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A Supplement for D. P. No. 713

by

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Summary.

This paper is added as a supplement for D. P. No. 713.

This paper is added as a supplement to the proof of UMP-unbiasedness for $\phi^*(y)$ given by (5) in D.P. No. 713.

Statement: Let X1, ..., Xn be a random sample from p.d.f. $f(x|\theta)=1$ for $\theta \leq x < \theta+1$; $=0$, otherwise. The test ϕ^* given by (5) in D. P. No. 713 for testing $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ is UMP-unbiased size-α.

Proof.) Let $Y \doteq (X_{(1)}+X_{(n)}-1)/2$. The p.d.f. of Y is given by

(1) $$g(y|\theta)=n(1-2|y-\theta|)^{n-1}I_{(-1/2,\ 1/2)}(y-\theta)\ (\doteq g(y-\theta)).$$

Let ϕ be a size-α test symmetric about θ. Let $\pi_\phi(\theta)\doteq E_\theta(\phi(Y))$ and we abbreviate $\phi_{\theta_0}(y)$ as $\phi(y)$. Then, it follows that for given α such that $0<\alpha<1$,

(2) $$\pi_\phi(\theta_0)=E_{\theta_0}(\phi(Y))=\alpha$$

and

(3) $$E_{\theta_0}(Y\phi(Y))=\theta_0 E_{\theta_0}(\phi(Y))=\theta_0\alpha.$$

Above (3) holds because $E_{\theta_0}((Y-\theta_0)\phi(Y))=0$ and (2) holds.

Hence, by generalized fundamental lemma, ϕ maximizes the integral

(4) $$\int \phi(y)g(y|\theta')dy \qquad \text{for } \theta'\neq\theta_0$$

when there exist k_1 and k_2 such that

(5) $$\phi(y)=\begin{cases} 1, & \text{if } (1-2|y-\theta'|)^{n-1}I_{(-1/2,\ 1/2)}(y-\theta') \\ & \quad \geq k_2(1-2|y-\theta_0|)^{n-1}I_{(-1/2,\ 1/2)}(y-\theta_0) \\ & \quad\quad +k_1 y(1-2|y-\theta_0|)^{n-1}I_{(-1/2,\ 1/2)}(y-\theta_0), \\ 0, & \text{if } (1-2|y-\theta'|)^{n-1}I_{(-1/2,\ 1/2)}(y-\theta') \\ & \quad < k_2(1-2|y-\theta_0|)^{n-1}I_{(-1/2,\ 1/2)}(y-\theta_0) \\ & \quad\quad +k_1 y(1-2|y-\theta_0|)^{n-1}I_{(-1/2,\ 1/2)}(y-\theta_0). \end{cases}$$

In order for $\pi_\phi(\theta)$ to be minimized at $\theta=\theta_0$, the region of y must have two-sided intervals.

Let $h(y)=(1-2|y-\theta'|)/(1-2|y-\theta_0|)$ (>0). The first inequality in (5) (we denote this by FI(5).) becomes

(6) $h(y)\geq(k_2+k_1y)^{1/(n-1)}$ $(\doteq z(y))$

(7) for $\theta_0<\theta'$ & $\theta'-1/2<y<\theta_0+1/2$, or for $\theta'<\theta_0$ & $\theta_0-1/2<y<\theta'+1/2$.

Let $\theta_0<\theta'$. Then, for $\theta_0-1/2<y<\theta'-1/2$, FI(5) holds when $k_1=k_2=0$, say. On the other hand, for $\theta_0+1/2<y<\theta'+1/2$, FI(5) always holds for any real k_1 and k_2. Let $\theta'<\theta_0$. Then, for $\theta'-1/2<y<\theta_0-1/2$, FI(5) always holds for any real k_1 and k_2. On the other hand, for $\theta'+1/2<y<\theta_0+1/2$, FI(5) holds when $k_1=k_2=0$, say.

Since $(0<)h(\theta_0)=1-2|\theta_0-\theta'|(<1)$, the graph of h(y) becomes as solid lines in the figure, below. We now examine (6). In order to get two-sided intervals for y according to (6), we take $k_1>0$ for $\theta_0<\theta'$ and $k_1<0$ for $\theta'<\theta_0$ and for simplicity we assume $z(\theta'-1/2)=0$ for $\theta_0<\theta'$ and $z(\theta'+1/2)=0$ for $\theta'<\theta_0$. Then, k_2 is taken as follows:

(8) $$k_2=\begin{cases} -k_1(\theta'-1/2), & \text{for } \theta_0<\theta', \\ -k_1(\theta'+1/2), & \text{for } \theta'<\theta_0. \end{cases}$$

Substituting these values into z(y) we have

(9) $$z(y)=\begin{cases} (k_1(y-\theta'+1/2))^{1/(n-1)}, & \text{for } \theta_0<\theta', \\ (k_1(y-\theta'-1/2))^{1/(n-1)}, & \text{for } \theta'<\theta_0, \end{cases}$$

which is drawn by stripe lines in the figure, below.

Since we need $h(\theta_0)<z(\theta_0)$, we take $k_1=2$ for $\theta_0<\theta'$ and $k_1=-2$ for $\theta'<\theta_0$. Then, $z^{n-1}(\theta_0)=h(\theta_0)$, for all n and hence, there exists an integer n_0 such that for all $n>n_0$, $h(\theta_0)<z(\theta_0)(<1)$.

Therefore, if we denote by y_1 and y_2, y-coordinates of the points of intersection of z(y) and h(y) for $\theta'<\theta_0$ and for $\theta'>\theta_0$, respectively, then our two-sided intervals are given as follows:

(10) $y<\theta'-1/2$ or $y>y_2$,for $\theta_0<\theta'$,

(11) $y<y_1$ or $y>\theta'+1/2$,for $\theta'<\theta_0$.

Now, we let $y_0=-k_2/k_1$. For $\theta_0<\theta'$, take $\theta'-1/2<y_0<\theta_0$ and for $\theta'<\theta_0$, take $\theta_0<y_0<\theta'+1/2$. Let p be a given number such that $0\leq p\leq 1$. Take $y_0=(1-p)\theta_0+p(\theta'-1/2)$, for $\theta_0<\theta'$ and $y_0=(1-p)\theta_0+p(\theta'+1/2)$, for $\theta'<\theta_0$. (Note that when p=1, we get the first case.) Then, k_2 is taken as follows:

(12) $$k_2=\begin{cases} -k_1((1-p)\theta_0+p(\theta'-1/2)), & \text{for } \theta_0<\theta' \\ -k_1((1-p)\theta_0+p(\theta'+1/2)), & \text{for } \theta'<\theta_0. \end{cases}$$

Substituting these values into z(y) we have

(13) $$z(y)=\begin{cases} [k_1\{y-((1-p)\theta_0+p(\theta'-1/2))\}]^{1/(n-1)}, & \text{for } \theta_0<\theta', \\ [k_1\{y-((1-p)\theta_0+p(\theta'+1/2))\}]^{1/(n-1)}, & \text{for } \theta'<\theta_0. \end{cases}$$

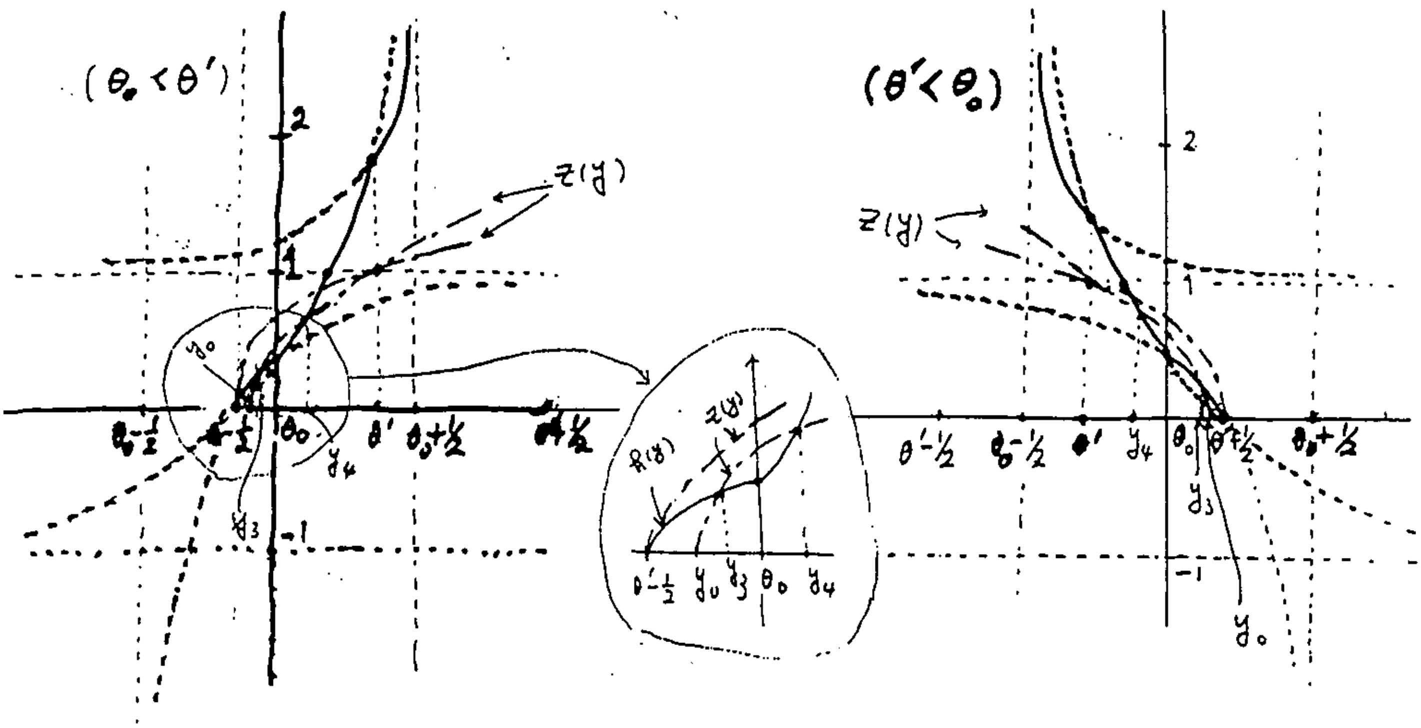
which is drawn by stripe lines in the figure, below.
As before, since we need $h(\theta_0)<z(\theta_0)$, we take $k_1=2/p$ for $\theta_0<\theta'$ and $k_1=-2/p$ for $\theta'<\theta_0$. Then, $(z(\theta_0))^{n-1}=h(\theta_0)$, for all n and hence, there exists an integer n_0 such that for all $n>n_0$, $h(\theta_0)<z(\theta_0)(<1)$.

Since $z(y)\geq 0$ and $dz(y)/dy>0$ for all $y\geq y_0$, we get two intersection points of z(y) and h(y) for $\theta'<\theta_0$ and for $\theta_0<\theta'$, respectively. Let y_3 and y_4 be such y-coordinates of those intersection points. Then, we finally have the UMP-test of form

(14) $$\phi(y)=\begin{cases} 1, & \text{if } y<y_3 \text{ or } y>y_4 \\ 0, & \text{if } y_3<y<y_4. \end{cases}$$

If we take $y_3=\theta_0-r$ and $y_4=\theta_0+r$ $(r\doteq(1-\alpha^{1/n})/2)$, we get ϕ^* in (5) in D. P. No. 713. This is unbiased size-α from its construction. Hence, ϕ^* is UMP-unbiased size-α. (q.e.d.)

FIGURE. The Graph of h(y).



Note that above result is immediately generalized to the case that the random sample of X_1, ... ,X_n is taken from the density $f(x|\theta)=c_0^{-1}I_{(\delta_1,\delta_2)}(y-\theta)$ $(\delta_1<\delta_2)$ where $c_0\doteq\delta_2-\delta_1$. Let $Y\doteq 2^{-1}(X_{(1)}+X_{(n)})-\delta_0$ $(\delta_0\doteq(\delta_1+\delta_2)/2)$. Then, the p.d.f. of Y is given of form

(15) $g(y|\theta)=nc_0^{-n}\{c_0-2|y-\theta|\}^{n-1}$, for $-c_0/2<y-\theta<c_0/2$.

Hence, (6) becomes

(16) $h(y)= (c_0-2|y-\theta'|)/(c_0-2|y-\theta_0|) \geq (k_2+k_1y)^{1/(n-1)}$,

for $\theta_0<\theta'$ & $\theta'-c_0/2<y<\theta_0+c_0/2$ or for $\theta'<\theta_0$ & $\theta_0-c_0/2<y<\theta'+c_0/2$,

so on.

So, we can proceed the proof in the same way as above and get the UMP-test of form (14).

Remark. The author is extending the results of D.P.'s No.507 (1992), No. 627 (1995), No.713 (1997) and this paper. Those will come up soon.