_i)-u(x_{i+1}))>0.$$

Since the range of ϕ is a finite set, we need to assume the richness of X to obtain a uniqueness property of a utility function u. Thus our axiom system developed in the next section requires that $\succeq$ on X be dense, i.e., $x\succ a\succ y$ for some $a\in X$ whenever $x\succ y$, so that we can construct u that is unique up to a positive linear transformation.

In general, ϕ on $\Gamma_S\times\Gamma_S$ cannot be unique as mentioned previously. However, we may obtain a uniqueness property (i.e., uniqueness up to a positive multiplicative transformation) of ϕ on a subset of $\Gamma_S\times\Gamma_S$. We say that ϕ on a subset of $\Gamma_S\times\Gamma_S$ is unique up to a positive multiplicative transformation if $\phi=\alpha\phi'$ on that subset for some positive number α whenever ϕ' is another nonseparable qualitative probability representation of $\succ^*$.

Consider the following simple example. Suppose that u is unique up to a positive linear transformation, and that $S=\{s_1,s_2\}$ and $S\succ^*\{s_i\}\succ^*\emptyset$ for $i=1,2$. For $i=1,2$, we define subsets of $\Gamma_S\times\Gamma_S$ by

$$\begin{aligned}\Gamma_i^2 &= \{(\emptyset,\emptyset),(\emptyset,\{s_i\}),(\{s_i\},\emptyset),(S,\{s_i\}),(\{s_i\},S),(S,S)\},\\ \Gamma_*^2 &= \Gamma_1^2\cup\Gamma_2^2,\end{aligned}$$

so that $\Gamma_*^2=\Gamma_S\times\Gamma_S\setminus\{(\emptyset,S),(S,\emptyset),(\{s_1\},\{s_2\}),(\{s_2\},\{s_1\})\}$.

Let $f_1(s_1)=f_2(s_2)=x$ and $f_1(s_2)=f_2(s_1)=y$. Then nontransitive CEU model requires that if $x\succ z_i\succ y$ and $f_i\sim z_i$, then

$$\frac{\phi(S,\{s_i\})}{\phi(\{s_i\},\emptyset)}=\frac{u(x)-u(z_i)}{u(z_i)-u(y)},$$

i.e., the ratio of ϕ is *uniquely* determined as a ratio of utility difference because of the uniqueness of u.

Let ϕ' be another nonseparable qualitative probability representation of $\succ^*$. Since skew-symmetry of ϕ implies that $\phi(A,A)=0$ for all $A \in \Gamma_S$, it follows that ϕ on each of Γ_i^2 is unique up to a positive multiplicative transformation, i.e., $\phi' = \alpha_i \phi$ on Γ_i^2 for some positive numbers α_1 and α_2. Let $\sim^*$ be the symmetric complement of $\succ^*$, i.e., $A \sim^* B \iff \text{not}(A \succ^* B)$ and $\text{not}(B \succ^* A)$. If $\{s_1\} \sim^* \{s_2\}$, then we may have $\alpha_1 \neq \alpha_2$, so that ϕ is not unique up to a positive multiplicative transformation on the entire domain $\Gamma_S \times \Gamma_S \setminus \{(S,\emptyset),(\emptyset,S)\}$. However, if $\{s_1\} \succ^* \{s_2\}$ or $\{s_2\} \succ^* \{s_1\}$, then we must have $\alpha_1 = \alpha_2$, so that ϕ on Γ_*^2 is unique up to a positive multiplicative transformation on that domain. To avoid the case that $\{s_1\} \sim^* \{s_2\}$, we shall assume that S includes at least three states.

Let Γ_i^2 for $i=1,2$ be two subsets of $\Gamma_S \times \Gamma_S$. Let ϕ and ϕ' be two nonseparable qualitative probability representation of $\succ^*$. Suppose that for $i=1,2$, $\phi' = \alpha_i \phi$ on Γ_i^2 for some positive number α_i. Then we have $\alpha_1 = \alpha_2$ if there are pairs of events, $(A_i, B_i) \in \Gamma_i^2$ for $i=1,2$ such that $A_1 \succ^* B_1$, $B_2 \succ^* A_2$, $B_2 \supseteq B_1$, and $A_2 \supseteq A_1$ or $A_2 \succ^* A_1$. To see this, assume $A_2 \supseteq A_1$. When $A_2 \succ^* A_1$, the desired conclusion similarly follows. Let

$$f(s) = \begin{cases} x & \text{for } s \in A_1 \\ y & \text{for } s \in A_2 \setminus A_1 \\ z & \text{for } s \in S \setminus A_2, \end{cases} \qquad g(s) = \begin{cases} x & \text{for } s \in B_1 \\ y & \text{for } s \in B_2 \setminus B_1 \\ z & \text{for } s \in S \setminus B_2. \end{cases}$$

Then nontransitive CEU model requires that if $x \succ y \succ z$ and $f \sim g$, then

$$\frac{\phi(A_1,B_1)}{\phi(A_2,B_2)} = \frac{u(y)-u(z)}{u(x)-u(y)}.$$

Thus we must have $\alpha_1 = \alpha_2$, i.e., ϕ is unique up to a positive multiplicative transformation on $\Gamma_1^2 \cup \Gamma_2^2$.

We say that an event A is *proper* when $A \neq S$ and $A \neq \emptyset$. The observation in the preceding paragraph suggests a condition that $S \succ^* A$ and $A \succ^* \emptyset$ for all proper events $A \in \Gamma_S$ to achieve the uniqueness of ϕ on $\Gamma_S \times \Gamma_S \setminus \{(S,\emptyset),(\emptyset,S)\}$, since we can always take $B_1 = \emptyset$ and $B_2 = S$. When S is finite, it seems plausible that we may exclude all events A which satisfy $\text{not}(A \succ^* \emptyset)$ or $\text{not}(S \succ^* A^c)$, because they can be regarded irrelevant in decision making.

3 Axioms for a Nontransitive CEU model

This section presents sufficient axioms for a nontransitive CEU model when S is finite. For $f,g \in \mathcal{F}$ and an event $A \in \Gamma_S$, let $f \bigcirc_A g$ denote the act h such that $h(s) = f(s)$ for all $s \in A$, and $h(s) = g(s)$ for all $s \in A^c$. Then any act f can be represented as follows: for some $A_1, \ldots, A_n \in \Gamma_S$ and some $x_1, \ldots, x_{n+1} \in X$,

$$(\cdots((x_1 \bigcirc_{A_1} x_2) \bigcirc_{A_2} x_3) \cdots \bigcirc_{A_{n-1}} x_n) \bigcirc_{A_n} x_{n+1}.$$

This means that for $i=1,\ldots,n+1$, $f(s) = x_i$ if $s \in A_i \setminus A_{i-1}$, where $A_0 = \emptyset$ and $A_{n+1} = S$. When $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$, we shall sometimes write

$$x_1 \bigcirc_{A_1} x_2 \cdots x_n \bigcirc_{A_n} x_{n+1}$$

without parentheses.

We say that an event A is *univeral* if $x \bigcirc_A y \sim x$ for all $x, y \in X$ with $x \succ y$, and *null* if $x \bigcirc_A y \sim y$ for all $x, y \in X$ with $x \succ y$. Then nontransitive CEU model requires that $\phi(A, S) = 0$ if A is universal, and $\phi(A, \emptyset) = 0$ if A is null. When either $x \succeq y \succeq z$ or $z \succeq y \succeq x$ for $x, y, z \in X$, we say that y is *between* x and z. For two subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$, when we write $\mathcal{F}_1 \succeq \mathcal{F}_2$, it means that $f_1 \succeq f_2$ for all $f_1 \in \mathcal{F}_1$ and all $f_2 \in \mathcal{F}_2$. We shall sometimes write $f \succeq g \succeq h$ when $f \succeq g$ and $g \succeq h$, $f \succ g \succ h$ when $f \succ g$ and $g \succ h$, and so forth. Notice that we may have $h \succ f$ (respectively, $h \succeq f$) when $f \succeq g \succeq h$ (respectively, $f \succ g \succ h$).

Structural assumptions for S, Γ_S, and X are stated as follows.

Assumption 1 *S is finite and includes at least three states. Γ_S is the set of all subsets of S. Every proper event is neither null nor universal. For some $a, b \in X$, $a \succ b$.*

We shall write 2^S in place of Γ_S. An implication of this assumption about the uniqueness of ϕ on $2^S \times 2^S \setminus \{(S, \emptyset), (\emptyset, S)\}$ was discussed in the preceding section.

We use eight axioms, five of which are stated as follows: for all $x, y, x_i, y_i, z_i \in X$ with $i = 1, 2$, all $A, B, C, D \in 2^S$, and all $f, g \in \mathcal{F}$,

Axiom 1 *$\succeq$ on X is a weak order.*

Axiom 2 *If $x \bigcirc_A f \succeq x \bigcirc_B g$ and $y \bigcirc_B g \succeq y \bigcirc_A f$, then there is an $a \in X$ such that a is between x and y, and $a \bigcirc_A f \sim a \bigcirc_B g$.*

Axiom 3 *If $x_i \succeq y_i \succeq z_i$ and $(x_i \bigcirc_A y_i) \bigcirc_B z_i \sim (x_i \bigcirc_C y_i) \bigcirc_D z_i$ for $i = 1, 2$, then*

$$x_1 \succeq x_2 \Longleftrightarrow y_1 \succeq y_2 \qquad \text{when } z_1 = z_2;$$
$$y_1 \succeq y_2 \Longleftrightarrow z_1 \succeq z_2 \qquad \text{when } x_1 = x_2.$$

Axiom 4 *If $A \supseteq B$ and $x \succ y$, then $x \bigcirc_A y \succeq x \bigcirc_B y$.*

Axiom 5 *If $a \bigcirc_{f^{-1}(z)} b \succeq a \bigcirc_{g^{-1}(z)} b$ for all $z \in X$ and some $a, b \in X$ with $a \succ b$, then $f \succeq g$. If, in addition, $a \bigcirc_{f^{-1}(w)} b \succ a \bigcirc_{g^{-1}(w)} b$ for some $w \in X$, then $f \succ g$.*

Axiom 1 says that the preference order $\succeq$ on the set of constant acts is a weak order, i.e., transitive (for $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$) and complete (for $x, y \in X$, $x \succeq y$ or $y \succeq x$), since nontransitive CEU model requires the existence of a utility function on X. Axiom 2 is a restricted solvability axiom that is slightly stronger than the usual one, and is not necessary for nontransitive CEU model. This axiom requires that $\succeq$ on X be dense. A monotonicity condition with respect to consequences is stated in Axiom 3. Axiom 4 is a monotonicity axiom with respect to events. Axiom 5 is a dominance axiom which is similar to the first stochastic dominance in decisions under risk with transitive preferences. It is easy to see that nontransitive CEU model satisfies Axioms 1, 3, 4, and 5.

Let K be any set of consecutive integers. Given $A, B, C, D \in 2^S$, we define a *standard sequence* as a set $\{x_k : x_k \in X \text{ and } k \in K\}$ for which there exist $a, b, a_k \in X$ for $k \in K$ such that $\text{not}(a \sim b)$, either, for all $k, k+1 \in K$,

$$\begin{aligned}(a \bigcirc_A a_k) \bigcirc_B x_k &\sim (a \bigcirc_C a_k) \bigcirc_D x_k, \\ (b \bigcirc_A a_k) \bigcirc_B x_{k+1} &\sim (b \bigcirc_C a_k) \bigcirc_D x_{k+1},\end{aligned}$$

and $\{a, b\} \succeq \{x_k\}$, or for all $k, k+1 \in K$,

$$\begin{aligned}(x_k \bigcirc_A a_k) \bigcirc_B a &\sim (x_k \bigcirc_C a_k) \bigcirc_D a, \\ (x_{k+1} \bigcirc_A a_k) \bigcirc_B b &\sim (x_{k+1} \bigcirc_C a_k) \bigcirc_D b,\end{aligned}$$

and $\{x_k\} \succeq \{a, b\}$. A standard sequence is strictly bounded if $a \succ \{x_k\} \succ b$ for some $a, b \in X$. Then our Archimedean axiom is stated as follows.

Axiom 6 *Every strictly bounded standard sequence is finite.*

Let $\succ^*$ be a qualitative probability relation on 2^S defined as follows: for all $A, B \in 2^S$,

$$A \succ^* B \text{ if and only if } x \bigcirc_A y \succ x \bigcirc_B y \text{ for } x, y \in X \text{ whenever } x \succ y.$$

Suppose that $x \succ y$, $z \succ w$, and $x \bigcirc_A y \succ x \bigcirc_B y$. Then by Axiom 5, $z \bigcirc_A w \succ z \bigcirc_B w$, so that $\succ^*$ is well defined. Let $\succeq^*$ and $\sim^*$ be defined in the usual way: $A \succeq^* B$ if $\text{not}(B \succ^* A)$; $A \sim^* B$ if $\text{not}(A \succ^* B)$ and $\text{not}(B \succ^* A)$. If $x \succ y$, then $x = x \bigcirc_S y \succ y = x \bigcirc_\emptyset y$. Thus by Assumption 1, we always have $S \succ^* \emptyset$. We shall sometimes write $A \succ^* B \succ^* C$ when $A \succ^* B$ and $B \succ^* C$, $A \succeq^* B \succeq^* C$ when $A \succeq^* B$ and $B \succeq^* C$, and so forth. Notice that we may have $C \succeq^* A$ (respectively, $C \succ^* A$) when $A \succ^* B \succ^* C$ (respectively, $A \succeq^* B \succeq^* C$).

In the rest of the paper, let Γ_S^* be the set of all proper events in Γ_S. Since $\Gamma_S = 2^S$, Γ_S^* is all proper subsets of S. By Assumption 1, $S \succ^* A$ and $A \succ^* \emptyset$ for all $A \in \Gamma_S^*$.

For $i = 1, 2$, let

$$\begin{aligned}f_i &= x_1 \bigcirc_{A_{i1}} x_2 \cdots x_{n-1} \bigcirc_{A_{i,n-1}} x_n, \\ g_i &= y_1 \bigcirc_{A_{i1}} y_2 \cdots x_{n-1} \bigcirc_{A_{i,n-1}} x_n, \\ h_i &= z_1 \bigcirc_{A_{i1}} z_2 \cdots x_{n-1} \bigcirc_{A_{i,n-1}} x_n,\end{aligned}$$

where $x_1 \succeq \cdots \succeq x_n$, $y_1 \succeq \cdots \succeq y_n$, and $z_1 \succeq \cdots \succeq z_n$. An ordered pair of acts (g_1, g_2) is said to be *proportionally shifted* from ordered pairs of acts (f_1, f_2) and (h_1, h_2) if $x_i \succeq y_i \succeq z_i$ for $i = 1, \ldots, n$, and there are $A, B, C, D \in 2^S$ such that $A \succ^* C$, $D \succ^* B$, and for $i = 1, \ldots, n$,

$$(x_i \bigcirc_A y_i) \bigcirc_B z_i \sim (x_i \bigcirc_C y_i) \bigcirc_D z_i.$$

The ordered pair of acts, (f_1, f_2), (g_1, g_2), and (h_1, h_2), are similar in the sense that the ranges of those pair of acts, i.e., $f_1(S) \cup f_2(S)$, $g_1(S) \cup g_2(S)$, and $h_1(S) \cup$

$h_2(S)$, are proportionally related on the utility space $u(X) = \{u(x) : x \in X\}$ when nontransitive CEU model applies. That is, since $f_1(S) \cup f_2(S) = \{x_1, \ldots, x_n\}$, $g_1(S) \cup g_2(S) = \{y_1, \ldots, y_n\}$, and $h_1(S) \cup h_2(S) = \{z_1, \ldots, z_n\}$, where some of the outcomes are allowed to be identical, nontransitive CEU model implies that for $i = 1, \ldots, n-1$,

$$u(y_i) - u(y_{i+1}) = \alpha(u(x_i) - u(x_{i+1})) + (1-\alpha)(u(z_i)) - u(z_{i+1})).$$

Thus preferences between acts in ordered pairs of acts, one of which is proportionally shifted from the others, are required to be related each other as shown in the following axiom, applying to all $f_i, g_i, h_i \in \mathcal{F}$ for $i = 1, 2$.

Axiom 7 *If (g_1, g_2) is proportionally shifted from (f_1, f_2) and (h_1, h_2), and if $f_1 \succeq f_2$ and $h_1 \succeq h_2$, then $g_1 \succeq g_2$. If, in addition, $f_1 \succ f_2$ or $h_1 \succ h_2$, then $g_1 \succ g_2$.*

To show the necessity of the axiom, suppose that nontransitive CEU model holds and that the hypotheses of the axioms hold. For $i = 1, 2$, let

$$\begin{aligned} f_i &= x_1 \bigcirc_{A_{i1}} x_2 \cdots x_{n-1} \bigcirc_{A_{i,n-1}} x_n, \\ g_i &= y_1 \bigcirc_{A_{i1}} y_2 \cdots x_{n-1} \bigcirc_{A_{i,n-1}} x_n, \\ h_i &= z_1 \bigcirc_{A_{i1}} z_2 \cdots x_{n-1} \bigcirc_{A_{i,n-1}} x_n. \end{aligned}$$

We assume that $x_1 \succeq \cdots \succeq x_n$, $y_1 \succeq \cdots \succeq y_n$, and $z_1 \succeq \cdots \succeq z_n$.

Let $\sigma_i = \{A_{i1}, \ldots, A_{i,n-1}\}$ and $A_{i0} = \emptyset$ for $i = 1, 2$. When $a_1 \succeq \cdots \succeq a_n$, let

$$\begin{aligned} &U_{\sigma_1\sigma_2}(a_1, \ldots, a_n) \\ &\quad = \sum_{i=1}^{n-1} (\phi(A_{1i}, A_{2i}) - \phi(A_{1,i-1}, A_{2,i-1}))u(a_i) - \phi(A_{1,n-1}, A_{2,n-1})u(a_n). \end{aligned}$$

Then we have

$$\begin{aligned} f_1 \succeq f_2 &\iff U_{\sigma_1\sigma_2}(x_1, \ldots, x_n) \geq 0, \\ g_1 \succeq g_2 &\iff U_{\sigma_1\sigma_2}(y_1, \ldots, y_n) \geq 0, \\ h_1 \succeq h_2 &\iff U_{\sigma_1\sigma_2}(z_1, \ldots, z_n) \geq 0. \end{aligned}$$

Since $(x_i \bigcirc_A y_i) \bigcirc_B z_i \sim (x_i \bigcirc_C y_i) \bigcirc_D z_i$ for $i = 1, \ldots, n$, we have

$$\phi(A, C)u(x_i) + (\phi(B, D) - \phi(A, C))u(y_i) + \phi(D, B)u(z_i) = 0$$

for $i = 1, \ldots, n$. Hence

$$\begin{aligned} &\phi(A, C)U_{\sigma_1\sigma_2}(x_1, \ldots, x_n) \\ &\quad + (\phi(B, D) - \phi(A, C))U_{\sigma_1\sigma_2}(y_1, \ldots, y_n) + \phi(D, B)U_{\sigma_1\sigma_2}(z_1, \ldots, z_n) = 0. \end{aligned}$$

Since $\phi(A, C) > 0$, $\phi(B, D) - \phi(A, C) < 0$, and $\phi(D, B) > 0$, the requirement of the axiom easily obtains.

To state the last axiom, we introduce two quadruple relations $E_{\succ^*}(A,B;C,D)$ and $E_{\supseteq}(A,B;C,D)$ on X as follows. Given $A,B,C,D \in 2^S$ with $A \succ^* B$, $C \succ^* D$, and $B \supseteq C$ or $B \succ^* C$, for all $x,y,z,w \in X$, define

$$
\begin{aligned}
&(x,y,z,w) \in E_{\succ^*}(A,B;C,D) \\
&\quad \Longleftrightarrow \quad B \succ^* C, A \supset C, B \supset D, x \succ y \succ z \succ w, \\
&\qquad (x \bigcirc_D y) \bigcirc_B z \sim x \bigcirc_C z, \text{ and } (y \bigcirc_C z) \bigcirc_A w \sim y \bigcirc_B w; \\
&(x,y,z,w) \in E_{\supseteq}(A,B;C,D) \\
&\quad \Longleftrightarrow \quad B \supseteq C, A \supset D, x \succ y \succeq z \succ w, (x \bigcirc_C a) \bigcirc_B w \sim (x \bigcirc_D a) \bigcirc_A w, \\
&\qquad \text{and } (y \bigcirc_C a) \bigcirc_B z \sim (y \bigcirc_D a) \bigcirc_A z \text{ for some } a \in X.
\end{aligned}
$$

To see meanings of those relations, we suppose that nontransitive CEU model holds. Since $A \succ^* B$ and $C \succ^* D$, $\phi(A,B) > 0$ and $\phi(C,D) > 0$. Then for $\sigma \in \{\succ^*, \supseteq\}$,

$$
(x,y,z,w) \in E_\sigma(A,B;C,D) \Longleftrightarrow \frac{\phi(A,B)}{\phi(C,D)} = \frac{u(x)-u(y)}{u(z)-u(w)},
$$

so that the ratio, $\phi(A,B)/\phi(C,D)$, is given by the ratio of utility differences, $(u(x)-u(y))/(u(z)-u(w))$. To see this, assume first that $B \succ^* C$, $A \supset C$, $B \supset D$, $(x \bigcirc_D y) \bigcirc_B z \sim x \bigcirc_C z$, and $(y \bigcirc_C z) \bigcirc_A w \sim y \bigcirc_B w$. Then

$$
\begin{aligned}
\phi(D,C)(u(x)-u(y)) + \phi(B,C)(u(y)-u(z)) &= 0, \\
\phi(C,B)(u(y)-u(z)) + \phi(A,B)(u(z)-u(w)) &= 0,
\end{aligned}
$$

which is rearranged to give the desired result.

Assume next that $B \supseteq C$, $A \supset D$, $(x \bigcirc_C a) \bigcirc_B w \sim (x \bigcirc_D a) \bigcirc_A w$, and $(y \bigcirc_C a) \bigcirc_B z \sim (y \bigcirc_D a) \bigcirc_A z$ for some $a \in X$. Then

$$
\begin{aligned}
\phi(C,D)(u(x)-u(a)) + \phi(B,A)(u(a)-u(w)) &= 0, \\
\phi(C,D)(u(y)-u(a)) + \phi(B,A)(u(a)-u(z)) &= 0,
\end{aligned}
$$

which is rearranged to give

$$
\frac{\phi(A,B)}{\phi(C,D)} = \frac{u(x)-u(a)}{u(a)-u(w)} = \frac{u(y)-u(a)}{u(a)-u(z)}.
$$

Then we have

$$
\begin{aligned}
\frac{u(x)-u(a)}{u(a)-u(w)} &= \frac{u(x)-u(y)+u(y)-u(a)}{u(a)-u(z)+u(z)-u(w)} \\
&= \frac{u(y)-u(a)}{u(a)-u(z)} \cdot \frac{1+\dfrac{u(x)-u(y)}{u(y)-u(a)}}{1+\dfrac{u(z)-u(w)}{u(a)-u(z)}},
\end{aligned}
$$

so that the desired result obtains.

The following axiom requires that the ratios of utility difference consistently determine the ratios of ϕ. The axiom applies to all $x_i, y_i, z_i, w_i \in X$, all $A, B \in 2^S$, and all $A_i, B_i \in \Gamma_S^*$ for $i = 1, \ldots, n$.

Axiom 8(n) *If $n \geq 1$, $A_i \succ^* B_i$, and $C_i \succ^* D_i$ for $i = 1, \ldots, n$, if $B_i \supseteq C_i$ or $B_i \succ^* C_i$ for $i = 1, \ldots, n$, and $B_j \supseteq C_{j+1}$ or $B_j \succ^* C_{j+1}$ for $j = 1, \ldots, n$ with $C_{n+1} = C_1$ and $D_{n+1} = D_1$, if $E_i, E_{i,i+1} \in \{E_{\succ^*}, E_{\supseteq}\}$ for $i = 1, \ldots, n$, and if*

$$\begin{array}{rcll} (x_i, y_i, z_i, w_i) & \in & E_i(A_i, B_i; C_i, D_i) & \text{for } i = 1, \ldots, n, \\ (x_{j+1}, y_{j+1}, z_j, w_j) & \in & E_{j,j+1}(A_j, B_j; C_{j+1}, D_{j+1}) & \text{for } j = 1, \ldots, n-1, \end{array}$$

then $(x_1, y_1, z_n, w_n) \in E_{n,n+1}(A_n, B_n; C_1, D_1)$.

We note that Axiom $8(n)$ implies Axiom $8(n-1)$. To see that the axiom is necessary for nontransitive CEU model, we suppose that the hypotheses of the axiom holds. Then

$$\begin{array}{rcll} \dfrac{\phi(A_i, B_i)}{\phi(C_i, D_i)} & = & \dfrac{u(x_i) - u(y_i)}{u(z_i) - u(w_i)} & \text{for } i = 1, \ldots, n, \\ \dfrac{\phi(A_j, B_j)}{\phi(C_{j+1}, D_{j+1})} & = & \dfrac{u(x_{j+1}) - u(y_{j+1})}{u(z_j) - u(w_j)} & \text{for } j = 1, \ldots, n-1, \end{array}$$

which are combined to give

$$\frac{\phi(A_n, B_n)}{\phi(C_1, D_1)} = \frac{u(x_1) - u(y_1)}{u(z_n) - u(w_n)}.$$

Thus $(x_1, y_1, z_n, w_n) \in E_{n,n+1}(A_n, B_n; C_1, D_1)$.

A representation theorem for nontransitive CEU model is given as follows. The proof of the existence of u and ϕ is given in the next section. Section 5 completes the proof.

Theorem 1 *Suppose that Assumption 1 holds. If Axioms 1–7 and Axiom 8(4) hold, then there exist a real valued function u on X and a skew-symmetric monotone bivariate function ϕ on $2^S \times 2^S$ such that $\phi(S, \emptyset) > 0$ and for all $f, g \in \mathcal{F}$,*

$$f \succ g \iff \int_{-\infty}^{+\infty} \phi(S_f^u(\tau), S_g^u(\tau)) d\tau > 0.$$

Moreover, u on X is unique up to a positive linear transformation, and ϕ on $2^S \times 2^S \setminus \{(\emptyset, S), (S, \emptyset)\}$ is unique up to a positive multiplicative transformation.

We note that all the axioms except Axiom 2 are necessary for the representation in Theorem 1, and that Axiom $8(n)$ for $n > 4$ are not required for the sufficiency of the representation.

4 A Nonseparable Qualitative Probability

Throughout the rest of the paper we assume that Assumption 1, Axioms 1–7, and Axiom 8(4) hold. The purpose of this section is to prove the existence of a utility function u on X and a nonseparable qualitative probability representation ϕ on $2^S \times 2^S$ such that ϕ is skew-symmetric and monotone, and for all $x, y \in X$ and all $A, B \in 2^S$,

$$\begin{aligned} x \succeq y &\iff u(x) \geq u(y); \\ A \succ^* B &\iff \phi(A, B) > 0. \end{aligned}$$

This is obtained in the following proposition.

Proposition 1 *There exist a real valued function u on X and a skew-symmetric monotone bivariate function ϕ on $2^S \times 2^S$ such that $\phi(S, \emptyset) > 0$, and for all $x, y, z \in X$ with $x \succeq y \succeq z$, and for all $A, B, C, D \in 2^S$ with $A \subseteq B$ and $C \subseteq D$,*

$$\begin{aligned} &(x \bigcirc_A y) \bigcirc_B z \succeq (x \bigcirc_C y) \bigcirc_D z \\ &\quad \iff \phi(A, C)(u(x) - u(y)) + \phi(B, D)(u(y) - u(z)) \geq 0. \end{aligned}$$

Moreover, u on X is unique up to a positive linear transformation and ϕ on $2^S \times 2^S \setminus \{(S, \emptyset), (\emptyset, S)\}$ is unique up to a positive multiplicative transformation.

Before providing the proof of the proposition, we need to prove four lemmas.

Lemma 1 *Suppose that $x \succ y$ for $x, y \in X$, and that $A \subseteq B$ and $C \subseteq D$ for $A, B, C, D \in 2^S$. If $A \succ^* C$ and $D \succ^* B$, then there is an $a \in X$ such that $x \succ a \succ y$ and for all $z \in X$,*

$$\begin{aligned} &(x \bigcirc_A z) \bigcirc_B y \succ (x \bigcirc_C z) \bigcirc_D y \quad \textit{when} \quad a \succ z \succeq y; \\ &(x \bigcirc_A z) \bigcirc_B y \sim (x \bigcirc_C z) \bigcirc_D y \quad \textit{when} \quad z \sim a; \\ &(x \bigcirc_C z) \bigcirc_D y \succ (x \bigcirc_A z) \bigcirc_B y \quad \textit{when} \quad x \succeq z \succ a. \end{aligned}$$

Proof. Suppose that the hypotheses of the lemma hold. Since $A \succ^* C$ and $D \succ^* B$, we have by definition that

$$\begin{aligned} x \bigcirc_A y &= y \bigcirc_{B \setminus A} (x \bigcirc_A y) \succ x \bigcirc_C y = y \bigcirc_{D \setminus C} (x \bigcirc_C y), \\ x \bigcirc_D y &= x \bigcirc_{D \setminus C} (x \bigcirc_C y) \succ x \bigcirc_B y = x \bigcirc_{B \setminus A} (x \bigcirc_A y). \end{aligned}$$

By Axiom 2, there is an $a \in X$ such that $x \succeq a \succeq y$ and $a \bigcirc_{B \setminus A} (x \bigcirc_A y) \sim a \bigcirc_{D \setminus C} (x \bigcirc_C y)$. Thus $(x \bigcirc_A a) \bigcirc_B y \sim (x \bigcirc_C a) \bigcirc_D y$. If $a \sim x$, then by Axiom 5, $(x \bigcirc_C a) \bigcirc_D y \succ (x \bigcirc_A a) \bigcirc_B y$, a contradiction. When $a \sim y$, a similar contradiction obtains, so that $x \succ a \succ y$.

By Axiom 3, $(x \bigcirc_A z) \bigcirc_B y \sim (x \bigcirc_C z) \bigcirc_D y$ when $z \sim a$. Suppose that $x \succeq z \succ a$. Assume $(x \bigcirc_A z) \bigcirc_B y \succeq (x \bigcirc_C z) \bigcirc_D y$. Note that $(x \bigcirc_A z) \bigcirc_B y = z \bigcirc_{B \setminus A} (x \bigcirc_A y)$ and $(x \bigcirc_C z) \bigcirc_D y = z \bigcirc_{D \setminus C} (x \bigcirc_C y)$. Since $x \bigcirc_{D \setminus C} (x \bigcirc_C y) \succ x \bigcirc_{B \setminus A} (x \bigcirc_A y)$,

Axiom 2 implies that there is a $b \in X$ such that $x \succeq b \succeq z$ and $b \bigcirc_{B\setminus A} (x \bigcirc_A y) \sim b \bigcirc_{D\setminus C} (x \bigcirc_C y)$, so that

$$(x \bigcirc_A b) \bigcirc_B y \sim (x \bigcirc_C b) \bigcirc_D y.$$

By Axiom 3, $a \sim b$. However, $b \succeq z \succ a$, so by Axiom 1, $b \succ a$, a contradiction. Hence we must have $(x \bigcirc_C z) \bigcirc_D y \succ (x \bigcirc_A z) \bigcirc_B y$. It similarly follows that $(x \bigcirc_A z) \bigcirc_B y \succ (x \bigcirc_C z) \bigcirc_D y$ when $a \succ z \succeq y$. □

The following lemma shows the existence of a utility function u on X. Throughout the rest of the paper, let $X^n = X \times \cdots \times X$ (n times) and $X^n_\downarrow = \{(x_1, \ldots, x_n) \in X^n : x_1 \succeq \cdots \succeq x_n\}$.

Lemma 2 *Suppose that $A_i \subseteq B_i$, $C_i \subseteq D_i$, $A_i \succ^* C_i$, and $D_i \succ^* B_i$ for $i = 1, 2$. Then there exist a real valued function u on X and real numbers $0 < \alpha_i < 1$ for $i = 1, 2$ such that for all $x, y, z \in X$,*

(1) $x \succeq y \iff u(x) \geq u(y)$,
(2) if $x \succ y \succ z$, then for $i = 1, 2$,

$$(x \bigcirc_{A_i} y) \bigcirc_{B_i} z \sim (x \bigcirc_{C_i} y) \bigcirc_{D_i} z \iff u(y) = \alpha_i u(x) + (1 - \alpha_i) u(z),$$

(3) the closure of $u(X)$ is an interval,
(4) α_i for $i = 1, 2$ are unique and u is unique up to a positive linear transformation.

Proof. Suppose that the hypotheses of the lemma hold. Let $\bigcirc_i$ for $i = 1, 2$ be binary operations on X that map $X^2_\downarrow$ into X such that for all $(x, y) \in X^2_\downarrow$, $x \bigcirc_i y \sim a$ if $(x \bigcirc_{A_i} a) \bigcirc_{B_i} y \sim (x \bigcirc_{C_i} a) \bigcirc_{D_i} y$. By Lemma 1, $\bigcirc_i$ are well defined. Let K be any set of consecutive integers. We define a *standard sequence* with respect to (w.r.t.) $\bigcirc_i$ as a set $\{x_k : x_k \in X, k \in K\}$ for which there exist $a, b \in X$ such that not$(a \sim b)$, either $\{a, b\} \succeq \{x_k\}$ and $a \bigcirc_i x_k \sim b \bigcirc_i x_{k+1}$ for all $k, k+1 \in K$, or $\{x_k\} \succeq \{a, b\}$ and $x_k \bigcirc_i a \sim x_{k+1} \bigcirc_i b$ for all $k, k+1 \in K$.

We show that the following five conditions hold for each $i = 1, 2$: for all $x, y, z, w \in X$,

C1. $\succeq$ on X is a weak order.
C2. If $\{x, y\} \succeq z$ and $x \bigcirc_i z \succeq w \succeq y \bigcirc_i z$, then $w \sim a \bigcirc_i z$ for some $a \in X$ with $a \succeq z$; if $z \succeq \{x, y\}$ and $z \bigcirc_i x \succeq w \succeq z \bigcirc_i y$, then $w \sim z \bigcirc_i a$ for some $a \in X$ with $z \succeq a$.
C3. Every strictly bounded standard sequence w.r.t. $\bigcirc_i$ is finite.
C4. If $\{x, y\} \succeq z$, then $x \succeq y$ iff $x \bigcirc_i z \succeq y \bigcirc_i z$; if $z \succeq \{x, y\}$, then $x \succeq y$ iff $z \bigcirc_i x \succeq z \bigcirc_i y$.
C5. If $x \succeq \{y, z\} \succeq w$, then $(x \bigcirc_1 y) \bigcirc_2 (z \bigcirc_1 w) \sim (x \bigcirc_2 z) \bigcirc_1 (y \bigcirc_2 w)$.

C1 follows from Axiom 1. To show C2, assume that $\{x, y\} \succeq z$ and $x \bigcirc_i z \succeq w \succeq y \bigcirc_i z$. The other case similarly follows. Let $b = x \bigcirc_i z$ and $c = y \bigcirc_i z$. Then

by definition,

$$\begin{aligned}(x \bigcirc_{A_i} b) \bigcirc_{B_i} z &\sim (x \bigcirc_{C_i} b) \bigcirc_{D_i} z,\\ (y \bigcirc_{A_i} c) \bigcirc_{B_i} z &\sim (y \bigcirc_{C_i} c) \bigcirc_{D_i} z.\end{aligned}$$

We note that $x \succeq b \succeq w \succeq c \succeq z$ and $y \succeq c$. By Axiom 3, $x \succeq y$.

Assume $y \succeq w$. Then by Lemma 1 and Axiom 5, we obtain that

$$\begin{aligned}(x \bigcirc_{A_i} w) \bigcirc_{B_i} z &\succeq (x \bigcirc_{C_i} w) \bigcirc_{D_i} z,\\ (y \bigcirc_{C_i} w) \bigcirc_{D_i} z &\succeq (y \bigcirc_{A_i} w) \bigcirc_{B_i} z.\end{aligned}$$

By Axiom 2, there is an $a \in X$ such that a is between x and y, and $(a \bigcirc_{A_i} w) \bigcirc_{B_i} z \sim (a \bigcirc_{C_i} w) \bigcirc_{D_i} z$. Hence $w \sim a \bigcirc_i z$.

Assume next $w \succ y$. Then $w \succ z$. By definition, $(w \bigcirc_C w) \bigcirc_D z \succ (w \bigcirc_A w) \bigcirc_B z$. Since $(x \bigcirc_A w) \bigcirc_B z \succeq (x \bigcirc_C w) \bigcirc_D z$, Axiom 2 implies that there is an $a \in X$ such that a is between x and w, and $(a \bigcirc_A w) \bigcirc_D z \sim (a \bigcirc_C w) \bigcirc_D z$. Hence $w \sim a \bigcirc_i z$.

C3 and C4 respectively follow from Axioms 6 and 3. To show C5, assume that $x \succeq \{y, z\} \succeq w$. It follows from Lemma 1 that for some $a_{xy}, a_{zw} \in X$,

$$\begin{aligned}f_1 &= (x \bigcirc_{A_1} a_{xy}) \bigcirc_{B_1} y &\sim\ f_2 &= (x \bigcirc_{C_1} a_{xy}) \bigcirc_{D_1} y,\\ h_1 &= (z \bigcirc_{A_1} a_{zw}) \bigcirc_{B_1} w &\sim\ h_2 &= (z \bigcirc_{C_1} a_{zw}) \bigcirc_{D_1} w.\end{aligned}$$

By Axiom 3, $a_{xy} \succeq a_{zw}$. Thus again Lemma 1 implies that for some $b_{xz}, b_{yw}, c \in X$,

$$\begin{aligned}(x \bigcirc_{A_2} b_{xz}) \bigcirc_{B_2} z &\sim (x \bigcirc_{C_2} b_{xz}) \bigcirc_{D_2} z,\\ (a_{xy} \bigcirc_{A_2} c) \bigcirc_{B_2} a_{zw} &\sim (a_{xy} \bigcirc_{C_2} c) \bigcirc_{D_2} a_{zw},\\ (y \bigcirc_{A_2} b_{yw}) \bigcirc_{B_2} w &\sim (y \bigcirc_{C_2} b_{yw}) \bigcirc_{D_2} w.\end{aligned}$$

By Axiom 3, $b_{xz} \succeq b_{yz} \succeq c$. Let

$$\begin{aligned}g_1 &= (b_{xz} \bigcirc_{A_1} c) \bigcirc_{B_1} b_{yw},\\ g_2 &= (b_{xz} \bigcirc_{C_1} c) \bigcirc_{D_1} b_{yw}.\end{aligned}$$

Then (g_1, g_2) is proportionally shifted from (f_1, f_2) and (h_1, h_2). Since $f_1 \sim f_2$ and $h_1 \sim h_2$, Axiom 7 implies that $g_1 \sim g_2$. Hence by the definition of $\bigcirc_i$ and C4, $c \sim b_{xz} \bigcirc_1 b_{yw} \sim (x \bigcirc_2 z) \bigcirc_1 (y \bigcirc_2 w)$, and $c \sim a_{xy} \bigcirc_2 a_{zw} \sim (x \bigcirc_1 y) \bigcirc_2 (z \bigcirc_1 w)$. Since $\sim$ on X is transitive, C5 obtains.

Since C1–C5 hold, it follows from Proposition 1 in Nakamura (1992) that there exist a real valued function u on X and real numbers $0 < \alpha_i < 1$ for $i = 1, 2$ such that for all $x, y \in X$,

(a) $x \succeq y \Longleftrightarrow u(x) \geq u(y)$.
(b) if $x \succeq y$, then $u(x \bigcirc_i y) = \alpha_i u(x) + (1 - \alpha_i) u(y)$.
(c) u is unique up to a positive linear transformation, and $\alpha_i, i = 1, 2$ are unique.

Hence (1)–(4) readily follows from (a)–(c). □

We say that $(y_1, \ldots, y_n) \in X_\downarrow^n$ is a *contraction* of $(x_1, \ldots, x_n) \in X_\downarrow^n$ if there are $A, B, C, D \in 2^S$ and $a \in X$ such that $A \succ^* C$, $D \succ^* B$, and either $a \succeq y_i \succeq x_i$ and $(a \bigcirc_A y_i) \bigcirc_B x_i \sim (a \bigcirc_C y_i) \bigcirc_D x_i$ for all $i = 1, \ldots, n$, or $x_i \succeq y_i \succeq a$ and $(x_i \bigcirc_A y_i) \bigcirc_B a \sim (x_i \bigcirc_C y_i) \bigcirc_D a$ for all $i = 1, \ldots, n$. Successive construction of contractions is said to be a *contraction process*. By $(x_1^*, \ldots, x_n^*)$, we shall denote an element of $X_\downarrow^n$ that is generated from $(x_1, \ldots, x_n)$ by a contraction process.

To see an implication of the contraction process, let $(x_1, \ldots, x_n) \in X_\downarrow^n$, and for $i = 1, 2$,

$$\begin{aligned} f_i &= x_1 \bigcirc_{A_{i1}} x_2 \cdots x_{n-1} \bigcirc_{A_{i,n-1}} x_n, \\ f_i^* &= x_1^* \bigcirc_{A_{i1}} x_2^* \cdots x_{n-1}^* \bigcirc_{A_{i,n-1}} x_n^*. \end{aligned}$$

If $(x_1^*, \ldots, x_n^*)$ is a contraction of $(x_1, \ldots, x_n)$, then there is a constant act $a \in X$ for which (f_1^*, f_2^*) is proportionally shifted from (f_1, f_2) and (a, a). Thus it easily follows from Axiom 7 that $f_1 \succeq f_2 \iff f_1^* \succeq f_2^*$. With no loss of generality, we assume $x_i \succeq x_i^* \succeq a$ for $i = 1, \ldots, n$. Then by Lemma 2, there is a number $0 < \alpha < 1$ such that, for $i = 1, \ldots, n$,

$$(x_i \bigcirc_A x_i^*) \bigcirc_B a \sim (x_i \bigcirc_C x_i^*) \bigcirc_D a \iff u(x_i^*) = \alpha u(x_i) + (1 - \alpha) u(a).$$

Therefore, for $i = 1, \ldots, n-1$,

$$u(x_i^*) - u(x_{i+1}^*) = \alpha(u(x_i) - u(x_{i+1})).$$

If $(x_1^*, \ldots, x_n^*)$ is contracted n times from $(x_1, \ldots, x_n)$ applying the same A, B, C, and D, then it easily follows that $f_1 \succeq f_2 \iff f_1^* \succeq f_2^*$, and

$$u(x_i^*) - u(x_{i+1}^*) = \alpha^n(u(x_i) - u(x_{i+1})).$$

Thus by a contraction process, we can obtain a contraction $(x_1^*, \ldots, x_n^*)$ whose utility difference, $u(x_1^*) - u(x_{n+1}^*)$, can be arbitrarily small.

Suppose that $A \subseteq C$ and $B \subseteq D$. When either $A \succ^* B$ and $D \succ^* C$, or $B \succ^* A$ and $C \succ^* D$, we define

$$\omega(AB; CD) = \frac{1 - \alpha}{\alpha},$$

where α is given as a unique number in Lemma 2 such that for all $x, y, z \in X$ with $x \succ y \succ z$,

$$(x \bigcirc_A y) \bigcirc_C z \sim (x \bigcirc_B y) \bigcirc_D z \iff u(y) = \alpha u(x) + (1 - \alpha) u(z).$$

Note that $\omega(AB; CD) = \omega(BA; DC)$. When either $A \succeq^* B$ and $C \succeq^* D$, or $B \succeq^* A$ and $D \succeq^* C$, $\omega(AB; CD)$ is undefined. Thus when we write $\omega(AB; CD)$, it always means that $A \subseteq C$, $B \subseteq D$, and either $A \succ^* B$ and $D \succ^* C$, or $B \succ^* A$ and $C \succ^* D$. The representation (2) in Lemma 2 is rewritten in terms of ω as follows. There exist a real valued function u on X, which is unique up to a positive linear

transformation, and a unique number $\omega(AB;CD)$ such that, for all $x,y,z \in X$ with $x \succeq y \succeq z$,

$$(x \bigcirc_A y) \bigcirc_C z \sim (x \bigcirc_B y) \bigcirc_D z \Longleftrightarrow u(x) - u(y) = \omega(AB;CD)(u(y) - u(z)).$$

Axiom 8(n) provides the following two chain rules of ω that will be used in the proof of Proposition 1.

Lemma 3 *Suppose that Axiom 8(n) holds. Let $n \geq 1$ and $\mathcal{N} = \{1,\ldots,n\}$. Given $A_i, B_i \in \Gamma_S^*$ for $i \in \mathcal{N}$, suppose that $A_i \supseteq B_i$ or $A_i \succ^* B_i$ for $i \in \mathcal{N}$, and $A_j \supseteq B_{j+1}$ or $A_j \succ^* B_{j+1}$ for $j \in \mathcal{N}$, where $B_{n+1} = B_1$. Let $\mathcal{N}^* = \{i \in \mathcal{N} : A_i \supseteq B_i\}$ and $\mathcal{N}_0^* = \{i \in \mathcal{N} : A_i \supseteq B_{i+1}\}$. Then*

$$\prod_{j \in \mathcal{N}^*} \omega(\emptyset B_i; SA_i) \prod_{i \in \mathcal{N} \setminus \mathcal{N}^*} \omega(\emptyset B_i; A_i B_i)\omega(B_i A_i; SA_i)$$
$$= \prod_{j \in \mathcal{N}_0^*} \omega(\emptyset B_{i+1}; SA_i) \prod_{i \in \mathcal{N} \setminus \mathcal{N}_0^*} \omega(\emptyset B_{i+1}; A_i B_{i+1})\omega(B_{i+1} A_i; SA_i).$$

Lemma 4 *Suppose that Axiom 8(3) holds. Given $A_1, A_2, A_3, B_1, B_2, B_3 \in \Gamma_S^*$ and $A, B \in 2^S$, suppose that $A \succ^* A_3$, $B_1 \succ^* B$, $A_i \supseteq B_i$ and $A_i \supseteq B_{i+1}$ for $i = 1,2,,3$, where $B_4 = B_1$. Then*

$$\omega(BB_1; SA_1)\omega(\emptyset B_2; SA_2)\omega(\emptyset B_3; AA_3) = \omega(\emptyset B_2; SA_1)\omega(\emptyset B_3; SA_2)\omega(BB_1; AA_3).$$

Proofs of Lemmas 3 and 4. We prove only Lemma 3, since the proof of Lemma 4 is similar. Let $X^* = \{x \in X : a \succ x \succ b \text{ for some } a, b \in X\}$. Suppose that $x, y, z \in X^*$, $A \succ^* B$, and $C \succ^* D$ for $A, B, C, D \in 2^S$. We first show the following implications of contraction processes:

(1) if $x \succ y$, then for some (x^*, y^*) and $a, b \in X$ with $y \succ a \succ b$,

$$(x^*, y^*, a, b) \in E_{\succ^*}(A, B; C, D),$$

(2) if $x \succ y$, then for some (x^*, y^*) and $a, b \in X$ with $a \succ b \succ x$,

$$(a, b, x^*, y^*) \in E_{\succ^*}(A, B; C, D),$$

(3) if $x \succ y \succeq z$, then for some (x^*, y^*, z^*) and $a \in X$,

$$(x^*, y^*, z^*, a) \in E_{\supseteq}(A, B; C, D),$$

(4) if $x \succeq y \succ z$, then for some (x^*, y^*, z^*) and $a \in X$,

$$(a, x^*, y^*, z^*) \in E_{\supseteq}(A, B; C, D).$$

We show (1). The proof of (2) is similar. Let $x, y \in X^*$ satisfy $x \succ y$. Since $B \succ^* C$, $C \succ^* D$, and $B \supseteq D$, it follows from Lemma 2 that there is a $c \in X$ such that $x \succ c \succ y$ and

$$(x \bigcirc_D c) \bigcirc_B y \sim x \bigcirc_C y.$$

Then let $x' = x$, $y' = c$, and $a = y$, so that (x', y') is a contraction of (x, y), and

$$(x^* \bigcirc_D y^*) \bigcirc_B a \sim x^* \bigcirc_C a.$$

Since $A \succ^* B$, $B \succ^* C$, and $A \supseteq C$, it follows from Lemma 2 that there are $d_1, d_2 \in X$ such that $x^* \succ d_1 \succ y^*$, $x^* \succ d_2 \succ a$, and

$$\begin{aligned}(x^* \bigcirc_C d_1) \bigcirc_A y^* &\sim x^* \bigcirc_B y^*, \\ (x^* \bigcirc_C d_2) \bigcirc_A a &\sim x^* \bigcirc_B a.\end{aligned}$$

By Axiom 3, $x^* \succ d_1 \succ^* d_2 \succ^* a$. By Lemma 1,

$$\begin{aligned}(d_1 \bigcirc_C d_2) \bigcirc_A a &\succ d_1 \bigcirc_B a, \\ d_1 \bigcirc_B d_2 &\succ (d_1 \bigcirc_C d_2) \bigcirc_A d_2,\end{aligned}$$

so that by Axiom 2, there is an $e \in X$ such that

$$(d_1 \bigcirc_C d_2) \bigcirc_A e \sim d_1 \bigcirc_B e.$$

Then Letting $x^* = x'$, $y^* = d_1$, $a = d_2$, and $b = e$, so (x^*, y^*) is a contraction of (x', y'). By Axiom 7, we get

$$(x^*, y^*, a, b) \in E_{\succ^*}(A, B; C, D).$$

Next we show (3). The proof of (4) is similar. Let $x, y, z \in X^*$ satisfy $x \succ y \succeq z$. By Lemma 2, there is a $c \in X$ with $y \succeq c \succeq z$ such that $(y \bigcirc_C c) \bigcirc_B z \sim (y \bigcirc_D c) \bigcirc_A z$. Again by Lemma 2, there are $d_1, d_2, d_3 \in d$ such that $x \succ d_1 \succ d_2 \succ d_3 \succ c$, and

$$\begin{aligned}(x \bigcirc_C d_1) \bigcirc_B y &\sim (x \bigcirc_D d_1) \bigcirc_A y, \\ (x \bigcirc_C d_2) \bigcirc_B c &\sim (x \bigcirc_D d_2) \bigcirc_A c, \\ (x \bigcirc_C d_3) \bigcirc_B z &\sim (x \bigcirc_D d_3) \bigcirc_A z.\end{aligned}$$

Then let $x^* = x$, $y^* = d_1$, $z^* = d_3$, and $a = c$, so (x^*, y^*, z^*) is a contraction of (x, y, z). By Axiom 7, we get

$$(x^*, y^*, z^*, a) \in E_{\supseteq}(A, B; C, D).$$

It follows from (1)–(4) and applying appropriate contraction processes that there are $x_i, y_i, z_i, w_i \in X$ for $i \in \mathcal{N}$ such that

$$\begin{aligned}(x_i, y_i, z_i, w_i) &\in E_{\supseteq}(S, A_i; B_i, \emptyset) && \text{for } i \in \mathcal{N}^*, \\ (x_i, y_i, z_i, w_i) &\in E_{\succ^*}(S, A_i; B_i, \emptyset) && \text{for } i \notin \mathcal{N}^*, \\ (x_{i+1}, y_{i+1}, z_i, w_i) &\in E_{\supseteq}(S, A_i; B_{i+1}, \emptyset) && \text{for } i \in \mathcal{N}_0^*, \\ (x_{i+1}, y_{i+1}, z_i, w_i) &\in E_{\succ^*}(S, A_i; B_{i+1}, \emptyset) && \text{for } i \notin \mathcal{N}_0^*.\end{aligned}$$

Hence the desired conclusion follows from Axiom $8(n)$ and the definition of ω. □

Proof of Proposition 1. We prove the proposition in four steps. The first step concerns with the construction of conditional monotone functions on 2^S. The second step constructs a monotone bivariate set function ϕ on $2^S \times 2^S$ such that for all $A, B \in 2^S$, $A \succ^* B \iff \phi(A,B) > 0$. The third step proves the representation. Finally, the third step shows that ϕ thus constructed is skew-symmetric. The uniqueness part of the proposition easily follow from the construction of ϕ. Let u be a real valued function on X obtained in Lemma 2. Recall that whenever $A \subseteq C$, $B \subseteq D$, $A \succ^* B$ and $D \succ^* C$, $\omega(AB;CD)$ and $\omega(BA;DC)$ are defined to be a unique number $(1-\alpha)/\alpha$ for which for all $a, y, z \in X$ with $x \succ y \succ z$,

$$(x \bigcirc_A y) \bigcirc_C z \sim (x \bigcirc_B y) \bigcirc_D z \iff u(y) = \alpha u(x) + (1-\alpha)u(z).$$

Step 1. Given $A \in \Gamma_S^*$, we construct a real valued function ϕ_A on 2^S such that for all $B \in 2^S$,

$$\begin{aligned} B \succ^* A &\iff \phi_A(B) > 0; \\ A \sim^* B &\iff \phi_A(B) = 0; \\ A \succ^* B &\iff \phi_A(B) < 0. \end{aligned}$$

Moreover, ϕ_A is unique up to a positive multiplicative transformation.

Since A is proper, we assign any negative number to $\phi_A(\emptyset)$. Then we define $\phi_A(B)$ for all $B \in 2^S$ as follows.

$$\phi_A(B) = \begin{cases} 0 & \text{if } B \sim^* A, \\ \omega(\emptyset A; BA)\phi_A(\emptyset) > 0 & \text{if } B \succ^* A, \\ -\dfrac{\omega(\emptyset A; SA)}{\omega(BA; SA)}\phi_A(\emptyset) < 0 & \text{if } A \succ^* B. \end{cases}$$

Since $\phi_A(S) = -\omega(\emptyset A; SA)\phi_A(\emptyset)$ by construction, we note

$$\phi_A(S) = -\omega(BA; SA)\phi_A(B).$$

The uniqueness of ϕ_A easily follows from the construction of ϕ_A, and the uniqueness of ω.

Step 2. Let $\{\phi_A : A \in \Gamma_S^*\}$ be the set of all real valued functions on 2^S obtained in Step 1. Given ϕ_A for all $A \in \Gamma_S^*$, it requires from the representation that for all $A \in \Gamma_S^*$ and some $a_A > 0$,

$$\begin{aligned} \phi(\cdot, A) &= a_A \phi_A(\cdot), \\ \phi(A, \cdot) &= -a_A \phi_A(\cdot). \end{aligned}$$

Throughout the rest of the proof, we shall fix $\{s\} \in \Gamma_S^*$ and write ϕ_s in place of $\phi_{\{s\}}$. Define for all $A \in 2^S$,

$$\begin{aligned} \phi(A, \{s\}) &= \phi_s(A), \\ \phi(\{s\}, A) &= -\phi_s(A), \end{aligned}$$

i.e., $a_{\{s\}} = 1$, so ϕ is skew-symmetric for the pairs $(A, \{s\})$. In the sequel, we shall construct a ϕ on $2^S \times 2^S \setminus \{(S,\emptyset),(\emptyset,S)\}$ by appropriately choosing constants a_A for all $A \in \Gamma_S^*$ except $\{s\}$.

Throughout the proof, we shall let $A_s = A \cup \{s\}$, and

$$\begin{aligned} P_s &= \{A \in \Gamma_S^* : \mathrm{not}(A \sim^* \{s\})\}, \\ R_s &= \{A \in \Gamma_S^* : A \supset \{s\}\}, \\ I_s &= \{A \in \Gamma_S^* : A \sim^* \{s\}\}. \end{aligned}$$

For every $A \in P_s \cup R_s$, scale ϕ_A by means of a_A so that if $A \in P_s$, then

$$\phi_A(\{s\}) = -\phi_s(A),$$

and if $A \in R_s$, then

$$\phi_A(S) = -\omega(\{s\}\emptyset; AS)\phi_s(\emptyset).$$

To see that the above scalings are consistent, we assume that $A \in P_s \cap R_s$, so $S \succ^* A \succ^* \{s\} \succ^* \emptyset$ and $A \supset \{s\}$. Then by Step 1,

$$\begin{aligned} \phi_s(A) &= -\omega(\{s\}\emptyset; \{s\}A)\phi_s(\emptyset), \\ \phi_A(S) &= -\omega(A\{s\}; AS)\phi_A(\{s\}). \end{aligned}$$

By scaling, $\phi_A(\{s\}) = -\phi_s(A)$. Thus we have

$$\phi_A(S) = -\omega(A\{s\}; AS)\omega(\{s\}\emptyset; \{s\}A)\phi_s(\emptyset).$$

Let $A_1 = A$, $B_1 = \{s\}$, $\mathcal{N} = \{1\}$, $\mathcal{N}^* = \emptyset$, and $\mathcal{N}_0^* = \{1\}$ in Lemma 3. Then

$$\omega(\{s\}\emptyset; AS) = \omega(A\{s\}; AS)\omega(\{s\}\emptyset; \{s\}A),$$

so that the scalings are consistent.

For every $A \in I_s$, scale ϕ_A by means of a_A so that if $A_s \neq S$, then

$$\phi_A(\emptyset) = \frac{\omega(\{s\}\emptyset; A_sS)}{\omega(A\emptyset; A_sS)}\phi_s(\emptyset),$$

and if $A_s = S$, then $\{s\} \cap A = \emptyset$, and by Assumption 1, $B \subseteq A$ and $A \neq B$ for some $B \in \Gamma_S^*$, so let

$$\phi_A(S) = \frac{\omega(B\emptyset; AS)\omega(\{s\}\emptyset; B_sS)}{\omega(B\emptyset; B_sS)}\phi_s(\emptyset).$$

To see that the above scalings are consistent, we first assume that $A \in I_s \cap R_s$. Since $S \succ^* A \supset \{s\} \succ^* \emptyset$ and $A \sim^* \{s\}$, we have

$$\phi_A(S) = \begin{cases} -\omega(\{s\}\emptyset; AS)\phi_s(\emptyset) & \text{(by scaling for } A \in P_s \cup R_s\text{)}, \\ -\omega(A\emptyset; AS)\phi_A(\emptyset) & \text{(by Step 1)}, \end{cases}$$

so that

$$\phi_A(\emptyset) = \frac{\omega(\{s\}\emptyset; AS)}{\omega(A\emptyset; AS)}\phi_s(\emptyset).$$

Let $(A_1, A_2) = (A_s, A)$, $(B_1, B_2) = (A, \{s\})$, and $\mathcal{N} = \mathcal{N}^* = \mathcal{N}_0^* = \{1, 2\}$ in Lemma 3. Then

$$\omega(\emptyset A; SA_s)\omega(\emptyset\{s\}; SA) = \omega(\emptyset\{s\}; SA_s)\omega(\emptyset A; SA),$$

so that the scaling for $A \in P_s \cup R_S$ and the scaling for $A \in I_s$ and $A_s \neq S$ are consistent.

Next we assume that $A_s = S$ and there is a $C \in \Gamma_S^*$ such that $C \subseteq A$, $C \neq A$ and $C \neq B$. Then by scaling for $A \in I_s$ and $A_s = S$,

$$\phi_A(S) = \frac{\omega(C\emptyset; AS)\omega(\{s\}\emptyset; C_sS)}{\omega(C\emptyset; C_sS)}\phi_s(\emptyset).$$

Let $(A_1, A_2, A_3) = (B_s, A, C_s)$, $(B_1, B_2, B_3) = (\{s\}, B, C)$, and $\mathcal{N} = \mathcal{N}^* = \mathcal{N}_0^* = \{1, 2, 3\}$ in Lemma 3. Then

$$\omega(\emptyset\{s\}; SB_s)\omega(\emptyset B; SA)\omega(\emptyset C; SC_s) = \omega(\emptyset B; SB_s)\omega(\emptyset C; SA)\omega(\emptyset\{s\}; SC_s),$$

so that the scaling for $A \in I_s$ and $A_s = S$ does not depend on choosing $B \supseteq A$.

Now we have $\{\phi_A : A \in \Gamma_S^*\}$ rescaled in terms of a_A in the preceding paragraphs. Then we define ϕ on $2^S \times 2^S$ as follows. Let $\phi(S, \emptyset) = -\phi(\emptyset, S)$ be given an arbitrary positive number, and for all $A \in \Gamma_S^*$ and all $B \in 2^S$, let

$$\begin{aligned} \phi(B, A) &= \phi_A(B), \\ \phi(A, S) &= -\phi_A(S), \\ \phi(A, \emptyset) &= -\phi_A(\emptyset). \end{aligned}$$

Monotonicity of ϕ follows from Axiom 4.

Step 3. We show that if $A \supseteq B$, $B \supseteq C$, $C \succ^* D$, and $A \supseteq D$, then

$$\phi(A, B) = -\omega(DC; AB)\phi(D, C).$$

Thus the representation follows from the definition of ω and Lemmas 1 and 2,

We have the following three cases to examine.

Case 1. $A = S$ and $D = \emptyset$,
Case 2. either $A = S$ and $D \neq \emptyset$ or $A \neq S$ and $D = \emptyset$,
Case 3. $A \neq S$ and $D \neq \emptyset$.

Case 1. $A = S$ and $D = \emptyset$. We have three subcases to examine: $s \in C$; $s \in B \setminus C$; $s \notin B$. Assume first that $s \in C$. When $C \succ^* \{s\}$, the proof is similar to the case of $C \sim^* \{s\}$. We assume that $C \sim \{s\}$. Let $(A_1, A_2) = (C, B)$, $(B_1, B_2) = (\{s\}, C)$, $\mathcal{N} = \mathcal{N}^* = \mathcal{N}_0^* = \{1, 2\}$ in Lemma 3. Then we obtain

$$\omega(\emptyset\{s\}; SC)\omega(\emptyset C; SB) = \omega(\emptyset\{s\}; SB)\omega(\emptyset C; SC).$$

By Steps 1 and 2,

$$\begin{aligned}
\phi(S,B) &= -\omega(\emptyset\{s\};SB)\phi(\emptyset,\{s\}) \quad \text{(Step 2)}\\
\phi(S,C) &= -\omega(\emptyset\{s\};SC))\phi(\emptyset,\{s\}) \quad \text{(Step 2)}\\
\phi(S,C) &= -\omega(\emptyset C;SC)\phi(\emptyset,C) \quad \text{(Step 1)}
\end{aligned}$$

Hence $\phi(S,B) = -\omega(\emptyset C;SB)\phi(\emptyset,C)$.

Assume next that $s \in B \setminus C$. Let $(A_1,A_2) = (C_s,B)$, $(B_1,B_2) = (\{s\},C)$, $\mathcal{N} = \mathcal{N}^* = \mathcal{N}_0^* = \{1,2\}$ in Lemma 3. Then we obtain

$$\omega(\emptyset\{s\};SC_s)\omega(\emptyset C;SB) = \omega(\emptyset\{s\};SB)\omega(\emptyset C;SC_s).$$

By Step 2,

$$\begin{aligned}
\phi(\emptyset,C) &= \frac{\omega(\emptyset\{s\};SC_s)}{\omega(\emptyset C;SC_s)}\phi(\emptyset,\{s\}),\\
\phi(S,B) &= -\omega(\emptyset\{s\};SB)\phi(\emptyset,\{s\}).
\end{aligned}$$

Hence $\phi(S,B) = -\omega(\emptyset C;SB)\phi(\emptyset,C)$.

Last we assume that $s \notin B$. When $C \sim^* \{s\}$ or $B \sim^* \{s\}$, the proofs are similar to the case of $C \sim^* \{s\}$ and $B \sim^* \{s\}$. We asusme that $C \sim^* \{s\}$ and $B \sim^* \{s\}$. Let $(A_1,A_2,A_3) = (C_s,B,B_s)$, $(B_1,B_2,B_3) = (\{s\},C,B)$, $\mathcal{N} = \mathcal{N}^* = \mathcal{N}_0^* = \{1,2,3\}$ in Lemma 3. Then

$$\omega(\emptyset\{s\};SC_s)\omega(\emptyset C;SB)\omega(\emptyset B;SB_s) = \omega(\emptyset C;SC_s)\omega(\emptyset B;SB)\omega(\emptyset\{s\};SB_s).$$

Bt Steps 1 and 2,

$$\begin{aligned}
\phi(\emptyset,C) &= \frac{\omega(\emptyset\{s\};SC_s)}{\omega(\emptyset C;SC_s)}\phi(\emptyset,\{s\}) \quad \text{(Step 2)}\\
\phi(\emptyset,B) &= \frac{\omega(\emptyset\{s\};SB_s)}{\omega(\emptyset B;SB_s)}\phi(\emptyset,\{s\}) \quad \text{(Step 2)}\\
\phi(S,B) &= -\omega(\emptyset B;SB)\phi(\emptyset,B) \quad \text{(Step 1)}
\end{aligned}$$

Hence $\phi(S,B) = -\omega(\emptyset C;SB)\phi(\emptyset,C)$.

Case 2. Suppose that $A = S$ and $D \neq \emptyset$. When $A \neq S$ and $D = \emptyset$, the proof is similar. Let $n = 2$, $(A_1,A_2) = (C,B)$, $(B_1,B_2) = (C,C)$ in Lemma 4. Then

$$\omega(DC;SC)\omega(\emptyset C;SB) = \omega(DC;SB)\omega(\emptyset C;SC).$$

By Case 1 and Step 1,

$$\begin{aligned}
\phi(S,B) &= -\omega(\emptyset C;SB)\phi(\emptyset,C),\\
\phi(S,C) &= -\omega(\emptyset C;SC)\phi(\emptyset,C),\\
\phi(S,C) &= -\omega(DC;SC)\phi(D,C).
\end{aligned}$$

Hence $\phi(S,B) - \omega(DC;SB)\phi(D,C)$.

Case 3. $A \neq S$ and $D \neq \emptyset$. Let $n = 3$, $(A_1, A_2, A_3) = (C, B, C)$, and $(B_1, B_2, B_3) = (C, C, B)$ in Lemma 4. Then

$$\omega(DC; SC)\omega(\emptyset C; SB)\omega(\emptyset B; AB) = \omega(\emptyset C; SC)\omega(\emptyset B; SB)\omega(DC; AB).$$

By Cases 1, 2 and Step 1,

$$\begin{aligned}
\phi(S, C) &= -\omega(DC; SC)\phi(D, C), \\
\phi(S, C) &= -\omega(\emptyset C; SC)\phi(\emptyset, C), \\
\phi(S, B) &= -\omega(\emptyset C; SB)\phi(\emptyset, C), \\
\phi(S, B) &= -\omega(\emptyset B; SB)\phi(\emptyset, B), \\
\phi(A, B) &= -\omega(\emptyset B; AB)\phi(\emptyset, B).
\end{aligned}$$

Hence $\phi(A, B) = -\omega(DC; AB)\phi(D, C)$.

Step 4. We show that ϕ is skew-symmetric, i.e., $\phi(A, B) = -\phi(B, A)$ for all $A, B \in 2^S$. If $A \sim^* B$, skew-symmetry of ϕ follows trivially. Thus assume $A \succ^* B$. We shall show here the proof of the case that $A \succ^* \{s\}$ and $B \succ^* \{s\}$, and defer other cases to Appendix, since the proofs are similar, but require Axiom 8(4) in some cases which does not apply in the proofs shown in the text.

Assume that $A \succ^* \{s\}$ and $B \succ^* \{s\}$. Let $(A_1, A_2) = (A, B)$, $(B_1, B_2) = (\{s\}, B)$, $\mathcal{N} = \{1, 2\}$, $\mathcal{N}^* = \{2\}$, and $\mathcal{N}_0^* = \emptyset$ in Lemma 3. Then

$$\begin{aligned}
&\omega(\emptyset\{s\}; A\{s\})\omega(\{s\}A; SA)\omega(\emptyset B; SB) \\
&\quad = \omega(\emptyset B; AB)\omega(BA; SA)\omega(\emptyset\{s\}; B\{s\})\omega(\{s\}B; SB),
\end{aligned}$$

It follows from Step 1 that

$$\begin{aligned}
\phi(S, A) &= -\omega(\{s\}A; SA)\phi(\{s\}, A) = -\omega(BA; SA)\phi(B, A), \\
\phi(S, B) &= -\omega(\{s\}B; SB)\phi(\{s\}, B) = -\omega(\emptyset B; SB)\phi(\emptyset, B), \\
\phi(B, A) &= -\omega(\emptyset B; AB)\phi(\emptyset, B),
\end{aligned}$$

so rearrangements give

$$\begin{aligned}
\phi(B, A) &= \frac{\omega(\{s\}A; SA)}{\omega(BA; SA)}\phi(\{s\}, A), \\
\phi(A, B) &= -\frac{\omega(\{s\}B; SB)\omega(\emptyset B; AB)}{\omega(\emptyset B; SB)}\phi(\{s\}, B).
\end{aligned}$$

We note by Steps 1 and 2 that

$$\begin{aligned}
\phi(\{s\}, A) &= -\phi(A, \{s\}) = \omega(\emptyset\{s\}; A\{s\})\phi(\emptyset, \{s\}), \\
\phi(\{s\}, B) &= -\phi(B, \{s\}) = \omega(\emptyset\{s\}; B\{s\})\phi(\emptyset, \{s\}).
\end{aligned}$$

Hence we obtain the following expressions of $\phi_A(B)$ and $\phi_B(A)$.

$$\begin{aligned}\phi(B,A) &= -\frac{\omega(\{s\}A;SA)\omega(\emptyset\{s\};A\{s\})}{\omega(BA;SA)}\phi(\emptyset,\{s\}),\\ \phi(A,B) &= \frac{\omega(\{s\}B;SB)\omega(\emptyset B;AB)\omega(\emptyset\{s\};B\{s\})}{\omega(\emptyset B;SB)}\phi(\emptyset,\{s\}).\end{aligned}$$

Hence by the chain rule, $\phi(B,A) = -\phi(A,B)$.

5 Proof of the Theorem

Let u on X and ϕ on $2^S \times 2^S$ be obtained in Proposition 1. We are to show by induction that, for all $f, g \in \mathcal{F}$, if $f(S) \cup g(S) = \{x_1, \ldots, x_{m+1}\}$ and $x_1 \succeq \ldots \succeq x_{m+1}$, then

$$f \succeq g \Longleftrightarrow \sum_{i=1}^{m} \phi(f^{-1}(x_i), g^{-1}(x_i))(u(x_i) - u(x_{i+1})) \geq 0.$$

Since the case for $m = 2$ has already been established in Proposition 1, we shall assume, in what follows, that the conclusion is true for all $m < n$, where $n > 2$.

Given two increasing sequences of events, $A_1 \subseteq \cdots \subseteq A_n$ and $B_1 \subseteq \cdots \subseteq B_n$, we define a subset $R(A_1, \ldots A_n; B_1, \ldots, B_n)$ of $X_{\downarrow}^{n+1}$ as follows: for all $(x_1, \ldots, x_{n+1}) \in X_{\downarrow}^{n+1}$,

$$\begin{aligned}&(x_1, \ldots, x_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n)\\ &\quad\Longleftrightarrow\quad x_1 \bigcirc_{A_1} x_2 \cdots x_n \bigcirc_{A_n} x_{n+1} \succeq x_1 \bigcirc_{B_1} x_2 \cdots x_n \bigcirc_{B_n} x_{n+1},\end{aligned}$$

and let

$$\begin{aligned}&I(A_1, \ldots, A_n; B_1, \ldots, B_n)\\ &\quad = \quad R(A_1, \ldots, A_n; B_1, \ldots, B_n) \cap R(B_1, \ldots, B_n; A_1, \ldots, A_n),\\ &P(A_1, \ldots, A_n; B_1, \ldots, B_n)\\ &\quad = \quad R(A_1, \ldots, A_n; B_1, \ldots, B_n) \setminus I(A_1, \ldots, A_n; B_1, \ldots, B_n).\end{aligned}$$

Lemma 5 *(1) Suppose that $(x_1, \ldots, x_{n+1})$, $(y_1, \ldots, y_{n+1})$, and $(z_1, \ldots, z_{n+1})$ are in $X_{\downarrow}^{n+1}$, and that $x_i \succeq y_i \succeq z_i$ and $(x_i, y_i, z_i) \in I(A, B; C, D)$ for $i = 1, \ldots, n+1$ and for $A, B, C, D \in 2^S$ with $A \succ^* C$ and $D \succ^* B$. If $(x_1, \ldots, x_{n+1})$ and $(z_1, \ldots, z_{n+1})$ are in $R(A_1, \ldots, A_n; B_1, \ldots, B_n)$, then $(y_1, \ldots, y_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n)$. If, in addition, $(x_1, \ldots, x_{n+1})$ or $(z_1, \ldots, z_{n+1})$ is in $P(A_1, \ldots, A_n; B_1, \ldots, B_n)$, then $(y_1, \ldots, y_{n+1}) \in P(A_1, \ldots, A_n; B_1, \ldots, B_n)$.*
(2) If $(x_1, \ldots, x_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n)$ and $x_k \sim x_{k+1}$ for some $1 \leq k \leq n$, then $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{n+1})$ is in

$$R(A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n; B_1, \ldots, B_{k-1}, B_{k+1}, \ldots, B_n).$$

(3) If $(x_1, \ldots, x_{n+1}) \in X_{\downarrow}^{n+1}$, *and*

$$\begin{aligned}(x_1, \ldots, x_{k+1}) &\in R(A_1, \ldots, A_k; B_1, \ldots, B_k),\\ (x_{k+1}, \ldots, x_{n+1}) &\in R(A_{k+1}, \ldots, A_n; B_{k+1}, \ldots, B_n),\end{aligned}$$

then $(x_1, \ldots, x_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n)$.

Proof. (1) is a restatement of Axiom 7. (2) follows from Axioms 1 and 5. We show (3). By the contraction processes, we can always take $a_1, \ldots, a_{k+1}$ to satisfy that $a_1 \succeq \ldots \succeq a_{k+1} \succeq x_1$ for which there are $b_1, \ldots, b_k$ such that, for $i = 1, \ldots, k$,

$$\begin{aligned}(a_{k+1}, b_i, x_i) &\in I(A, B; C, D),\\ (a_i, b_i, x_{k+1}) &\in I(A, B; C, D).\end{aligned}$$

Let $b_{k+1}, \ldots, b_{n+1}$ satisfy that, for $i = k+1, \ldots, n+1$,

$$(a_{k+1}, b_i, x_i) \in I(A, B; C, D).$$

Applying (1) each step below, we get

$$\begin{aligned}&(x_1, \ldots, x_{k+1}) \in R(A_1, \ldots, A_k; B_1, \ldots, B_k)\\ &\Longrightarrow\ (b_1, \ldots, b_{k+1}) \in R(A_1, \ldots, A_k; B_1, \ldots, B_k)\\ &\Longrightarrow\ (a_1, \ldots, a_{k+1}) \in R(A_1, \ldots, A_k; B_1, \ldots, B_k)\\ &\Longrightarrow\ (a_1, \ldots, a_{k+1}, \ldots, a_{k+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n)\\ &\Longrightarrow\ (b_1, \ldots, b_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n)\\ &\Longrightarrow\ (x_1, \ldots, x_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n).\end{aligned}$$

This completes the proof. □

In what follows, we are to show that

$$\begin{aligned}&(x_1, \ldots, x_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n)\\ &\Longleftrightarrow\ \sum_{i=1}^{n} \phi(A_i, B_i)(u(x_i) - u(x_{i+1})) \geq 0.\end{aligned}$$

If $x_k \sim x_{k+1}$ for some $1 \leq k \leq n$, then by Lemma 5(2) and the induction hypothesis, the desired representation immediately follows. Thus we assume hereafter that $x_1 \succ \cdots \succ x_{n+1}$.

Suppose that $A_k \sim^* B_k$ for some $1 \leq k \leq n$. Then $\phi(A_k, B_k) = 0$. By the contraction process, we can take $a, b, a_1, \ldots, a_n \in X$ and $A, B, C, D \in 2^S$ such that $x_k \succeq a$, $x_{n+1} \succeq b$, and for $i = 1, \ldots, k$ and $j = k, \ldots, n$,

$$\begin{aligned}(a_i, x_i, a) &\in I(A, B; C, D),\\ (a_j, x_{j+1}, b) &\in I(A, B; C, D).\end{aligned}$$

By Proposition 1, we obtain that for $i = 1, \ldots, k$ and $j = k, \ldots, n$,

$$\begin{aligned} u(a_i) &= \frac{1}{\alpha}u(x_i) - \frac{1-\alpha}{\alpha}u(a), \\ u(a_j) &= \frac{1}{\alpha}u(x_{j+1}) - \frac{1-\alpha}{\alpha}u(b). \end{aligned}$$

Since $(a, b) \in I(A_k; B_k)$, it follows from Lemma 5(1) and the induction hypothesis that

$$\begin{aligned} &(x_1, \ldots, x_{n+1}) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n) \\ &\iff (a_1, \ldots, a_{k-1}, a_k, a_k, a_{k+1}, \ldots, a_n) \in R(A_1, \ldots, A_n; B_1, \ldots, B_n) \\ &\iff (a_1, \ldots, a_n) \in R(A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n; B_1, \ldots, B_{k-1}, B_{k+1}, \ldots, B_n) \\ &\iff \sum_{i=1}^{k-1} \phi(A_i, B_i)(u(a_i) - u(a_{i+1})) \\ &\qquad + \sum_{i=k}^{n-1} \phi(A_{i+1}, B_{i+1})(u(a_i) - u(a_{i+1})) \geq 0. \end{aligned}$$

Substituting the expressions of $u(a_i)$ for $i = 1, \ldots, n+1$ for the above, we get the desired result.

If either $A_k \succ^* B_k$ for $k = 1, \ldots, n$ or $B_k \succ^* A_k$ for $k = 1, \ldots, n$, then the desired result readily follows from Axiom 5. Hence we assume that not$(A_k \sim^* B_k)$ for $k = 1, \ldots, n$, and that $A_i \succ^* B_i$ and $B_j \succ^* A_j$ for some distinct $i, j \in \{1, \ldots, n\}$.

We have two cases to examine.

Case 1.	either $A_1 \succ^* B_1$ and $B_n \succ^* A_n$
	or $B_1 \succ^* A_1$ and $A_n \succ^* B_n$.
Case 2.	either $A_1 \succ^* B_1$ and $A_n \succ^* B_n$
	or $B_1 \succ^* A_1$ and $B_n \succ^* A_n$.

Case 1. Suppose that $A_1 \succ^* B_1$ and $B_n \succ^* A_n$. A similar proof applies to another case. We assume that

$$(x_2, \ldots, x_n) \in R(A_2, \ldots, A_{n-1}; B_2, \ldots, B_{n-1}).$$

When $(x_2, \ldots, x_n) \in R(B_2, \ldots, B_{n-1}; A_2, \ldots, A_{n-1})$, the proof is similar. It follows from the contraction process that there are $A, B, C, D \in 2^S$ and $a_0, b_0 \in X$ such that

$$\begin{aligned} (a_0, x_1, x_{n+1}) &\in I(A, B; C, D), \\ (x_1, x_{n+1}, b_0) &\in I(A, B; C, D). \end{aligned}$$

Then take $x \in X$ to satisfy that $x_1 \succ x \succ x_2$, and define $a, a_1, \ldots, a_n \in X$ as follows: for $i = 1, \ldots, n$,

$$\begin{aligned} (x, x_{i+1}, a_i) &\in I(A, B; C, D), \\ (a, x_1, a_1) &\in I(A, B; C, D). \end{aligned}$$

If $(x_2,\ldots,x_{n+1}) \in R(A_2,\ldots,A_n;B_2,\ldots,B_n)$, then it follows from Lemma 5(3) that $(x_1,\ldots,x_{n+1}) \in R(A_1,\ldots,A_n;B_1,\ldots,B_n)$, since $(x_1,x_2) \in R(A_1;B_1)$. Therefore, the desired representation trivially obtians.

In the sequel, we shall assume that $(x_2,\ldots,x_{n+1}) \notin R(A_2,\ldots,A_n;B_2,\ldots,B_n)$. Thus by Lemma 5(1),

$$(a_1,\ldots,a_n) \notin R(A_2,\ldots,A_n;B_2,\ldots,B_n).$$

On the other hand, Lemma 5(1) implies

$$(a_1,\ldots,a_{n-1},a_{n-1}) \in R(A_2,\ldots,A_n;B_2,\ldots,B_n),$$

since $(x_2,\ldots,x_n) \in R(A_2,\ldots,A_{n-1};B_2,\ldots,B_{n-1})$. Hence Axiom 2 implies that there is a $b \in X$ such that

$$(a_1,\ldots,a_{n-1},b) \in I(A_2,\ldots,A_n;B_2,\ldots,B_n).$$

Let $y \in X$ be such that $(y,x_{n+1},b) \in I(A,B;C,D)$. Then by Lemma 5(1) and Proposition 1, we obtain that

$$\begin{aligned}
&(x_1,\ldots,x_{n+1}) \in R(A_1,\ldots,A_n;B_1,\ldots,B_n)\\
&\iff (a,x,\ldots,x,y) \in R(A_1,\ldots,A_n;B_1,\ldots,B_n)\\
&\iff \phi(A_1,B_1)(u(a)-u(x))+\phi(A_n,B_n)(u(x)-u(y)) \geq 0.
\end{aligned}$$

Let $(1-\alpha)/\alpha = \phi(D,C)/\phi(A,B)$. By Propositon 1, we have that for $k = 1,\ldots,n-1$,

$$\begin{aligned}
u(a) &= \frac{1}{\alpha}u(x_1) - \frac{1-\alpha}{\alpha}u(a_1),\\
u(a_k) &= \frac{1}{1-\alpha}u(x_{k+1}) - \frac{\alpha}{1-\alpha}u(x),\\
u(y) &= \frac{1}{\alpha}u(x_{n+1}) - \frac{1-\alpha}{\alpha}u(b),\\
\phi(A_n,B_n)(u(b)-u(a_{n-1})) &= \sum_{i=1}^{n-2}\phi(A_{i+1},B_{i+1})u(a_{i+1})-u(a_i)).
\end{aligned}$$

Substituting the expressions of $u(a_k), k = 1,\ldots,n-1$ for the last equation gives

$$\phi(A_n,B_n)(u(b)-u(a_{n-1})) = \sum_{i=1}^{n-2}\phi(A_{i+1},B_{i+1})(u(x_i)-u(x_{i+1})).$$

Note that $u(x) = \frac{1}{\alpha}u(x_2) - \frac{1-\alpha}{\alpha}u(a_1) = \frac{1}{\alpha}u(x_n) - \frac{1-\alpha}{\alpha}u(a_{n-1})$. Then we have

$$\begin{aligned}
u(a)-u(x) &= \frac{1}{\alpha}(u(x_1)-u(x_2)),\\
u(x)-u(y) &= \frac{1}{\alpha}(u(x_n)-u(x_{n+1})) + \frac{1-\alpha}{\alpha}(u(b)-u(a_{n-1})).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\phi(A_1,B_1)(u(a)-u(y))-\phi(A_n,B_n)(u(x)-u(b))\\
&= \frac{1}{\alpha}\phi(A_1,B_1)(u(x_1)-u(x_2))+\frac{1}{\alpha}(u(x_n)-u(x_{n+1}))+\frac{1-\alpha}{\alpha}(u(b)-u(a_{n-1}))\\
&= \frac{1}{\alpha}\sum_{i=1}^{n}\phi(A_i,B_i)(u(x_i)-u(x_{i+1})).
\end{aligned}$$

Hence the conclusion obtains.

Case 2. Suppose that $A_1 \succ^* B_1$ and $A_n \succ^* B_n$. A similar proof applies to show another case. We assume $(x_1,\ldots,x_n) \notin R(A_1,\ldots,A_{n-1};B_1,\ldots,B_{n-1})$. Otherwise, the desired result follows from Lemma 5(3). Then it is easy to see that there is a $1 \le k < n$ such that

$$\begin{aligned}
(x_1,\ldots,x_k) &\in R(A_1,\ldots,A_{k-1};B_1,\ldots,B_{k-1}),\\
(x_1,\ldots,x_{k+1}) &\notin R(A_1,\ldots,A_k;B_1,\ldots,B_k).
\end{aligned}$$

By the contraction process, take $a_0,a_1,\ldots,a_k,b_0$ to satisfy that $a_0 \succeq x_1$, $x_{n+1} \succeq a_k$, and

$$\begin{aligned}
(a_0,x_i,a_i) &\in I(A,B;C,D) \quad \text{for } i=1,\ldots,k,\\
(a_0,x_{k+1},b_0) &\in I(A,B;C,D).
\end{aligned}$$

By Lemma 5(1), we have

$$\begin{aligned}
(a_1,\ldots,a_k) &\in R(A_1,\ldots,A_{k-1};B_1,\ldots,B_{k-1}),\\
(a_1,\ldots,a_k,b_0) &\notin R(A_1,\ldots,A_k;B_1,\ldots,B_k).
\end{aligned}$$

Thus by Axiom 2, $(a_1,\ldots,a_k,a) \in I(A_1,\ldots,A_k;B_1,\ldots,B_k)$ for some $a \in X$. Then for $i=k+1,\ldots,n+1$, let a_i satisfy $(a_i,x_i,a) \in I(AB;CD)$. Since $(a_1,\ldots,a_k,a) \in I(A_1,\ldots,A_k;B_1,\ldots,B_k)$, it follows from Lemma 5(1) and the induction hypothesis that

$$\begin{aligned}
&(x_1,\ldots,x_{n+1}) \in R(A_1,\ldots,A_n;B_1,\ldots,B_n)\\
&\iff (a_0,a_{k+1},\ldots,a_{n+1}) \in R(A_k,\ldots,A_n;B_k,\ldots,B_n)\\
&\iff \phi(A_k,B_k)(u(a_0)-u(a_{k+1}))+\sum_{i=k+1}^{n}\phi(A_i,B_i)(u(a_i)-u(a_{i+1})) \ge 0.
\end{aligned}$$

Let $(1-\alpha)/\alpha = \phi(D,C)/\phi(A,B)$. By Proposition 1, we have that for $i=1,\ldots,k$ and $j=k+1,\ldots,n+1$,

$$\begin{aligned}
u(a_i) &= \frac{1}{1-\alpha}u(x_i)-\frac{\alpha}{1-\alpha}u(a_0),\\
u(a_j) &= \frac{1}{\alpha}u(x_j)-\frac{1-\alpha}{\alpha}u(a),\\
u(a)-u(a_k) &= \frac{1}{\phi(A_k,B_k)}\sum_{i=1}^{k-1}\phi(A_i,B_i)(u(a_i)-u(a_{i+1})).
\end{aligned}$$

Substituting the expressions of $u(a_i)$ for $i = 1, \ldots, k-1$ for the last equation gives

$$(1-\alpha)\phi(A_k, B_k)(u(a) - u(a_k)) = \sum_{i=1}^{k-1} \phi(A_i, B_i)(u(x_i) - u(x_{i+1})).$$

Substituting the expressions of $u(a_j)$ for $j = 1, \ldots, n$ for the following we get

$$\sum_{i=k+1}^{n} \phi(A_i, B_i)(u(a_i) - u(a_{i+1})) = \frac{1}{\alpha} \sum_{i=k+1}^{n} \phi(A_i, B_i)(u(x_i) - u(x_{i+1})).$$

Since $u(a_0) = \frac{1}{\alpha}u(x_{k+1}) - \frac{1-\alpha}{\alpha}u(a)$ and $u(a_{k+1}) = \frac{1}{\alpha}u(x_k) - \frac{1-\alpha}{\alpha}u(a_k)$, we have

$$\begin{aligned} & u(a_0) - u(a_{k+1}) \\ &= \frac{1}{\alpha}(u(x_k) - u(x_{k+1}) - \frac{1-\alpha}{\alpha}(u(a_k) - u(a)) \\ &= \frac{1}{\alpha}(u(x_k) - u(x_{k+1})) + \frac{1}{\alpha\phi(A_k, B_k)} \sum_{i=1}^{k-1} \phi(A_i, B_i)(u(x_i) - u(x_{i+1})). \end{aligned}$$

Hence

$$\begin{aligned} & (a_0, a_{k+1}, \ldots, a_{n+1}) \in R(A_k, \ldots, A_n; B_k, \ldots, B_n) \\ &= \frac{1}{\alpha} \sum_{i=1}^{n} \phi(A_i, B_i)(u(x_i) - u(x_{i+1}) \geq 0, \end{aligned}$$

so that the desired result obtains.

6 Conclusions

The purpose of this paper has been to propose a generalization of subjective expected utility without assuming transitivity of preferences and additivity of subjective state probabilities referred to nontransitive CEU model. While preferences for acts may not be transitive, our model requires that preferences for constant acts must be weakly ordered, so that a utility function over the consequence space exists. Also, our qualitaive probability relation causes nontransitivity, so that a numerical representation for that relation is given by a nonseparable bivariate set function. Eight axioms are identified to be sufficient for that model when the state space is finite.

Appendix

This appendix shows the proofs of Step 4 of Proposition 1. However, we simply show in each case how to apply Lemma 3, since the proof method exactly follows the proof in the text.

We have following cases to examine.

Case 1. not($A \sim^* \{s\}$) and not($B \sim^* \{s\}$),
Case 2. not($A \sim^* \{s\}$) and $B \sim^* \{s\}$,
Case 3. $A \sim^* \{s\}$ and not($B \sim^* \{s\}$),
Case 4. $A \sim^* \{s\}$ and $B \sim^* \{s\}$.

Case 1. Suppose that not($A \sim^* \{s\}$) and not($B \sim^* \{s\}$). We have the following four subcases to examine:

Subcase 1.1. $A \succ^* \{s\}$ and $B \succ^* \{s\}$,
Subcase 1.2. $A \succ^* \{s\}$ and $\{s\} \succ^* B$,
Subcase 1.3. $\{s\} \succ^* A$ and $B \succ^* \{s\}$,
Subcase 1.4. $\{s\} \succ^* A$ and $\{s\} \succ^* B$.

Subcase 1.1 is given in the text.

Subcase 1.2. $A \succ^* \{s\} \succ^* B$. Let $(A_1, A_2) = (A, \{s\})$, $(B_1, B_2) = (\{s\}, B)$, $\mathcal{N} = \{1, 2\}$, $\mathcal{N}^* = \emptyset$, and $\mathcal{N}_0^* = \{2\}$ in Lemma 3.

By Steps 1 and 2, we get

$$\begin{aligned}\phi(B, A) &= \frac{\omega(\{s\}A; SA)\omega(\emptyset\{s\}; A\{s\})}{\omega(BA; SA)}\phi(\emptyset, \{s\}),\\ \phi(A, B) &= -\frac{\omega(\emptyset B; AB)\omega(\emptyset\{s\}; S\{s\})}{\omega(\emptyset B; \{s\}B)\omega(B\{s\}; S\{s\})}\phi(\emptyset, \{s\}),\end{aligned}$$

so the chain rule implies that $\phi(B, A) = -\phi(A, B)$.

Subcase 1.3. $B \succ^* \{s\} \succ^* A$. Let $(A_1, A_2, A_3) = (B, A, \{s\})$, $(B_1, B_2, B_3) = (\{s\}, B, A)$, $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{N}^* = \emptyset$, and $\mathcal{N}_0^* = \{1, 2, 3\}$ in Lemma 3.

By Steps 1 and 2, we get

$$\begin{aligned}\phi(B, A) &= \frac{\omega(\emptyset A; SA)\omega(\emptyset\{s\}; S\{s\})}{\omega(BA; SA)\omega(\emptyset A; \{s\}A)\omega(A\{s\}; S\{s\})}\phi(\emptyset, \{s\}),\\ \phi(A, B) &= -\frac{\omega(\emptyset B; AB)\omega(\{s\}B; SB)\omega(\emptyset\{s\}; B\{s\})}{\omega(\emptyset B; SB)}\phi(\emptyset, \{s\}),\end{aligned}$$

so by the chain rule, $\phi(B, A) = -\phi(A, B)$.

Subcase 1.4. $\{s\} \succ^* A$ and $\{s\} \succ^* B$. Let $(A_1, A_2) = (A, \{s\})$, $(B_1, B_2) = (B, A)$, $\mathcal{N} = \{1, 2\}$, $\mathcal{N}^* = \emptyset$ and $\mathcal{N}_0^* = \{1\}$ in Lemma 3.

By Steps 1 and 2,

$$\begin{aligned}\phi(B, A) &= -\frac{\omega(\emptyset A; SA)}{\omega(BA; SA)\omega(\emptyset A; \{s\}A)\omega(A\{s\}; S\{s\})}\phi(S, \{s\}),\\ \phi(A, B) &= \frac{\omega(\emptyset B; AB)}{\omega(\emptyset B; \{s\}B)\omega(B\{s\}; S\{s\})}\phi(S, \{s\}),\end{aligned}$$

so by the chain rule, $\phi(B,A) = -\phi(A,B)$.

Case 2. not$(A \sim^* \{s\})$ and $B \sim^* \{s\}$. We have two subcases to examine:

$$\begin{array}{ll} \text{Subcase 2.1} & A \succ^* \{s\} \text{ and } B \sim^* \{s\}, \\ \text{Subcase 2.2} & \{s\} \succ^* A \text{ and } B \sim^* \{s\}, \end{array}$$

Subcase 2.1. $A \succ^* \{s\}$ and $B \sim^* \{s\}$. If $B_s \neq S$, then let $(A_1, A_2) = (A, B_s)$, $(B_1, B_2) = (\{s\}, B)$, $\mathcal{N} = \{1,2\}$, $\mathcal{N}^* = \{2\}$ and $\mathcal{N}_0^* = \{2\}$ in Lemma 3. If $B_s = S$, then $s \notin B$ and $C \subset B$ for some $C \in \Gamma^*$, so let $(A_1, A_2, A_3) = (A, B, C_s)$, $(B_1, B_2, B_3) = (\{s\}, B, C)$, $\mathcal{N} = \{1,2,3\}$, $\mathcal{N}^* = \{2,3\}$, and $\mathcal{N}_0^* = \{2,3\}$ in Lemma 3.

By Steps 1 and 2,

$$\begin{aligned} \phi(B,A) &= \frac{\omega(\{s\}A; SA)\omega(\emptyset\{s\}; A\{s\})}{\omega(BA; SA)}\phi(\emptyset\{s\}), \\ \phi(A,B) &= -\frac{\omega(\emptyset B; AB)\omega(\emptyset\{s\}; SB_s)}{\omega(\emptyset B; SB_s)}\phi(\emptyset, \{s\}) \quad \text{if } B_s \neq S, \\ \phi(A,B) &= -\frac{\omega(\emptyset B; AB)\omega(\emptyset C; SB)}{\omega(\emptyset B; SB)\omega(\emptyset C; SC_s)}\phi(\emptyset, \{s\}) \quad \text{if } B_s = S, \end{aligned}$$

so by the chain rule, $\phi(B,A) = -\phi(A,B)$.

Subcase 2.2. $\{s\} \succ^* A$ and $B \sim^* \{s\}$. If $B_s \neq S$, then let $(A_1, A_2, A_3) = (A, \{s\}, B_s)$, $(B_1, B_2, B_3) = (B, A, \{s\})$, $\mathcal{N} = \{1,2,3\}$, $\mathcal{N}^* = \{3\}$ and $\mathcal{N}_0^* = \{1,2,3\}$ in Lemma 3. If $B_s = S$, then $s \notin B$ and $C \subset B$ for some $C \in \Gamma^*$, so let $(A_1, A_2, A_3, A_4) = (B, C_s, \{s\}, A)$, $(B_1, B_2, B_3, B_4) = (B, C, \{s\}, A)$, $\mathcal{N} = \{1,2,3,4\}$, $\mathcal{N}^* = \{1,2,3,4\}$ and $\mathcal{N}_0^* = \{1,2\}$ in Lemma 3.

By Steps 1 and 2,

$$\begin{aligned} \phi(B,A) &= \frac{\omega(\emptyset A; SA)\omega(\emptyset\{s\}; S\{s\})}{\omega(BA; SA)\omega(\emptyset A; \{s\}A)\omega(A\{s\}; S\{s\})}\phi(\emptyset, \{s\}), \\ \phi(A,B) &= -\frac{\omega(\emptyset B; AB)\omega(\emptyset\{s\}; SB_s)}{\omega(\emptyset B; SB_s)}\phi(\emptyset, \{s\}) \quad \text{if } B_s \neq S, \\ \phi(A,B) &= -\frac{\omega(\emptyset B; AB)\omega(\emptyset C; SB)\omega(\emptyset\{s\}; SC_s)}{\omega(\emptyset B; SB)\omega(\emptyset C; SC_s)}\phi(\emptyset, \{s\}) \quad \text{if } B_s = S, \end{aligned}$$

so by the chain rule, $\phi(B,A) = -\phi(A,B)$.

Case 3. $A \sim^* \{s\}$ and not$(B \sim^* \{s\})$. We have the following two subcases to examine:

$$\begin{array}{ll} \text{Subcase 3.1} & A \sim^* \{s\} \text{ and } B \succ^* \{s\}, \\ \text{Subcase 3.2} & A \sim^* \{s\} \text{ and } \{s\} \succ^* B. \end{array}$$

Subcase 3.1. $A \sim^* \{s\}$ and $B \succ^* \{s\}$. If $A_s \neq S$, then let $(A_1, A_2, A_3) = (A, A_s, B)$, $(B_1, B_2, B_3) = (B, A, \{s\})$, $\mathcal{N} = \{1,2,3\}$, $\mathcal{N}^* = \{2\}$, and $\mathcal{N}_0^* = \{1,2,3\}$ in Lemma 3. If $A_s = S$, then $s \notin A$ and $C \subset A$ for some $C \in \Gamma^*$, so let $(A_1, A_2, A_3) =$

(A, C_s, B), $(B_1, B_2, B_3) = (B, C, \{s\})$, $\mathcal{N} = \{1,2,3\}$, $\mathcal{N}^* = \{2\}$, and $\mathcal{N}_0^* = \{1,2,3\}$ in Lemma 3.

By Steps 1 and 2,

$$\begin{aligned}
\phi(A,B) &= -\frac{\omega(\emptyset B; AB)\omega(\{s\}B; SB)\omega(\emptyset\{s\}; B\{s\})}{\omega(\emptyset B; SB)}\phi(\emptyset, \{s\}), \\
\phi(B,A) &= \frac{\omega(\emptyset A; SA)\omega(\emptyset\{s\}; SA_s)}{\omega(BA; SA)\omega(\emptyset A; SA_s)}\phi(\emptyset, \{s\}) && \text{if } A_s \neq S, \\
\phi(B,A) &= \frac{\omega(\emptyset C; SA)\omega(\emptyset\{s\}; SC_s)}{\omega(BA; SA)\omega(\emptyset C; SC_s)}\phi(\emptyset, \{s\}) && \text{if } A_s = S,
\end{aligned}$$

so by the chain rule, $\phi(B,A) = -\phi(A,B)$.

Subcase 3.2. $A \sim^* \{s\}$ and $\{s\} \succ^* B$. If $A_s \neq S$, then let $(A_1, A_2, A_3) = (A, A_s, \{s\})$, $(B_1, B_2, B_3) = (B, A, \{s\})$, $\mathcal{N} = \{1,2,3\}$, $\mathcal{N}^* = \{2,3\}$, and $\mathcal{N}_0^* = \{1,2\}$ in Lemma 3. If $A_s = S$, then $s \notin A$ and $C \subset A$ for some $C \in \Gamma^*$, so let $(A_1, A_2, A_3) = (A, C_s, \{s\})$, $(B_1, B_2, B_3) = (B, C, \{s\})$, $\mathcal{N} = \{1,2,3\}$, $\mathcal{N}^* = \{2,3\}$, and $\mathcal{N}_0^* = \{1,2\}$ in Lemma 3.

By Steps 1 and 2,

$$\begin{aligned}
\phi(A,B) &= -\frac{\omega(\emptyset B; AB)\omega(\emptyset\{s\}; S\{s\})}{\omega(\emptyset B; \{s\}B)\omega(B\{s\}; S\{s\})}\phi(\emptyset, \{s\}) \\
\phi(B,A) &= \frac{\omega(\emptyset A; SA)\omega(\emptyset\{s\}; SA_s)}{\omega(BA; SA)\omega(\emptyset A; SA_s)}\phi(\emptyset, \{s\}) && \text{if } A_s \neq S, \\
\phi(B,A) &= \frac{\omega(\emptyset C; SA)\omega(\emptyset\{s\}; SC_s)}{\omega(BA; SA)\omega(\emptyset C; SC_s)}\phi(\emptyset, \{s\}) && \text{if } A_s = S,
\end{aligned}$$

so by the chain rule, $\phi(B,A) = -\phi(A,B)$.

Case 4. $A \sim^* \{s\}$ and $B \sim^* \{s\}$. If $A_s \neq S$, then by Subcase 3.1,

$$\phi(B,A) = \frac{\omega(\emptyset A; SA)\omega(\emptyset\{s\}; SA_s)}{\omega(BA; SA)\omega(\emptyset A; SA_s)}\phi(\emptyset, \{s\}).$$

If $A_s = S$, then $s \notin A$ and $C \subset A$ for some $C \in \Gamma^*$, so by Subcase 3.1,

$$\phi(B,A) = \frac{\omega(\emptyset C; SA)\omega(\emptyset\{s\}; SC_s)}{\omega(BA; SA)\omega(\emptyset C; SC_s)}\phi(\emptyset, \{s\}).$$

If $B_s \neq S$, then by Subcase 2.2,

$$\phi(A,B) = \frac{\omega(\emptyset B; AB)\omega(\emptyset\{s\}; SB_s)}{\omega(\emptyset B; SB_s)}\phi(\emptyset, \{s\}).$$

If $B_s = S$, then $s \notin B$ and $C' \subset B$ for some $C' \in \Gamma^*$, soby Subcase 2.2,

$$\phi(A,B) = -\frac{\omega(\emptyset B; AB)\omega(\emptyset C'; SB)\omega(\emptyset\{s\}; SC'_s)}{\omega(\emptyset B; SB)\omega(\emptyset C'; SC'_s)}\phi(\emptyset, \{s\}).$$

If $A_s \neq S$ and $B_s \neq S$, then let $(A_1, A_2, A_3) = (A, B_s, A_s)$, $(B_1, B_2, B_3) = (A, B, \{s\})$, $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{N}^* = \{1, 2, 3\}$, and $\mathcal{N}_0^* = \{2, 3\}$ in Lemma 3. If $A_s = S$ and $B_s \neq S$, then let $(A_1, A_2, A_3) = (A, B_S, C_s)$, $(B_1, B_2, B_3) = (C, B, \{s\})$, $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{N}^* = \{1, 2, 3\}$, and $\mathcal{N}_0^* = \{2, 3\}$ in Lemma 3. If $A_s \neq S$ and $B_s = S$, then let $(A_1, A_2, A_3, A_4) = (A, B, C_s', A_s)$, $(B_1, B_2, B_3, B_4) = (A, B, C', \{s\})$, $\mathcal{N} = \{1, 2, 3, 4\}$, $\mathcal{N}^* = \{1, 2, 3, 4\}$, and $\mathcal{N}_0^* = \{2, 3, 4\}$ in Lemma 3. The case of $A_s = B_s = S$ is impossible.

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